General Physics II: Tutorial Material

Lecture 13 (Chapter 10 & 11, Thermodynamic potentials and Heat transfer)

1) Isothermal heat of surface expansion: A system consists of a thin film of surface area A, of internal energy $E_{int}(S, A)$, where $dE_{int} = TdS + \gamma dA$. γ is the surface tension given by:

$$\gamma(S, A) = \frac{\partial E_{int}(S, A)}{\partial A} \tag{1}$$

Express the heat Q_{if} to provide to the film for a variation of $\Delta A_{if} = A_f - A_i$ of the surface of the film through an isothermal process at temperature T, that brings the film from an initial state i to a final state f, in terms of $\gamma(T, A)$ and its partial derivatives.

Solution:

Perform a Legendre transformation on the internal energy $E_{int}(S, A)$ with respect to the entropy to define the free energy and derive its differential:

$$\begin{aligned} &F(T, A) = E_{int} - TS \\ &dF(T, A) = dE_{int} - TdS - SdT = TdS + \gamma dA - TdS - SdT = -SdT + \gamma dA, \end{aligned} \tag{2}$$

where,

$$S = -\frac{\partial F}{\partial T} \quad \text{and} \quad \gamma = \frac{\partial F}{\partial A}. \tag{3}$$

For an isothermal process, we can compute the heat Q_{if} as

$$Q_{if} = T\Delta S_{if} = T\frac{\partial S}{\partial A}\Delta A_{if}. \tag{4}$$

The Schwartz theorem applied to the free energy $F(T, A)$ yields:

$$\frac{\partial}{\partial A}\left(\frac{\partial F}{\partial T}\right) = \frac{\partial}{\partial T}\left(\frac{\partial F}{\partial A}\right) \tag{5}$$

which leads to the Maxwell relation of this system:

$$\frac{\partial S}{\partial A} = \frac{\partial \gamma}{\partial T}. \tag{6}$$

Hence, the heat in terms of γ and its partial derivatives is given by:

$$Q_{if} = -T\frac{\partial \gamma}{\partial T}\Delta A_{if}. \tag{7}$$

2) Two questions related to thermal conductivity and Fourier's law for the heat flow rate

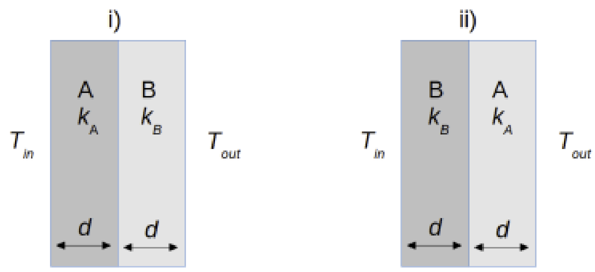
$$\dot{Q} \equiv \frac{dQ}{dt} = -kA\frac{dT}{dx} \tag{8}$$

I) Let us consider a wall consisting of two plates, A and B: both plates have a thickness d with thermal conductivities, k_A and k_B, respectively, and $k_A > k_B$. We use this wall for a house and can make i) surface A facing inside of the house and ii) surface B facing inside of the house. In winter when the outside temperature, T_{out}, is lower than the room temperature T_{in}, i.e., $T_{in} > T_{out}$.

a) Calculate the heat flow rate from the room to outside through wall with configuration i).

b) Calculate the heat flow rate from the room to outside through wall with configuration ii).

c) Are the temperature profiles through the wall from the inside to the outside surface for the two configurations same or different? Which configuration loses more thermal energy to outside?



