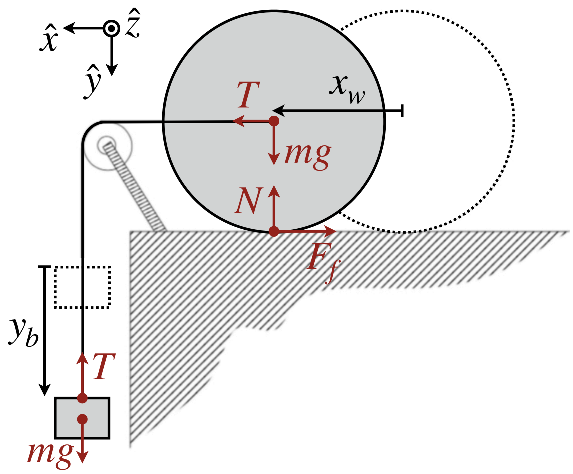
Solutions to Problem Set 14

Rotation, translation, and rolling

PHYS-101(en)

1. Wheel pulled by a block



1. Consider the system composed of the wheel and the block (i.e. the two objects in the problem that move and have mass). Since we are asked about the state of the system at a particular *position* (i.e. after the center of the wheel has traveled a distance d), it is best to use conservation of energy. If we were asked about the system after a given amount of time t, it would be easiest to use Newton's laws.

   The forces on the wheel are gravity $\vec{F}_{gw} = mg\hat{y}$, the normal force from the table $\vec{N} = -N\hat{y}$, and the tension from the wire $\vec{T} = T\hat{x}$. Additionally, we can infer that there must be a static friction force $F_f \leq \mu_s N$ in the $-\hat{x}$ direction, in order for the wheel to roll without slipping. The forces on the block are gravity $\vec{F}_{gb} = mg\hat{y}$ and the tension force from the wire $\vec{T} = -T\hat{y}$. The tension in the wire at the block must be equal in magnitude to the tension at the wheel as both the rope and pulley are massless.

   Since tension is an internal force that does not dissipate any energy, the only nonconservative forces that could be doing work on the system are the normal force $\vec{N}$ and the friction force $\vec{F}_f$. However, since the wheel is rolling without slipping, both forces are not being applied over a distance, just at a changing but always stationary point. This is analogous to a box sitting on a table. You can push the box slightly and static friction will keep it at rest by opposing the applied force. In this case, friction does no work as the box is stationary the entire time, so the force is applied over zero distance. Another way to see this is by considering the power $P = \vec{F} \cdot \vec{v}$. Since the velocity of the point of the wheel in contact with the ground is $\vec{v} = 0$, the power is clearly $P = 0$. Since the power $P = dW/dt$ is also the time derivative of the work W, the work must be a constant value throughout the motion of the block. Since the amount of work done by the friction and normal forces on the wheel is zero initially (as the system has yet to start to move), this means they never do any work. Thus mechanical energy is conserved throughout the motion of the system.

The total mechanical energy of the system is given by

$$E_m = K_w^{rot} + K_w^{trans} + K_b^{trans} + U_{gb} \quad \Rightarrow \quad E_m = \frac{I_w}{2}\omega_w^2 + \frac{m}{2}v_w^2 + \frac{m}{2}v_b^2 - mgy_b \tag{1}$$

$$\Rightarrow \quad E_m = \frac{m}{4}R^2\omega_w^2 + \frac{m}{2}v_w^2 + \frac{m}{2}v_b^2 - mgy_b, \tag{2}$$

where K_w^{rot} is the rotational kinetic energy of the wheel, K_w^{trans} is the translational kinetic energy of the wheel, K_b^{trans} is the translational kinetic energy of the falling block, U_{gb} is the gravitational potential energy of the block,

$$I_w = \frac{m}{2}R^2 \tag{3}$$

is the moment of inertia of the uniform solid wheel rotating about its center, ω_w is the angular speed of the wheel, v_w is the speed of the center of mass of the wheel, v_b is the speed of the block, and we have defined a Cartesian coordinate system as shown in the figure above. Using the no-slip condition, we know that the speed of the center of the wheel is

$$v_w = R\omega_w \quad \Rightarrow \quad \omega_w = \frac{v_w}{R}. \tag{4}$$

Substituting this into equation (2) gives

$$E_m = \frac{m}{4}R^2\left(\frac{v_w}{R}\right)^2 + \frac{m}{2}v_w^2 + \frac{m}{2}v_b^2 - mgy_b \quad \Rightarrow \quad E_m = \frac{3}{4}mv_w^2 + \frac{m}{2}v_b^2 - mgy_b. \tag{5}$$

