Solutions to Problem Set 12

Angular momentum

PHYS-101(en)

1. Planetary survey

After the instrument is launched, the only force it will experience is its gravitational attraction to the planet. This force is given by

$$\vec{F}_G = -\frac{G m_p m_i}{r^2}\hat{r}, \tag{1}$$

where r is the distance between the instrument and the center of the planet, $\hat{r}$ is the radial unit vector pointing from the center of the planet towards the instrument, and G is the universal gravitational constant. Since this is the only force acting on the instrument, it experiences a total external torque about the center of the planet of

$$\sum \vec{\tau}_{ext} = \vec{\tau}_G = \vec{r} \times \vec{F}_G = \vec{r} \times \left(-\frac{G m_p m_i}{r^2}\hat{r}\right) = -\frac{G m_p m_i}{r^2}\vec{r} \times \hat{r} = 0, \tag{2}$$

where we have used equation (1) and the fact that the cross product of parallel vectors is zero. Thus, since the total external torque on the instrument about the center of the planet is zero, its angular momentum about the center of the planet must be conserved throughout its motion. Conservation of angular momentum is expressed as

$$\vec{L}_i = \vec{L}_f, \tag{3}$$

where the angular momentum is

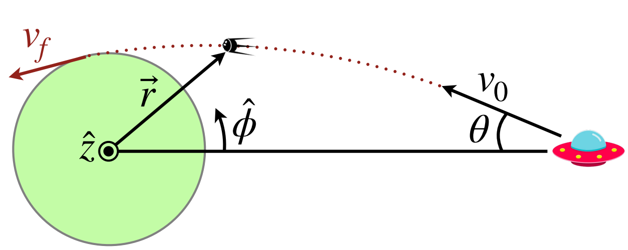
$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m_i \vec{v}. \tag{4}$$

Here $\vec{p}$ and $\vec{v}$ are the momentum and velocity of the instrument respectively. It is natural to define a cylindrical coordinate system with its origin at the center of the planet. The initial velocity can be found from trigonometry and the figure below to be

$$\vec{v}_i = -v_0 \cos\theta \hat{r} + v_0 \sin\theta \hat{\phi}. \tag{5}$$

Substituting this and the initial position vector of the instrument into equation (4) gives

$$\vec{L}_i = (5 r_p \hat{r}) \times m_i \left(-v_0 \cos\theta \hat{r} + v_0 \sin\theta \hat{\phi}\right) = 5 r_p m_i v_0 \sin\theta \left(\hat{r} \times \hat{\phi}\right) = 5 r_p m_i v_0 \sin\theta \hat{z}. \tag{6}$$



We will consider the final state to occur when the instrument just grazes the surface of the planet. At this instant, the position vector is $\vec{r}_f = r_p \hat{r}$ and the final velocity $\vec{v}_f = v_f \hat{\phi}$ is in the direction exactly tangent to the surface. Thus, the final angular momentum is

$$\vec{L}_f = (r_p \hat{r}) \times m_i \left(v_f \hat{\phi}\right) = r_p m_i v_f \left(\hat{r} \times \hat{\phi}\right) = r_p m_i v_f \hat{z}. \tag{7}$$

Plugging equations (6) and (7) into equation (3) yields

$$5r_p m_i v_0 \sin\theta \hat{z} = r_p m_i v_f \hat{z} \quad \Rightarrow \quad v_f = 5v_0 \sin\theta. \tag{8}$$

However, this equation still has two unknowns v_f and θ, so we require another condition.

