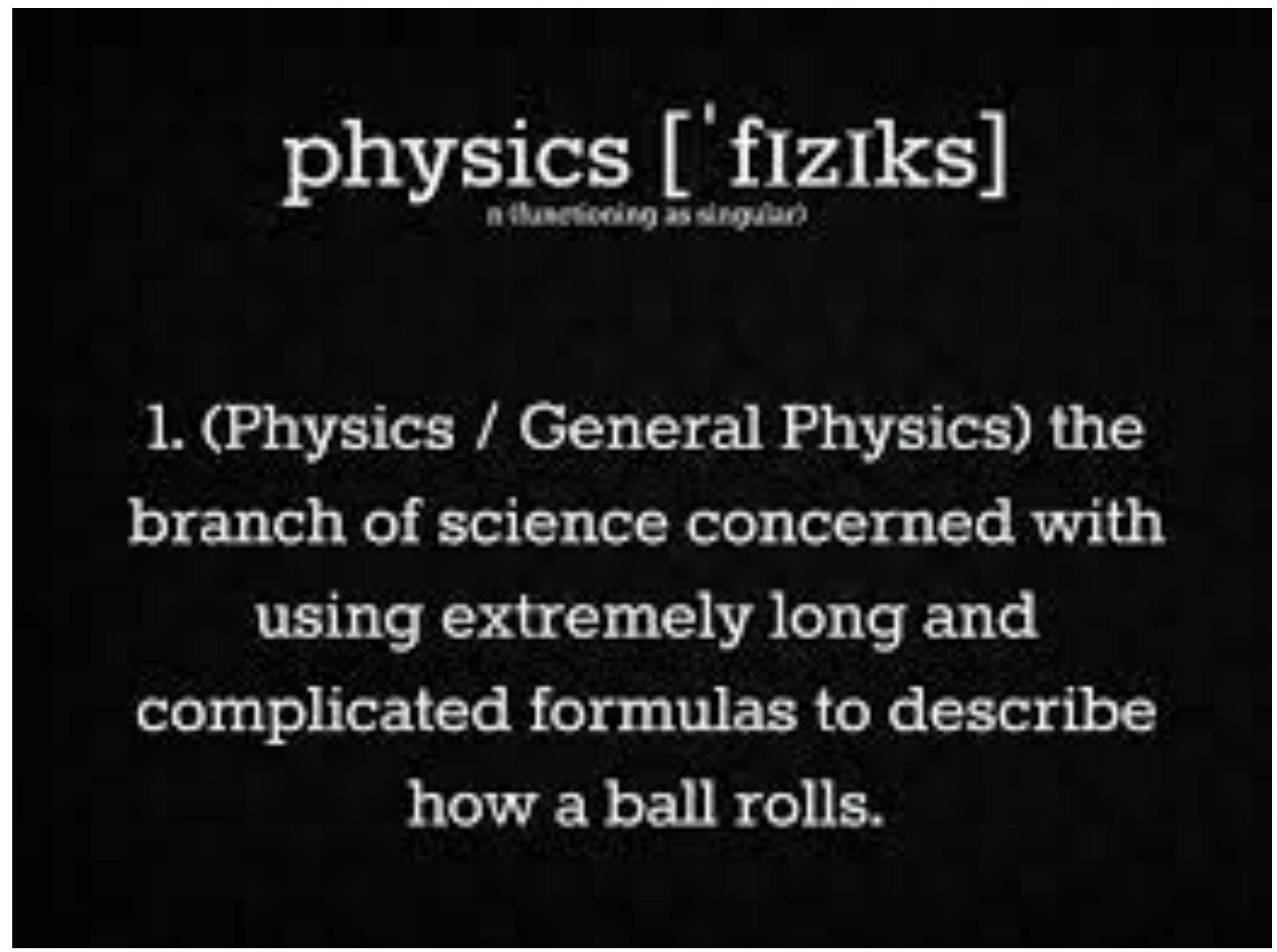
General Physics: Mechanics

PHYS-101(en)

Lecture 4a:
Circular motion



Dr. Marcelo Baquero
marcelo.baquero@epfl.ch
September 30th, 2024

Today's agenda (Serway 6, MIT 6 and 9)