Since the wire is inextensible and we have carefully defined our coordinate system such that $x_w = 0$ and $y_b = 0$ initially, we can see the constraint condition is $y_b = x_w$. Taking a derivative in time gives

$$v_b = v_w. \tag{6}$$

Substituting these two relations into equation (5) gives

$$E_m = \frac{3}{4}mv_w^2 + \frac{m}{2}v_w^2 - mgx_w = \frac{5}{4}mv_w^2 - mgx_w. \tag{7}$$

Now we can apply conservation of mechanical energy between the initial state (when everything is at rest) and the final state (when the center of the wheel has moved by a distance $x_w = d$) to find

$$0 - 0 = \frac{5}{4}mv_w^2 - mgd, \tag{8}$$

where we have take the reference point for the gravitational potential energy of each object to be their initial location. Solving for the final velocity of the center of the wheel gives

$$v_w = \sqrt{\frac{4}{5}gd}. \tag{9}$$

2. For the wheel not to slip, the static friction force must not exceed its maximum magnitude of

$$F_f \leq \mu_s N. \tag{10}$$

Drawing a free body diagram for the wheel and looking at Newton's second law in the vertical direction shows that the normal force has a magnitude of $N = mg$. Thus, we find

$$F_f \leq \mu_s mg. \tag{11}$$

By rearranging we find that the coefficient of static friction must be sufficiently high to satisfy

$$\mu_s \geq \frac{F_f}{mg}, \tag{12}$$

otherwise the wheel will slip.

In order to determine F_f, we must analyze the entire system. We can apply Newton's second law to the wheel to get the equation

$$\sum F_{wx} = ma_w \quad \Rightarrow \quad T - F_f = m\frac{dv_w}{dt} \quad \Rightarrow \quad T = F_f + m\frac{dv_w}{dt} \tag{13}$$

in the horizontal direction, where a_w is the acceleration of the center of the wheel. Applying Newton's second law to the falling block and substituting equation (13) gives

$$\sum F_{by} = ma_b \quad \Rightarrow \quad mg - T = m\frac{dv_b}{dt} \quad \Rightarrow \quad mg - F_f - m\frac{dv_w}{dt} = m\frac{dv_b}{dt}. \tag{14}$$

From the constraint condition of equation (6), we find

$$mg - F_f - m\frac{dv_w}{dt} = m\frac{dv_w}{dt} \quad \Rightarrow \quad mg - F_f = 2m\frac{dv_w}{dt} \quad \Rightarrow \quad \frac{dv_w}{dt} = \frac{g}{2} - \frac{F_f}{2m}. \tag{15}$$

Lastly, applying Newton's second law for the rotational motion of the wheel about its center gives

$$\sum \vec{\tau}_w = I_w\vec{\alpha}_w \quad \Rightarrow \quad R\hat{y} \times \vec{F}_f = I_w\frac{d\omega_w}{dt}\hat{z} \quad \Rightarrow \quad R\hat{y} \times (-F_f\hat{x}) = I_w\frac{d\omega_w}{dt} \quad \Rightarrow \quad F_f = \frac{I_w}{R}\frac{d\omega_w}{dt} \tag{16}$$

in the $\hat{z}$ direction, where $\vec{\alpha}_w$ is the angular acceleration of the wheel (which is the derivative of the angular velocity $\vec{\omega}_w$). There are no other torques on the wheel because tension and gravity act at the pivot point and the normal force is anti-parallel to the position vector to its point of application (i.e. the cross product in the torque evaluates to zero). Substituting equations (3), (4), and (15) into equation (16) gives

