

Solutions to Problem Set 13

Harmonic motion and gyroscopes

PHYS-101(en)

1. Gyroscope

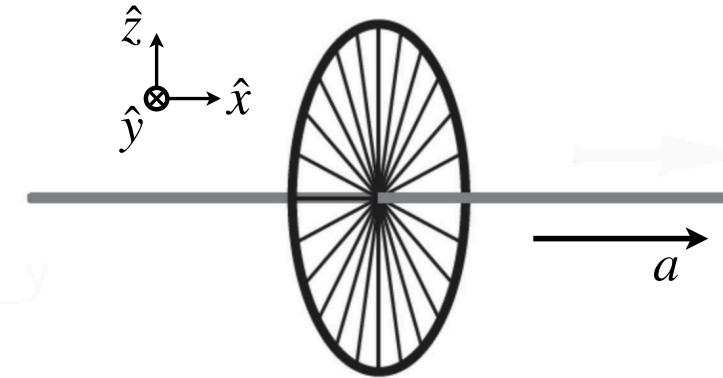
1. Since we are interesting in finding the *additional* force required to achieve particular motion, we can ignore the force of gravity. The only forces we must consider are the additional force from the demonstrator's hand on the *right* end of the bar \vec{F}_r and the additional force from the demonstrator's hand on the *left* end of the bar \vec{F}_l . Note that here the subscript r indicate the right end of the bar, while the subscript l indicates the left. If the gyroscope is displaced in a direction parallel to its axis of rotation, there is no change in angular momentum. Since $d\vec{L}/dt = 0$, the net torque can be found to be

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt} = 0 \quad (1)$$

from Newton's second law for rotation, where we recall that a torque is defined to be $\vec{\tau} = \vec{r} \times \vec{F}$. Thus, since \vec{F}_r is applied at $\vec{r}_r = \ell\hat{x}$ and \vec{F}_l is applied at $\vec{r}_l = -\ell\hat{x}$, they must satisfy

$$\vec{\tau}_{net} = \vec{\tau}_r + \vec{\tau}_l = \vec{r}_r \times \vec{F}_r + \vec{r}_l \times \vec{F}_l = \ell\hat{x} \times \vec{F}_r - \ell\hat{x} \times \vec{F}_l = 0 \Rightarrow \hat{x} \times (\vec{F}_r - \vec{F}_l) = 0. \quad (2)$$

Note that we have defined the origin to be the initial position of the center of the wheel.



To accelerate a mass M , we need a net force \vec{F}_{net} given by Newton's second law of

$$\vec{F}_{net} = M\vec{a} = Ma\hat{x}. \quad (3)$$

Thus, the additional forces from the demonstrator's hands must also satisfy

$$\vec{F}_r + \vec{F}_l = Ma\hat{x} \Rightarrow \vec{F}_l = Ma\hat{x} - \vec{F}_r. \quad (4)$$

Substituting equation (4) into equation (2) gives

$$\hat{x} \times (\vec{F}_r - (Ma\hat{x} - \vec{F}_r)) = 0 \Rightarrow \hat{x} \times \vec{F}_r = 0. \quad (5)$$

as $\hat{x} \times \hat{x} = 0$. Writing out $\vec{F}_r = F_{rx}\hat{x} + F_{ry}\hat{y} + F_{rz}\hat{z}$ component-by-component in Cartesian coordinates allows us to reformulate the equation as

$$\hat{x} \times (F_{rx}\hat{x} + F_{ry}\hat{y} + F_{rz}\hat{z}) = 0 \Rightarrow F_{ry}\hat{z} - F_{rz}\hat{y} = 0. \quad (6)$$

If we take the dot product of this equation with \hat{z} , we see that $F_{ry} = 0$ and if we take the dot product of this equation with \hat{y} , we see that $F_{rz} = 0$. This implies that

$$\vec{F}_r = F_{rx}\hat{x} = F_r\hat{x} \quad (7)$$

must be purely in the \hat{x} direction. Through substitution into equation (4), we see that the same is true for

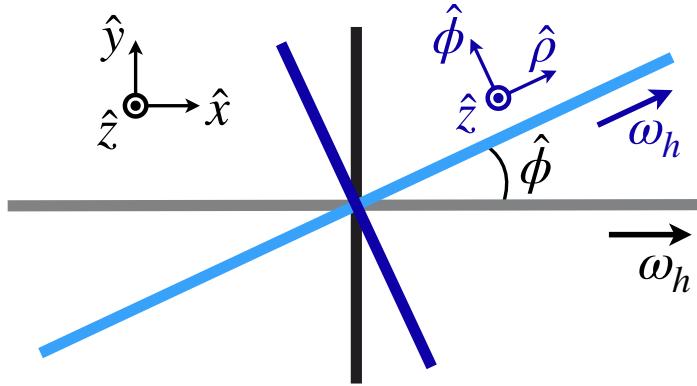
$$\vec{F}_l = Ma\hat{x} - F_r\hat{x} = (Ma - F_r)\hat{x} = F_l\hat{x}. \quad (8)$$

Taking the dot product of both sides of this equation with \hat{x} , we find

$$Ma - F_r = F_l \Rightarrow F_r + F_l = Ma. \quad (9)$$

Thus, the additional forces from each hand must only have an \hat{x} component and satisfy $F_r + F_l = Ma$.