To determine the final velocity of the instrument, we can think about the situation physically. We realize that the instrument will be accelerated as it falls into the gravitational potential of the planet. The change in speed of the instrument can be found from conservation of mechanical energy because there are no nonconservative forces acting on the instrument. Thus, we have

$$E_{mi} = E_{mf} \quad \Rightarrow \quad K_i + U_{Gi} = K_f + U_{Gf}. \tag{9}$$

In previous problem sets, we've found the universal gravitational potential (with a reference point infinitely far away) to be

$$U_G(R) = -\frac{Gm_i m_p}{R}. \tag{10}$$

Plugging this and the form of the kinetic energy into equation (9) gives

$$\frac{m_i}{2}v_0^2 - \frac{Gm_i m_p}{5r_p} = \frac{m_i}{2}v_f^2 - \frac{Gm_i m_p}{r_p}. \tag{11}$$

Substituting equation (8) allows us to find the final answer of

$$\frac{m_i}{2}v_0^2 - \frac{Gm_i m_p}{5r_p} = \frac{m_i}{2}\left(5v_0\sin\theta\right)^2 - \frac{Gm_i m_p}{r_p} \quad \Rightarrow \quad (5v_0\sin\theta)^2 = v_0^2 + \frac{8Gm_p}{5r_p} \tag{12}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{1}{5}\sqrt{1 + \frac{8Gm_p}{5v_0^2 r_p}}\right). \tag{13}$$

2. Toy locomotive

We begin by choosing our system to consist of the locomotive and the track. Because there are no *external* torques about the central vertical axis, the angular momentum of our system must remain constant about that axis

$$\vec{L}^i_{sys} = \vec{L}^f_{sys}. \tag{1}$$

The initial angular momentum of the system is

$$\vec{L}^i_{sys} = 0 \tag{2}$$

because both the locomotive and the track are at rest. The final angular momentum will be composed of the angular momentum of the locomotive and the track according to

$$\vec{L}^f_{sys} = \vec{L}^f_L + \vec{L}^f_T, \tag{3}$$

where the subscripts L and T refer to the locomotive and track respectively.

The final angular momentum of the locomotive (which can be considered to be a point mass) is given by

$$\vec{L}^f_L = \vec{R}_L \times \vec{p}_L = R_T\hat{r} \times m_L\vec{v}_L, \tag{4}$$

where $\vec{r}_L$ is the position vector from the pivot point at the top of the vertical axis to the locomotive, $\vec{p}_L$ is momentum of the locomotive in the ground reference frame, and $\vec{v}_L$ is the velocity of the locomotive relative to the ground. From the figure we see that the locomotive is moving only in the $\hat{\phi}$ direction, so $\vec{v}_L = v_f\hat{\phi}$

(where v_f is the final speed relative to the floor that we are trying to determine). Plugging this into equation (4) produces

$$\vec{L}_L^f = R_T\hat{r} \times m_L v_f \hat{\phi} = R_T m_L v_f \hat{z}, \tag{5}$$

where the $\hat{\phi}$ unit vector points counter-clockwise (when viewed from above) and the $\hat{z}$ unit vector points upwards.

Next we must determine the final angular momentum of the track (which is a continuous system). We know that the moment of inertia of a thin uniform ring about an axis passing through its center is

$$I_T = m_T R_T^2. \tag{6}$$

This can also be calculated from the definition of the moment of inertia according to

$$I_T = \int_{ring} \rho^2 dm = \int_{ring} \rho^2 \lambda dl_\phi = \int_{ring} \rho^2 \lambda \rho d\phi = \lambda \rho^3 \int_0^{2\pi} d\phi = \lambda R_T^3 (2\pi - 0) = \left(\frac{m_T}{2\pi R_T}\right) R_T^3 2\pi = m_T R_T^2 \tag{7}$$

using the linear mass density $\lambda = m_T/(2\pi R_T) = dm/dl_\phi$ and the arc length along the track $l_\phi = \rho\phi$. Using the definition of the angular momentum of a continuous system, we know that the final angular momentum of the track is

$$\vec{L}_T^f = I_T \vec{\omega}_f, \tag{8}$$

where $\vec{\omega}_f$ is the final angular velocity of the track. The angular velocity can be related to the tangential velocity of any point on the track through

$$\vec{\omega}_f = \frac{\vec{\rho} \times \vec{v}_\phi}{\rho^2} = \frac{R_T \hat{\rho} \times \left(-v_T \hat{\phi}\right)}{R_T^2} = -\frac{v_T \hat{\rho} \times \hat{\phi}}{R_T} = -\frac{v_T}{R_T}\hat{z}, \tag{9}$$

where v_T is the final tangential speed of the track relative to the ground and we have deduced its direction from imagining the physical situation (i.e. by Newton's third law, if the locomotive goes one way, the track must go the other). Substituting this and equation (6) into equation (8) gives

$$\vec{L}_T^f = \left(m_T R_T^2\right)\left(-\frac{v_T}{R_T}\hat{z}\right) = -m_T R_T v_T \hat{z}. \tag{10}$$