$$F_f = \frac{1}{R}\left(\frac{m}{2}R^2\right)\left(\frac{1}{R}\frac{dv_w}{dt}\right) = \frac{m}{2}\left(\frac{g}{2} - \frac{F_f}{2m}\right) = \frac{1}{4}mg - \frac{1}{4}F_f, \tag{17}$$

which can be rearranged to show

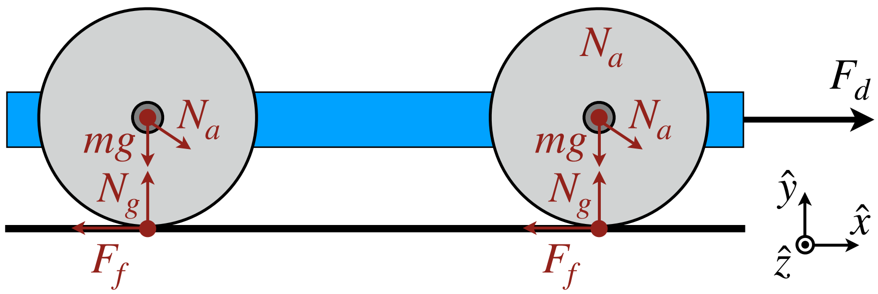
$$F_f = \frac{1}{5}mg. \tag{18}$$

Substituting this into equation (12) reveals the condition on the static coefficient of friction to be

$$\mu_s \geq \frac{1}{5} = 0.2. \tag{19}$$

This is quite a fundamental solution as we see that it is independent of the mass m and radius R. The main requirement for it to be applicable is that the wheel and block must have the same mass.

2. Donkey cart



1. Due to the symmetry of the system, the free body diagram for all of the wheels is identical and is shown above. There are four forces on each wheel: gravity acting at the center of mass of the wheel (i.e. the center of the wheel), the normal force from the axle of the cart, the normal force from the ground, and the static friction force from the ground.

2. To approach this problem, it is best to first think about which objects are moving together and the different types of motion that are occurring. Since the wheels and the body of the cart are attached via the axles, the translational motion of the centers of the wheels and the cart must be identical. Thus, we can consider the cart-wheels system as a whole and write Newton's second law in the horizontal direction as

$$\sum \vec{F} = (M + 4m)\,\vec{a} \quad \Rightarrow \quad F_d - 4F_f = (M + 4m)\,a, \tag{1}$$

where a is the acceleration of the cart as well as the centers of the wheels. Note that all other forces are internal forces within the cart-wheels system, so they cancel due to Newton's third law.

However, there is one other type of motion, which is the rotation of the wheels. Enforcing the no-slip condition will allow us to relate the frictional force causing the rotation of the wheels with the translational motion of the whole cart. We start by writing Newton's second law for the rotation of a single wheel in the $\hat{z}$ direction as

$$\sum \vec{\tau} = I_w \vec{\alpha} \quad \Rightarrow \quad \vec{R}_f \times \vec{F}_f = \frac{m}{8} d^2 \vec{\alpha} \quad \Rightarrow \quad \left(-\frac{d}{2}\hat{y}\right) \times (-F_f \hat{x}) = -\frac{m}{8} d^2 \frac{d\omega}{dt}\hat{z} \quad \Rightarrow \quad F_f = \frac{m}{4} d \frac{d\omega}{dt}, \tag{2}$$

where $I_w = m(d/2)^2/2 = md^2/8$ is the moment of inertia of the wheel about its axle, $\vec{\alpha}$ is the angular acceleration of the wheel, $\vec{R}_f$ is the position vector from the axle to the point of application of the friction force, and ω is the angular velocity of the wheel in the $\hat{z}$ direction. The condition for rolling without slipping is given by

$$\omega = \frac{v}{d/2}, \tag{3}$$

where v is the translational velocity of the center of the wheel (as well as the cart as a whole).