2. This problem is tricky because there are two different types of rotation. There is the rotation of the hoop about the axis, which has a constant magnitude ω_h that starts out in the \hat{x} direction at $t = 0$. However, it changes direction because the bar is made to rotate about its center with a constant angular velocity $\omega_b\hat{z}$. The second type of rotation (i.e. the rotation of the bar in the x - y plane) causes the angular momentum to change. To quantify this second type of rotation, we will use the cylindrical coordinate system shown in the figure below (which shows the wheel from a top view relative to the figure in the problem statement).



We can write the total angular momentum of the system (including both types of rotation) as

$$\vec{L}_{sys} = \vec{L}_h + \vec{L}_b = I_h\omega_h\hat{\rho} + I_b\omega_b\hat{z}, \quad (10)$$

where $\hat{\rho}$ is changing with time and the subscripts h and b indicate the hoop and bar respectively. The hoop is thin and rotating about its center, so its moment of inertia is $I_h = MR^2$. The bar is massless, but its rotation also rotates the hoop about its diameter. Thus, it has a moment of inertia of $I_b = MR^2/2$, which we have found from a table noting that the hoop is thin (meaning it has zero width). Substituting these values into equation (10), we find

$$\vec{L}_{sys} = MR^2\omega_h\hat{\rho} + \frac{M}{2}R^2\omega_b\hat{z}. \quad (11)$$

A change in angular momentum is always caused by a net torque, which can be found from Newton's second law for rotation

$$\vec{\tau}_{net} = \frac{d\vec{L}_{sys}}{dt}. \quad (12)$$

Substituting equation (11) gives

$$\vec{\tau}_{net} = \frac{d}{dt} \left(MR^2 \omega_h \hat{\rho} + \frac{M}{2} R^2 \omega_b \hat{z} \right) = MR^2 \omega_h \frac{d\hat{\rho}}{dt}. \quad (13)$$

In lecture 4a, we calculated the derivatives of the cylindrical unit vectors, finding that $d\hat{\rho}/dt = (d\phi/dt)\hat{\phi}$. Since the bar is changing angular position ϕ at a constant angular speed $\omega_b = d\phi/dt$, we can write $d\hat{\rho}/dt = \omega_b \hat{\phi}$. Substituting this into equation (13) produces

$$\vec{\tau}_{net} = MR^2 \omega_h \omega_b \hat{\phi}. \quad (14)$$

This torque must arise from the forces applied at the end of the bar by the demonstrator's hands. Thus, analogously to equation 2, we have

$$\vec{\tau}_{net} = \vec{r}_r + \vec{r}_l = \vec{r}_r \times \vec{F}_r + \vec{r}_l \times \vec{F}_l = \ell \hat{\rho} \times \vec{F}_r - \ell \hat{\rho} \times \vec{F}_l = MR^2 \omega_h \omega_b \hat{\phi} \quad (15)$$

$$\Rightarrow \hat{\rho} \times (\vec{F}_r - \vec{F}_l) = \frac{MR^2}{\ell} \omega_h \omega_b \hat{\phi}, \quad (16)$$

where we must take into account the rotation of the bar into our expression for the radial position of the demonstrator's hands by using $\hat{\rho}$ instead of \hat{x} .

Additionally, we know that the center of the bar (i.e. the center of mass of the system) has no translational motion. Thus, by Newton's first law, the net force must be equal to

$$\vec{F}_{net} = \vec{F}_r + \vec{F}_l = 0 \Rightarrow \vec{F}_l = -\vec{F}_r. \quad (17)$$

Substituting this into equation (16) gives

$$\hat{\rho} \times (\vec{F}_r + \vec{F}_l) = \frac{MR^2}{\ell} \omega_h \omega_b \hat{\phi} \Rightarrow \hat{\rho} \times \vec{F}_r = \frac{MR^2}{2\ell} \omega_h \omega_b \hat{\phi}. \quad (18)$$

Writing out $\vec{F}_r = F_{r\rho} \hat{\rho} + F_{r\phi} \hat{\phi} + F_{rz} \hat{z}$ component-by-component in cylindrical coordinates shows that

$$\hat{\rho} \times (F_{r\rho} \hat{\rho} + F_{r\phi} \hat{\phi} + F_{rz} \hat{z}) = \frac{MR^2}{2\ell} \omega_h \omega_b \hat{\phi} \Rightarrow F_{r\phi} \hat{z} - F_{rz} \hat{\phi} = \frac{MR^2}{2\ell} \omega_h \omega_b \hat{\phi}. \quad (19)$$