Solution:

Fourier's law for a surface area A, can be written as

$$\dot{Q}dx = -kAdT. \tag{9}$$

Place the inner surface of the wall at $x = 0$ and denote the temperature at the boarder of the two layers to be T_m.

a) For Configuration i), integrations of the above equation over x for the left side and T for the right side give, for the first layer and second layer, to be

$$\dot{Q}\int_0^d dx = -k_A A \int_{T_{in}}^{T_m} dT \quad \text{and} \quad \dot{Q}\int_d^{2d} dx = -k_B A \int_{T_m}^{T_{out}} dT \tag{10}$$

and leads to

$$\dot{Q} = -\frac{k_A A(T_m - T_{in})}{d} \quad \text{and} \quad \dot{Q} = -\frac{k_B A(T_{out} - T_m)}{d} \tag{11}$$

Since the two heat rates must be identical, T_m can be obtained as

$$k_A(T_m - T_{in}) = k_B(T_{out} - T_m) \tag{12}$$

$$T_m(k_A + k_B) = k_A T_{in} + k_B T_{out} \tag{13}$$

$$T_m = \frac{k_A T_{in} + k_B T_{out}}{k_A + k_B} \tag{14}$$

Inserting this to the expression of Q, we obtain

$$\dot{Q} = -\frac{k_A k_B}{k_A + k_B}\frac{A}{d}(T_{out} - T_{in}) \tag{15}$$

Note that for $k_A = k_B = k$, it gives

$$\dot{Q} = -\frac{kA(T_{out} - T_{in})}{2d} \tag{16}$$

i.e. identical to a $2d$ thick wall with a thermal conductivity k.

b) In a similar manner, we obtain for the Configuration ii)

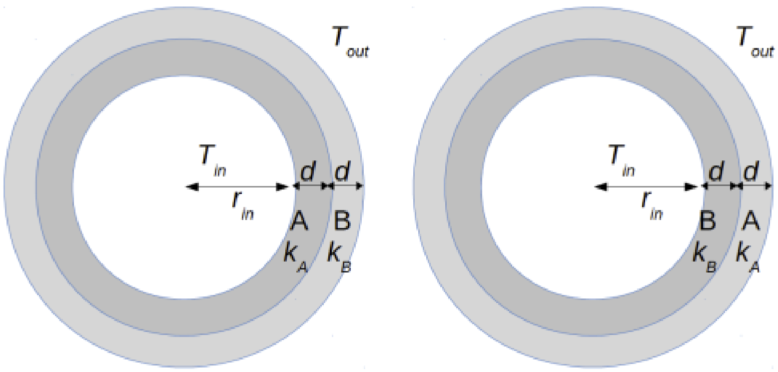
$$T_m = \frac{k_b T_{in} + k_a T_{out}}{k_a + k_b} \tag{17}$$

and

$$\dot{Q} = -\frac{k_a k_b}{k_a + k_b}\frac{A}{d}(T_{out} - T_m) \tag{18}$$

c) The temperature at the border of the two layers, T_m, is different for the two configurations, thus the temperature profiles in the wall are different. Since $k_a > k_b$ and $T_{in} > T_{out}$, the T_m is higher for the Configuration i), i.e. the temperature drops slower in the first layer in Configuration i) than in ii). However, the heat rates are identical for the two configurations. Therefore, the loss of thermal energy in the two configurations are identical.

II) A pipe consists of the two layers of material with a same thickness d. The inner radius of the pipe is r_{in}. Two material A and B with thermal conductivities, k_A and k_B, respectively, are available for the layers where $k_A > k_B$. This pipe is used to transport hot water with a temperature T_{in} through cold outside with a temperature of T_{out}, where $T_{in} > T_{out}$. Figures below show the cross-sections of the pipes.



a) Calculate the heat rate from the water to outside for a pipe where the inner layer with material A.

b) Calculate the heat rate from the water to outside for a pipe where the inner layer with material B.

c) Are the radial temperature profiles different between the two configurations? Which configuration loses more thermal energy to outside?