Substituting this into equation (2) gives

$$F_f = \frac{m}{4} d \left(\frac{2}{d}\frac{dv}{dt}\right) = \frac{m}{2} a. \tag{4}$$

Substituting this into equation (1) allows us to find

$$F_d - 4\left(\frac{m}{2} a\right) = (M + 4m)\,a \quad \Rightarrow \quad F_d = (M + 6m)\,a \quad \Rightarrow \quad a = \frac{F_d}{M + 6m}. \tag{5}$$

Since the acceleration is constant, it is straightforward to integrate this to find that the velocity is

$$v(t) = at + v_0 = at = \frac{F_d}{M+6m}t, \tag{6}$$

where the problem statement tells us that the initial velocity is $v_0 = 0$. Plugging in the numerical values into equations (5) and (6) gives $a = 0.94$ m/s^2 and $v = 2.8$ m/s after $t = 3$ seconds respectively.

3. To calculate the required coefficient of static friction to enable rolling without slipping, we first must calculate the magnitude of the static friction force. This can be found by substituting equation (5) into equation (4) to see

$$F_f = \frac{m}{2}\frac{F_d}{M+6m}. \tag{7}$$

We then require that this force be less than the maximum possible static friction force

$$F_f \leq \mu_s N_g \quad \Rightarrow \quad \frac{m}{2}\frac{F_d}{M+6m} \leq \mu_s N_g \quad \Rightarrow \quad \mu_s \geq \frac{m}{2}\frac{1}{N_g}\frac{F_d}{M+6m}. \tag{8}$$

The normal force can be found through the vertical component of Newton's second law on the entire cart-wheels system according to

$$\sum \vec{F} = (M+4m)\,\vec{a} \quad \Rightarrow \quad 4N_g - F_g = 0 \quad \Rightarrow \quad 4N_g - (M+4m)\,g = 0 \quad \Rightarrow \quad N_g = \frac{1}{4}(M+4m)\,g \tag{9}$$

as the acceleration in the $\hat{y}$ direction is zero. Substituting this into equation (8) gives

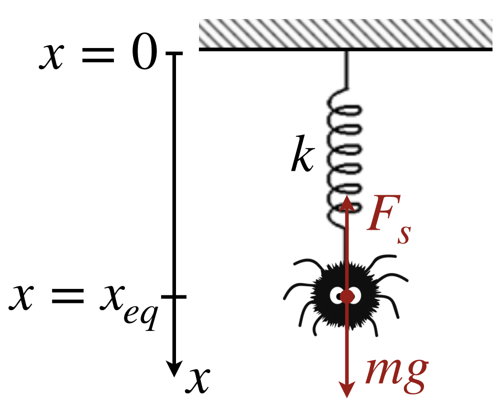
$$\mu_s \geq \frac{m}{2}\frac{4}{(M+4m)\,g}\frac{F_d}{M+6m} \quad \Rightarrow \quad \mu_s \geq \frac{2m}{M+4m}\frac{F_d}{(M+6m)\,g}. \tag{10}$$

Plugging in the numerical values from the problem statements yields

$$\mu_s \geq 0.014. \tag{11}$$

Remarkably, we see that the diameter of the wheel has no influence on the answer.

3. The hanging spider



1. We will choose to use a one-dimensional Cartesian coordinate system with the origin defined to be the location of the ceiling. If the spider were located at $x = 0$, the spring would have a length of zero.

Thus, the equilibrium position *of the spring* is at $x = L$, but this is different than the equilibrium position *of the spider* (due to the presence of the gravitational force). We show a free body diagram above, but note that the spring force can point either up or down depending on the location of the spider. The equilibrium position of the spider x_{eq} is where the sum of the forces are equal to zero (i.e. the acceleration is $a = 0$). This is the definition of "equilibrium." Using Newton's second law and the form of the spring force

$$F_s = k\left(x - L\right), \tag{1}$$

we find

$$mg - F_s = 0 \quad \Rightarrow \quad mg - k\left(x_{eq} - L\right) = 0 \quad \Rightarrow \quad x_{eq} = L + \frac{m}{k}g, \tag{2}$$

where we know that for equilibrium the spring force must be pointing up to counteract gravity.