Unfortunately, we do not know the tangential speed of the track v_T. However, we can find it from information given in the problem statement. Specifically, we know the speed of the locomotive relative to the track v. The formula for converting velocities between different reference frames is

$$\vec{v}_{gL} = \vec{v}_{gT} + \vec{v}_{TL} \quad \Rightarrow \quad v_f \hat{\phi} = -v_T \hat{\phi} + v\hat{\phi} \quad \Rightarrow \quad v_T = v - v_f, \tag{11}$$

where $\vec{v}_{gL} = v_f \hat{\phi}$ is the velocity of the locomotive in the reference frame of the ground, $\vec{v}_{gT} = -v_T \hat{\phi}$ is the velocity of the track in the reference frame of the ground, and $\vec{v}_{TL} = v\hat{\phi}$ is the velocity of the locomotive in the reference frame of the *track*. Plugging this into equation (10) gives

$$\vec{L}_T^f = -m_T R_T (v - v_f)\hat{z}, \tag{12}$$

which is now composed exclusively of known quantities and the parameter we are trying to find v_f.

Finally, we can substitute equations (2), (3), (5), and (12) into equation (1) to find the final answer of

$$0 = \vec{L}_L^f + \vec{L}_T^f \quad \Rightarrow \quad 0 = R_T m_L v_f \hat{z} - m_T R_T (v - v_f)\hat{z} \quad \Rightarrow \quad v_f = \frac{m_T}{m_L + m_T} v. \tag{13}$$

3. Particle-rod collision revisited

1. The motion of any rigid body can be represented as the motion of the center of mass, plus a rotation about the center of mass. In problem set 6, we found the position of the center of mass after the collision to be

$$\vec{R}_{CM}(t) = \frac{V_0}{2} t\hat{x} + \frac{\ell}{4}\hat{y}. \quad (1)$$

However, to completely specify the motion of the particle-rod system, we must also calculate the angular velocity of rotation of the system *about its center of mass*.

To calculate the rotation after the collision from the information just before the collision, we will use conservation of angular momentum about the center of mass (as there are no external torques acting on the particle-rod system). This is expressed as

$$\vec{L}_b = \vec{L}_a. \quad (2)$$

where the subscript "b" indicates that the quantity is evaluated just before the collision and the subscript "a" indicates just after. Just before the collision, the angular momentum of the system about the center of mass is the sum of the angular momenta of all the objects i in the system. This is

$$\vec{L}_b = \sum_i \vec{L}_{ib} = \sum_i \vec{r}_{ib} \times m_i \vec{v}_{ib} = \vec{r}_{pb} \times M\vec{v}_{pb}, \quad (3)$$

where the subscript "p" indicates the particle. Note that there is no contribution to the angular momentum from the rod as it is completely stationary before the collision. From inspecting the problem statement we see that, just before the collision, the particle is moving with $\vec{v}_{pb} = V_0\hat{x}$ at a position $\vec{r}_{pb} = -(\ell/4)\hat{y}$ relative to the center of mass of the particle-rod system at $t = 0$. Substituting these values, equation (3) becomes

$$\vec{L}_b = \frac{M}{4}\ell V_0 \hat{z}. \quad (4)$$

After the collision, the rod and particle form a combined object that rotates at a common angular velocity $\vec{\omega}$ about its center of mass. The angular momentum of such an rotating extended object is

$$\vec{L}_a = I_{CM}\vec{\omega}, \quad (5)$$

where I_{CM} is the momentum of inertia of the particle-rod system about its center of mass. Substituting this and equation (4) into equation (2) allows us to find

$$\vec{\omega} = \frac{M}{4I_{CM}}\ell V_0 \hat{z}. \quad (6)$$

This is almost the final solution, but we don't yet know I_{CM}. To calculate it, we start from the definition of the center of mass

$$I_{CM} = \int_M \rho^2 dm, \quad (7)$$

where the integral is taken over the entire mass of the combined object. Because integrals are just summations of infinitesimally small differential elements, we can separate it into the contributions from the two objects

$$I_{CM} = \int_{rod} \rho^2 dm + \int_{particle} \rho^2 dm = I_{rod} + I_{particle}. \quad (8)$$