Solution:

For a pipe, i.e., a cylindrical geometry, Fourier's law becomes

$$\dot{Q}\frac{dr}{r} = -2\pi k l dT \tag{19}$$

where l is the length of the pipe.

a) For configuration i), the heat rate for the first layer is given by the integration of the above formula as

$$\begin{aligned} \dot{Q} \int_{r_{in}}^{r_{in}+d} \frac{dr}{r} &= -2\pi k_A l \int_{T_{in}}^{T_m} dT \\ \dot{Q} &= \frac{-2\pi k_A l}{\ln[(r_{in}+d)/r_{in}]} (T_m - T_{in}) \end{aligned} \tag{20}$$

and for the second layer

$$\begin{aligned} \dot{Q} \int_{r_{in}+d}^{r_{in}+2d} \frac{dr}{r} &= -2\pi k_B l \int_{T_m}^{T_{out}} dT \\ \dot{Q} &= \frac{-2\pi k_B l}{\ln[(r_{in}+2d)/(r_{in}+d)]} (T_{out} - T_m) \end{aligned} \tag{21}$$

where T_m is the temperature at the border of the two layers. Since the two heat rates must be identical:

$$\frac{k_A}{\ln[(r_{in}+d)/r_{in}]} (T_m - T_{in}) = \frac{k_B}{\ln[(r_{in}+2d)/r_{in}+d]} (T_{out} - T_m) \tag{22}$$

and T_m can be obtained as

$$T_m = \frac{\frac{k_a T_{in}}{\ln(1+\delta)} + \frac{k_b T_{out}}{\ln[(1+2\delta)/(1+\delta)]}}{\frac{k_a}{\ln(1+\delta)} + \frac{k_b}{\ln[(1+2\delta)/(1+\delta)]}} \tag{23}$$

$$T_m = \frac{k_a T_{in} \ln[(1+2\delta)] + (k_b T_{out} - k_a T_{in}) \ln(1+\delta)}{k_a \ln[(1+2\delta)] + (k_b - k_a) \ln(1+\delta)} \tag{24}$$

where $\delta = d/r_{in}$. By inserting T_m to the expression of $\dot{Q}$, we obtain

$$\dot{Q} = -\frac{2\pi l k_a k_b}{k_a \ln[(1+2\delta)] + (k_b - k_a) \ln(1+\delta)} (T_{out} - T_{in}) \tag{25}$$

b) Similarly for the Configuration ii), we obtain

$$T_m = \frac{k_b T_{in} \ln(1+2\delta) + (k_a T_{out} - k_b T_{in}) \ln(1+\delta)}{k_b \ln(1+2\delta) + (k_a - k_b) \ln(1+\delta)} \tag{26}$$

$$\dot{Q} = -\frac{2\pi l k_a k_b}{k_b \ln[(1+2\delta)] + (k_a - k_b) \ln(1+\delta)} (T_{out} - T_{in}) \tag{27}$$

c) Two configurations have different T_m. Therefore, the temperature profiles in the radial direction are different for the two different configurations of pipes. The difference in the thermal energy lost to outside is given by

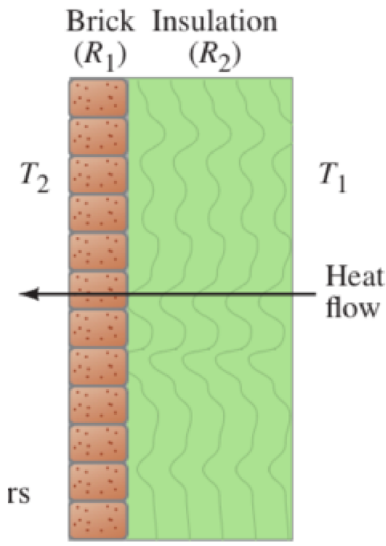
$$\begin{aligned} |\dot{Q}_1| - |\dot{Q}_2| &= \frac{2\pi l k_a k_b}{k_a \ln(1+2\delta) + (k_b - k_a) \ln(1+\delta)} (T_{in} - T_{out}) \\ &\quad - \frac{2\pi l k_a k_b}{k_b \ln(1+2\delta) + (k_a - k_b) \ln(1+\delta)} (T_{in} - T_{out}) \\ &= 2\pi l k_a k_b (T_{in} - T_{out}) \\ &\quad \times \left[\frac{k_b \ln(1+2\delta) + (k_a - k_b) \ln(1+\delta) - k_a \ln(1+2\delta) - (k_b - k_a) \ln(1+\delta)}{[k_a \ln(1+2\delta) + (k_b - k_a) \ln(1+\delta)][k_b \ln(1+2\delta) + (k_a - k_b) \ln(1+\delta)]} \right] \\ &= \frac{2\pi l k_a k_b (T_{in} - T_{out})(k_a - k_b)[2\ln(1+\delta) - \ln(1+2\delta)]}{[k_a \ln(1+2\delta) + (k_b - k_a) \ln(1+\delta)][k_b \ln(1+2\delta) + (k_a - k_b) \ln(1+\delta)]} \end{aligned}$$