2. The equation of motion is also found from Newton's second law, but allowing the spider to be at other positions than the equilibrium position (such that the acceleration a is not necessarily zero). At any arbitrary x location, Newton's second law is

$$mg - F_s = ma \quad \Rightarrow \quad mg - k\left(x - L\right) = m\frac{d^2x}{dt^2} \quad \Rightarrow \quad \frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{k}{m}L + g. \tag{3}$$

This is the equation of motion for the spider.

3. The problem statement tells us that the general solution to equation (3) has the form

$$x(t) = A\cos\left(\omega_0 t + \varphi\right) + \overline{x}. \tag{4}$$

We can find the values of the constants by substituting equation (4) into equation (3). We start by taking the first and second derivatives to find the velocity and acceleration of the spider to be

$$v(t) = \frac{dx}{dt} = -A\omega_0 \sin\left(\omega_0 t + \varphi\right) \tag{5}$$

$$a(t) = \frac{d^2x}{dt^2} = -A\omega_0^2 \cos\left(\omega_0 t + \varphi\right) \tag{6}$$

respectively. Substituting equations (4) and (6) into equation (3), we find

$$-A\omega_0^2\cos\left(\omega_0 t + \varphi\right) + \frac{k}{m}\left(A\cos\left(\omega_0 t + \varphi\right) + \overline{x}\right) = \frac{k}{m}L + g \tag{7}$$

$$\Rightarrow \quad \left(\frac{k}{m} - \omega_0^2\right)A\cos\left(\omega_0 t + \varphi\right) = \frac{k}{m}L + g - \frac{k}{m}\overline{x}. \tag{8}$$

To solve this equation, we note that the time t is an independent parameter and equation (8) must be satisfied for all values of t. This means that we can separate the equation based on terms with different functional dependences on time (i.e. the terms proportional to $\cos\left(\omega_0 t + \varphi\right)$ and those constant with t) and require them to zero independently (e.g. equations (9) and (11) below). Equivalently, we can simply evaluate equation (8) at $t = (\pi/2 - \varphi)/\omega_0$ to find

$$\left(\frac{k}{m} - \omega_0^2\right)A\cos\left(\omega_0\left(\frac{\pi/2 - \varphi}{\omega_0}\right) + \varphi\right) = \frac{k}{m}L + g - \frac{k}{m}\overline{x} \quad \Rightarrow \quad 0 = \frac{k}{m}L + g - \frac{k}{m}\overline{x} \tag{9}$$

$$\Rightarrow \quad \overline{x} = L + \frac{m}{k}g, \tag{10}$$

where we note from equation (2) that $\overline{x} = x_{eq}$. Then we can substitute this result into equation (8) at any arbitrary time t to get

$$\left(\frac{k}{m} - \omega_0^2\right)A\cos\left(\omega_0 t + \varphi\right) = \frac{k}{m}L + g - \frac{k}{m}\left(L + \frac{m}{k}g\right) \quad \Rightarrow \quad \left(\frac{k}{m} - \omega_0^2\right)A\cos\left(\omega_0 t + \varphi\right) = 0 \tag{11}$$

$$\Rightarrow \quad \frac{k}{m} - \omega_0^2 = 0 \quad \Rightarrow \quad \omega_0 = \sqrt{\frac{k}{m}}. \tag{12}$$

4. The problem states that at time t_0 the velocity of the spider is $v(t_0) = 0$ and the spring does not exert a force on the spider. We can use equation (5) to see that the first condition is equivalent to

$$v(t_0) = 0 = -A\omega_0 \sin(\omega_0 t_0 + \varphi) \quad \Rightarrow \quad 0 = \sin(\omega_0 t_0 + \varphi). \tag{13}$$

The sine function evaluates to zero when its argument equals $n\pi$ for some integer $n \in \mathbb{Z}$. We are free to arbitrarily choose any integer value of n as they all satisfy the equation. This implies that

$$\omega_0 t_0 + \varphi = n\pi \quad \Rightarrow \quad \varphi = n\pi - \omega_0 t_0. \tag{14}$$