Looking at each component of this equation, we see that we must have $F_{r\phi} = 0$ and

$$-F_{rz} = \frac{MR^2}{2\ell} \omega_h \omega_b \Rightarrow F_{rz} = -\frac{MR^2}{2\ell} \omega_h \omega_b. \quad (20)$$

Plugging this into equation (17) shows that $F_{r\phi} = 0$ and

$$F_{rz} = \frac{MR^2}{2\ell} \omega_h \omega_b. \quad (21)$$

Thus, our final answer is

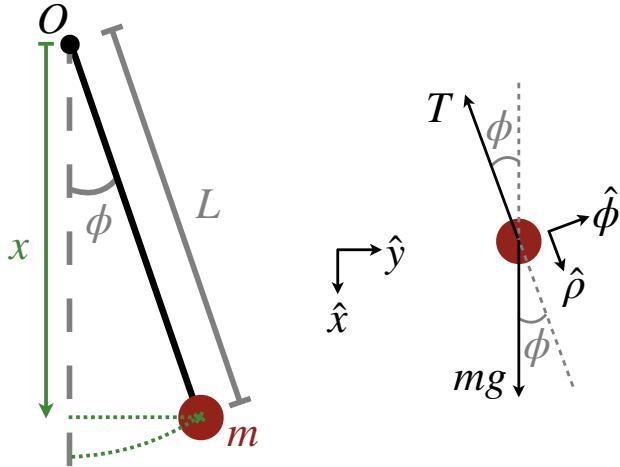
$$\vec{F}_r = F_{r\rho} \hat{\rho} - \frac{MR^2}{2\ell} \omega_h \omega_b \hat{z} \quad (22)$$

$$\vec{F}_l = -F_{r\rho} \hat{\rho} + \frac{MR^2}{2\ell} \omega_h \omega_b \hat{z}. \quad (23)$$

We see that the radial component of the force is unconstrained as long as it is equal and opposite according to equation (17). This is intuitive as the tension in the bar will transmit the force along the bar, while not affecting its motion. Less intuitive is the fact that, to get the bar to rotate in the $\hat{\phi}$ direction, you must apply forces in the \hat{z} direction. This is the nature of a gyroscope — applying a force creates motion in a perpendicular direction.

2. Simple pendulum

Below is the free body diagram for the point mass of the simple pendulum in polar coordinates at an arbitrary angular position of ϕ . We define a polar coordinate system, which is natural for describing the rotational motion of the mass. Additionally, we define a Cartesian coordinate system such that the polar angle ϕ measures the angle from the x axis towards the positive y axis (as is conventional).



1. We can solve part one in two ways: using Newton's second law or conservation of energy.

Using Newton's second law: At a given angular position, the gravitational force on the point mass is given by

$$\vec{F}_g = mg\hat{x} = mg \left(\cos \phi \hat{\rho} - \sin \phi \hat{\phi} \right), \quad (1)$$

where we have related the Cartesian and polar unit vectors using the formula $\hat{x} = \cos \phi \hat{\rho} - \sin \phi \hat{\phi}$ from lecture 4a. We can calculate the tension from the radial component of Newton's second law using the centripetal acceleration $\vec{a}_{cent} = -(v_\phi^2/L)\hat{\rho}$ to get

$$mg \cos \phi - T = ma_{cent} = -m \frac{v_\phi^2}{L} \quad \Rightarrow \quad T = mg \cos \phi + m \frac{v_\phi^2}{L}, \quad (2)$$

where v_ϕ is the speed of the point mass (which is purely tangential). However, this will not be needed for to solve the problem. Instead, the tangential component of Newton's second law is useful

$$-mg \sin \phi = ma_\phi \quad \Rightarrow \quad -g \sin \phi = a_\phi. \quad (3)$$

The form of the acceleration in polar coordinates is given by

$$\vec{a} = \left(\ddot{\rho} - \rho \dot{\phi}^2 \right) \hat{\rho} + \left(\rho \ddot{\phi} + 2\dot{\rho}\dot{\phi} \right) \hat{\phi}. \quad (4)$$

Since $\rho = L$ is a constant, we have $\dot{\rho} = 0$ and we can substitute the tangential component of equation (4) into equation (3) to find

$$-g \sin \phi = L \frac{d^2\phi}{dt^2} \Rightarrow \frac{d^2\phi}{dt^2} + \frac{g}{L} \sin \phi = 0. \quad (5)$$