Let us consider a function $f(\delta) = 2\ln(1+\delta) - \ln(1+2\delta)$. At $\delta = 0$ we have $f(0) = 0$. The first derivative of $f(\delta)$ is given by

$$
\begin{aligned}
\frac{df(\delta)}{d\delta} &= \frac{2}{1+\delta} - \frac{2}{1+2\delta} \\
&= \frac{2\delta}{(1+\delta)(1+2\delta)}
\end{aligned}
$$

which is > 0 for $\delta > 0$, thus $f(\delta)$ is a monotonically increasing function for $\delta > 0$ and we have $f(\delta) > 0$ for $\delta > 0$. In conclusion, $|\dot{Q}_1| - |\dot{Q}_2| > 0$, i.e. more thermal energy is lost to outside for the configuration 1.

3) Suppose the insulating qualities of the wall of a house come mainly from a 4.0 inch layer of brick and an R-19 layer of insulation as shown below. What is the total rate of heat loss through such a wall if its total area is 195 feet^2 and the temperature difference across it is 35 F?



Definition of R values: the R value, i.e. the thermal resistance, specifies insulation properties of a building material, defined for a given thickness l of the material: R = l/k with k being the thermal conductivity.
Larger R means a better insulation from heat or cold. Here, we exceptionally work with R units in British units [ft2 h F/Btu]; [Btu] is the British unit for heat with one BTU equal to about $1,055J$. Below you can see a table giving some R values for common building materials.

TABLE 14–5 *R*-values

Material	Thickness	*R*-value ($ft^2 \cdot h \cdot F°/Btu$)
Glass	$\frac{1}{8}$ inch	1
Brick	$3\frac{1}{2}$ inches	0.6–1
Plywood	$\frac{1}{2}$ inch	0.6
Fiberglass insulation	4 inches	12

Solution:

The conduction rates through the two materials must be equal as in the previous exercise. If they were not, the temperatures in the materials would be changing. Call the temperature at the boundary between the materials T_x.

$$fractQt = k_1 A \frac{T_1 - T_x}{\ell_1} = k_2 A \frac{T_x - T_2}{\ell_2} \quad \rightarrow \quad \frac{Q\ell_1}{tk_1 A} = T_1 - T_x; \quad \frac{Q\ell_2}{tk_2 A} = T_x - T_2 \tag{28}$$

Add these two equations together, and solve for the heat conduction rate.

$$\frac{Q\ell_1}{tk_1 A} + \frac{Q\ell_2}{tk_2 A} = T_1 - T_x + T_x - T_2 \quad \rightarrow \quad \frac{Q}{t}\left(\frac{\ell_1}{k_1} + \frac{\ell_2}{k_2}\right)\frac{1}{A} = T_1 - T_2 \tag{29}$$

$$\rightarrow \frac{Q}{t} = A\frac{(T_1 - T_2)}{\left(\frac{\ell_1}{k_1} + \frac{\ell_2}{k_2}\right)} = A\frac{(T_1 - T_2)}{(R_1 + R_2)} \tag{30}$$

Then the R-value for the insulating layer is given as $R_2 = 19$. For the brick layer, it needs to be calculated. The table gives a range of values for 3.5 inches of brick, so in our case with 4 inches, we must multiply the value by 8/7.

$$R_1 = \frac{8}{7} \times 0.8 = \frac{7}{10} ft^2\ h\ F\ Btu^{-1} \tag{31}$$

Then the heat loss rate is

$$\frac{Q}{t} = 195\ ft^2\ 35^\circ F \frac{1}{7/10 + 19} = 346.45 \approx 350\ Btu\ h^{-1} \tag{32}$$

This is about 100 watts.