Using equations (1) and (4), we see that the second condition corresponds to

$$F_s(t_0) = 0 = k(x(t_0) - L) \quad \Rightarrow \quad x(t_0) = L \quad \Rightarrow \quad A\cos(\omega_0 t_0 + \varphi) + \overline{x} = L. \tag{15}$$

The condition $x(t_0) = L$ makes sense. It is saying that, if the force exerted by the spring is zero at $t = t_0$, the spider must be starting at the equilibrium position *of the spring* L. Substituting equations (10) and (14) gives

$$A\cos(\omega_0 t_0 + (n\pi - \omega_0 t_0)) + \left(L + \frac{m}{k}g\right) = L \quad \Rightarrow \quad A\cos(n\pi) = -\frac{m}{k}g \tag{16}$$

$$\Rightarrow \quad A(-1)^n = -\frac{m}{k}g \quad \Rightarrow \quad A = -(-1)^n \frac{m}{k}g \quad \Rightarrow \quad A = \pm\frac{m}{k}g. \tag{17}$$

This indicates that there are two possible solutions: the plus sign (corresponding to odd values of n) or the minus sign (corresponding to even values of n). However, we can show that these two solutions are physically identical. If we take the solution with the positive sign (i.e. choosing any odd value of n) and substitute it into equation (4) along with equation (14), we can determine the position to be

$$x(t) = \left(\frac{m}{k}g\right)\cos(\omega_0 t + (n\pi - \omega_0 t_0)) + \overline{x} = \frac{m}{k}g\cos(\omega_0(t - t_0) + n\pi) + \overline{x}. \tag{18}$$

In the same way, we can determine the position corresponding to the solution with the negative sign (i.e. choosing any even value of n) to be

$$x(t) = \left(-\frac{m}{k}g\right)\cos(\omega_0 t + (n\pi - \omega_0 t_0)) + \overline{x} = -\frac{m}{k}g\cos(\omega_0(t - t_0) + n\pi) + \overline{x} \tag{19}$$

$$= \frac{m}{k}g\cos(\omega_0(t - t_0) + n\pi + \pi) + \overline{x} = \frac{m}{k}g\cos(\omega_0(t - t_0) + (n+1)\pi) + \overline{x} \tag{20}$$

using a trigonometric identity to remove the negative sign out in front. The only difference between equation (18) and equation (20) is the choice of n, which is arbitrary. Thus, without loss of generality, we can take $n = 1$ meaning that equations (14) and (17) become

$$\varphi = \pi - \omega_0 t_0 \tag{21}$$

$$A = \frac{m}{k}g. \tag{22}$$

5. The maximum speed of the spider can be found by substituting equation (22) into equation (5) to get

$$v(t) = -\left(\frac{m}{k}g\right)\omega_0 \sin(\omega_0 t + \varphi). \tag{23}$$

The maximum of this function occurs at the time that $\sin(\omega_0 t + \varphi) = -1$. This leads to a maximum speed of

$$v(t) = \frac{m}{k}g\omega_0 = \frac{m}{k}g\sqrt{\frac{k}{m}} = g\sqrt{\frac{m}{k}}, \tag{24}$$

where we have used equation (12).

6. We can substitute equations (4), (5), and (12) into the expression for E in the problem statement to find

$$E = \frac{k}{2}\left((A\cos(\omega_0 t + \varphi) + \overline{x}) - \overline{x}\right)^2 + \frac{m}{2}\left(-A\omega_0 \sin(\omega_0 t + \varphi)\right)^2 \tag{25}$$

$$= \frac{k}{2}A^2\cos^2(\omega_0 t + \varphi) + \frac{m}{2}A^2\left(\frac{k}{m}\right)\sin^2(\omega_0 t + \varphi) \tag{26}$$

$$= \frac{k}{2}A^2\left(\cos^2(\omega_0 t + \varphi) + \sin^2(\omega_0 t + \varphi)\right) = \frac{k}{2}A^2 \tag{27}$$

using the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$. In this formula E, represents the total mechanical energy of the system, which is a constant as there are no nonconservative forces acting on the system.