Using the small angle approximation $\sin \phi \approx \phi$ gives the differential equation of a simple harmonic oscillator

$$\frac{d^2\phi}{dt^2} + \frac{g}{L} \phi = 0. \quad (6)$$

Using conservation of energy: Since all of the forces acting on the pendulum are conservative, we can impose conservation of mechanical energy

$$E_{m0} = E_{mf} \quad (7)$$

between the initial state described in the problem (denoted by the subscript 0) and the final state when the pendulum is at any arbitrary angular position ϕ (denoted by the subscript f). The only forces involved are gravity, so equation (7) is

$$K_0 + U_{g0} = K_f + U_{gf}. \quad (8)$$

The system is released from rest, so $K_0 = 0$. Additionally, we will define the reference point for the gravitational potential energy to be the origin O of our polar coordinate system (i.e. the location of the pivot point at the top of the string). Given the Cartesian coordinate system shown above, the gravitational potential energy of the point mass is given by

$$U_g = -mgx = -mg\rho \cos \phi = -mgL \cos \phi, \quad (9)$$

where we have expressed the Cartesian coordinate x in polar coordinates using the formula $x = \rho \cos \phi$ from lecture 4a. Substituting this, $K_0 = 0$, and the form of the rotational kinetic energy $K = (I/2)\omega^2$ into equation (8) gives

$$0 - mgL \cos \phi_0 = \frac{I}{2}\omega^2 - mgL \cos \phi, \quad (10)$$

where ω is the angular speed. For a point mass located a distance $\rho = L$ away from the axis of rotation, the moment of inertia is $I = mL^2$. Employing this and taking the time derivative of equation (10) using the chain rule yields

$$0 = \frac{mL^2}{2} \left(2\omega \frac{d\omega}{dt} \right) + mgL \sin \phi \frac{d\phi}{dt} \Rightarrow 0 = L\omega \frac{d\omega}{dt} + g \sin \phi \frac{d\phi}{dt}. \quad (11)$$

Identifying that $\omega = d\phi/dt$ produces

$$0 = L\omega \frac{d^2\phi}{dt^2} + g \omega \sin \phi \Rightarrow \frac{d^2\phi}{dt^2} + \frac{g}{L} \sin \phi = 0, \quad (12)$$

which is identical to equation (5). Thus, energy conservation gives the same answer as Newton's second law.

2. The differential equation for a simple harmonic oscillator has the form

$$\frac{d^2\phi}{dt^2} + \omega_0^2 \phi = 0, \quad (13)$$

where ω_0 is the angular frequency of oscillation. Thus, by comparison with equation (6), we see that

$$\omega_0 = \sqrt{\frac{g}{L}}. \quad (14)$$

To get the frequency f_0 (i.e. the number of oscillations per second) from the angular frequency (i.e. the average number of radians the object completes in its oscillation per second), we use the fact that one oscillation corresponds to 2π radians. This means that $\omega_0 = 2\pi f_0$, so

$$f_0 = \frac{1}{2\pi} \sqrt{\frac{g}{L}}. \quad (15)$$

3. The period is the time it takes for the object to complete a full oscillation (i.e. the number of seconds per oscillation). This is simply the inverse of the number of oscillations per second (i.e. the frequency). Thus, using $T_0 = 1/f_0$ we find the period to be

$$T_0 = 2\pi \sqrt{\frac{L}{g}}. \quad (16)$$

4. The angular velocity of the point mass at a given location is easiest to calculate from conservation of energy. We can take equation (10) and let $\phi = 0$ to choose the final state to be when the pendulum is at the bottom of its swing

$$-mgL \cos \phi_0 = \frac{I}{2} \omega^2 - mgL. \quad (17)$$

Given that $I = mL^2$, we can rewrite this as

$$-mgL \cos \phi_0 = \frac{mL^2}{2} \omega^2 - mgL \quad \Rightarrow \quad \omega = \sqrt{2 \frac{g}{L} (1 - \cos \phi_0)}. \quad (18)$$

The translational speed $v = v_\phi$ is purely tangential, which is related to the angular speed ω through

$$v_\phi = \rho \omega. \quad (19)$$

Substituting equation (18) and the fact that $\rho = L$ gives

$$v_\phi = L \sqrt{2 \frac{g}{L} (1 - \cos \phi_0)} \quad \Rightarrow \quad v_\phi = \sqrt{2gL (1 - \cos \phi_0)}. \quad (20)$$

5. No, these are not the same. The angular speed $\omega = d\phi/dt$ is the rate of change in the angular position of the pendulum (where 2π radians corresponds to a *full circle* in our polar coordinate ϕ). The angular frequency $\omega_0 = \sqrt{g/L}$ represents the average number of radians the object completes per second (where 2π radians corresponds to a *full oscillation*). Moreover, the angular speed ω will change throughout the pendulum's motion, while the angular frequency ω_0 is defined such that it is always a constant.

6. No, as shown by equation (16). Since the mass appears on both sides in Newton's second law (see equation (3)), it cancels out.