Since the particle is well represented by a point mass, all of its mass is located at the same distance $\rho = \ell/4$ from the center of mass. Thus,

$$I_{particle} = \int_{particle} \rho^2 dm = \int_{particle} \left(\frac{\ell}{4}\right)^2 dm = \frac{\ell^2}{16} \int_{particle} dm = \frac{1}{16} M\ell^2. \tag{9}$$

There are two ways to find the contribution to the moment of inertia from the rod. This first is simpler and uses the parallel axis theorem. From the table of moments of inertia presented in lecture, we know that a uniform thin rod rotated about its center of mass (i.e. its geometric center) has a moment of inertia of $I_{rod}^{center} = M\ell^2/12$. However, we are interested in the rotation of the rod about the center of mass of the particle-rod system, not the center of mass of the rod alone. From equation (1), we see that the center of mass of the rod (which is at $(\ell/2)\hat{y}$) is a distance of $h = \ell/4$ away from the center of mass of the particle-rod system at $t = 0$. Thus, we will use the parallel axis theorem to see that

$$I_{rod} = I_{rod}^{center} + Mh^2 = \frac{1}{12} M\ell^2 + \frac{1}{16} M\ell^2 = \frac{7}{48} M\ell^2. \tag{10}$$

The second way to find the moment of inertia of the rod is to directly evaluate the integral in equation (8). This approach is more challenging, but applies to a wider variety of situations. To convert from an integral over mass to an integral in space, we use the linear mass density $\lambda = dm/d\rho$ and the fact that the density is uniform $\lambda = M/\ell$ to see that $dm = (M/\ell)d\rho$. Substituting this we see that

$$I_{rod} = \int_{rod} \rho^2 dm = \frac{M}{\ell} \int_{rod} \rho^2 d\rho. \tag{11}$$

To determine the bounds of the integral, we must think about the geometry of the problem. Here ρ represents the distance from the center of mass of the particle-rod system, which is at $y = \ell/4$. Thus, to integrate over the full object we must consider the part of the rod above and below the center of mass, which is tricky as some of these points have the same value of ρ. This can be handled by splitting the integral into the contributions above and below, which are given by

$$I_{rod} = \frac{M}{\ell} \int_0^{3\ell/4} \rho^2 d\rho + \frac{M}{\ell} \int_0^{\ell/4} \rho^2 d\rho. \tag{12}$$

respectively. Evaluting the integrals is straightforward and yields

$$I_{rod} = \frac{M}{\ell} \left(\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=3\ell/4} + \frac{M}{\ell} \left(\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=\ell/4} = \frac{M}{3\ell} \left(\frac{3}{4}\ell\right)^3 + \frac{M}{3\ell} \left(\frac{1}{4}\ell\right)^3 = \frac{1}{3}\left(\frac{27}{64} + \frac{1}{64}\right) M\ell^2 = \frac{7}{48} M\ell^2, \tag{13}$$

which is identical to the solution using the parallel axis theorem (i.e. equation (10)).

Substituting equation (9) and (13) into equation (8) gives the total moment of inertia of the particle-rod system around its center of mass, which is

$$I_{CM} = \frac{7}{48} M\ell^2 + \frac{1}{16} M\ell^2 = \frac{5}{24} M\ell^2. \tag{14}$$

Substituting this into equation (6) gives the final answer of

$$\vec{\omega} = \frac{6}{5} \frac{V_0}{\ell} \hat{z}. \tag{15}$$

Importantly, since there are no additional forces acting at later times, we have conservation of angular momentum. Thus, the angular velocity of the particle-rod system remains the same at all times $t \geq 0$.

2. We know that the particle-rod system moves based on the combination of two types of motion. Its center of mass translates, which has been calculated in equation (1). Additionally, in the center of mass reference frame, all points in the system rotate about the center of mass with a constant angular velocity $\vec{\omega} = (6/5)(V_0/\ell)\hat{z}$. This rotation is uniform circular motion, so the angular velocity corresponds to a velocity of

$$\vec{v} = \vec{\omega} \times \vec{\rho} = \omega\hat{z} \times \rho\hat{\rho} = \rho\omega\hat{\phi}. \tag{16}$$

We are asked about the position of the particle, which is located at a distance $\rho = \ell/4$ away from the center of mass. Thus, it has a velocity of

$$\vec{v} = \frac{\ell}{4}\omega\hat{\phi} \tag{17}$$

after the collision. Note that if we substitute the value for ω, we find $\vec{v} = (6/20)V_0\hat{\phi}$, which shows that the particle is slowed down substantially as a result of the collision.

Ultimately, we want to express the position in Cartesian coordinates, so we will convert the cylindrical unit vector $\hat{\phi}$ according to

$$\vec{v} = \frac{\ell}{4}\omega\left(-\sin\phi\hat{x} + \cos\phi\hat{y}\right) \tag{18}$$

(using the table of coordinate system conversions given in lecture 4a). Given that ω is constant, we can integrate the definition of the angular speed $\dot{\phi} = \omega$ to find

$$\phi(t) = \omega t + C, \tag{19}$$

where C is an integration constant. Since ϕ is the angle from the $+x$-axis and increases towards the $+y$-axis, at $t = 0$ the particle is at $\phi(0) = -\pi/2$. Using this initial condition, we find that $C = -\pi/2$. Substituting this and equation (19) into equation (18) gives

$$\vec{v}(t) = \frac{\ell}{4}\omega\left(-\sin\left(\omega t - \frac{\pi}{2}\right)\hat{x} + \cos\left(\omega t - \frac{\pi}{2}\right)\hat{y}\right) = \frac{\ell}{4}\omega\left(\cos\left(\omega t\right)\hat{x} + \sin\left(\omega t\right)\hat{y}\right), \tag{20}$$

where in the second step we have used trigonometric identities that one can find in a table. This result is consistent with our intuition – the bottom of the rod should start rotating to the right, in the same direction the particle strikes it.

To find the position, we simply integrate equation (20) to find

$$\vec{r}(t) = \frac{\ell}{4}\left(\sin\left(\omega t\right)\hat{x} - \cos\left(\omega t\right)\hat{y}\right). \tag{21}$$

However, we must remember that the position $\vec{r}(t)$ is in the reference frame moving with the center of mass of the particle-rod system. Thus, we must change back to the reference frame given in the problem statement using $\vec{R}_p(t) = \vec{R}_{CM}(t) + \vec{r}(t)$. This yields the final answer of

$$\vec{R}_p(t) = \vec{R}_{CM}(t) + \frac{\ell}{4}\left(\sin(\omega t)\hat{x} - \cos(\omega t)\hat{y}\right). \tag{22}$$

4. Elliptic Orbit

1. As in problem 1, the motion of the satellite will conserve both angular momentum and mechanical energy according to

$$\vec{L}_f = \vec{L}_c \tag{1}$$
$$E_{mf} = E_{mc}. \tag{2}$$

We will choose to evaluate angular momentum about the center of the planet and take the reference point for the gravitational potential energy to be infinitely far away. Thus, conservation of angular momentum and mechanical energy become

$$\vec{r}_f \times m_s \vec{v}_f = \vec{r}_c \times m_s \vec{v}_c \tag{3}$$
$$K_f + U_{Gf} = K_c + U_{Gc} \tag{4}$$

respectively. Taking a cylindrical coordinate system and substituting the forms of the kinetic and gravitational potential energy gives

$$(r_f \hat{r}) \times m_s \left(v_f \hat{\phi}\right) = (r_c \hat{r}) \times m_s \left(v_c \hat{\phi}\right) \quad \Rightarrow \quad m_s r_f v_f \hat{z} = m_s r_c v_c \hat{z} \quad \Rightarrow \quad r_c = \frac{v_f}{v_c} r_f \tag{5}$$
$$\frac{m_s}{2} v_f^2 - \frac{G m_s m_p}{r_f} = \frac{m_s}{2} v_c^2 - \frac{G m_s m_p}{r_c} \quad \Rightarrow \quad \frac{2Gm_p}{r_c} - \frac{2Gm_p}{r_f} = v_c^2 - v_f^2. \tag{6}$$

Substituting equation (5) into equation (6) gives

$$\frac{2Gm_p}{r_f}\frac{v_c}{v_f} - \frac{2Gm_p}{r_f} = v_c^2 - v_f^2 \quad \Rightarrow \quad \frac{2Gm_p}{r_f v_f}\left(v_c - v_f\right) = \left(v_c + v_f\right)\left(v_c - v_f\right) \quad \Rightarrow \quad v_c = \frac{2Gm_p}{r_f v_f} - v_f. \tag{7}$$

We can plug this into equation (5) to find

$$r_c = \frac{v_f}{\frac{2Gm_p}{r_f v_f} - v_f} r_f = r_f \left(\frac{2Gm_p}{r_f v_f^2} - 1\right)^{-1}. \tag{8}$$

2. Since the satellite is not burning any fuel, the gravitational attraction to the planet must be causing the centripetal acceleration enabling the uniform circular motion. This condition is expressed through Newton's second law for the satellite as

$$\vec{F}_G = m_s \vec{a}_{cent}. \tag{9}$$

The centripetal acceleration is given by $\vec{a}_{cent} = -r_0 \omega^2 \hat{r} = -r_0 (v_0/r_0)^2 \hat{r} = -(v_0^2/r_0)\hat{r}$, where ω is the angular speed of the satellite. Substituting this and the form of the gravitational force into equation (9) gives

$$-\frac{G m_s m_p}{r_0^2}\hat{r} = -m_s \frac{v_0^2}{r_0}\hat{r} \quad \Rightarrow \quad \frac{Gm_p}{r_0} = v_0^2 \quad \Rightarrow \quad v_0 = \sqrt{\frac{Gm_p}{r_0}}. \tag{10}$$

This solution can be checked by taking the circular case in our solution to part 1. If we let $r_f = r_c = r_0$, $v_f = v_c = v_0$, and substitute equation (10), we find that both equations (7) and (8) are satisfied.

Now let's compare v_0 with v_c, given that $r_0 = r_c$. Even though we have a solution for both, given by equations (7) and (10), this turns out to be surprisingly tricky. To make the comparison easier, we want to eliminate the velocity v_f from equation (7). Thus, we rearrange equation (5) to find

$$v_f = \frac{r_c}{r_f} v_c. \tag{11}$$

We substitute this into equation (7) to get

$$v_c = \frac{2Gm_p}{r_c v_c} - \frac{r_c}{r_f} v_c \quad \Rightarrow \quad v_c^2 = \frac{2Gm_p}{r_c} - \frac{r_c}{r_f} v_c^2 \quad \Rightarrow \quad \left(\frac{r_f + r_c}{r_f} \right) v_c^2 = \frac{2Gm_p}{r_c} \quad \Rightarrow \quad v_c = \sqrt{\frac{Gm_p}{r_c}} \sqrt{\frac{2r_f}{r_f + r_c}}. \tag{12}$$

We can now evaluate equation (10) at $r_0 = r_c$ to get

$$v_0 = \sqrt{\frac{Gm_p}{r_c}} \tag{13}$$

and compare with equation (12). Since $r_c < r_f$, we know that $\sqrt{2r_f/(r_f + r_c)} > 1$. Thus, we find that

$$v_c > v_0 \tag{14}$$

and the speed throughout the circular orbit is less than the speed at the point of closest approach in an elliptical orbit.