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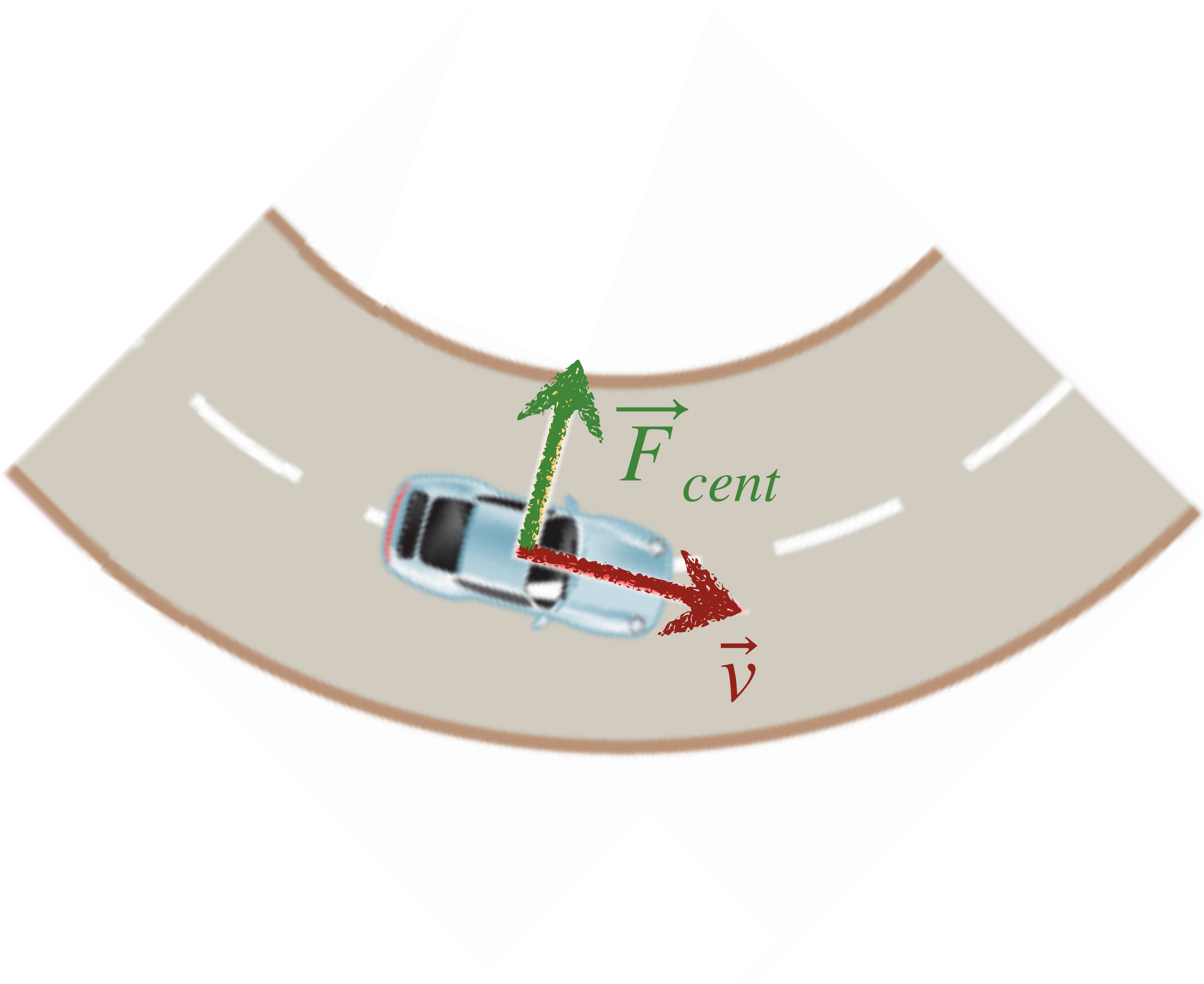
1. Circular motion
   - Polar, cylindrical, and spherical coordinate systems
   - Centripetal acceleration and centripetal force

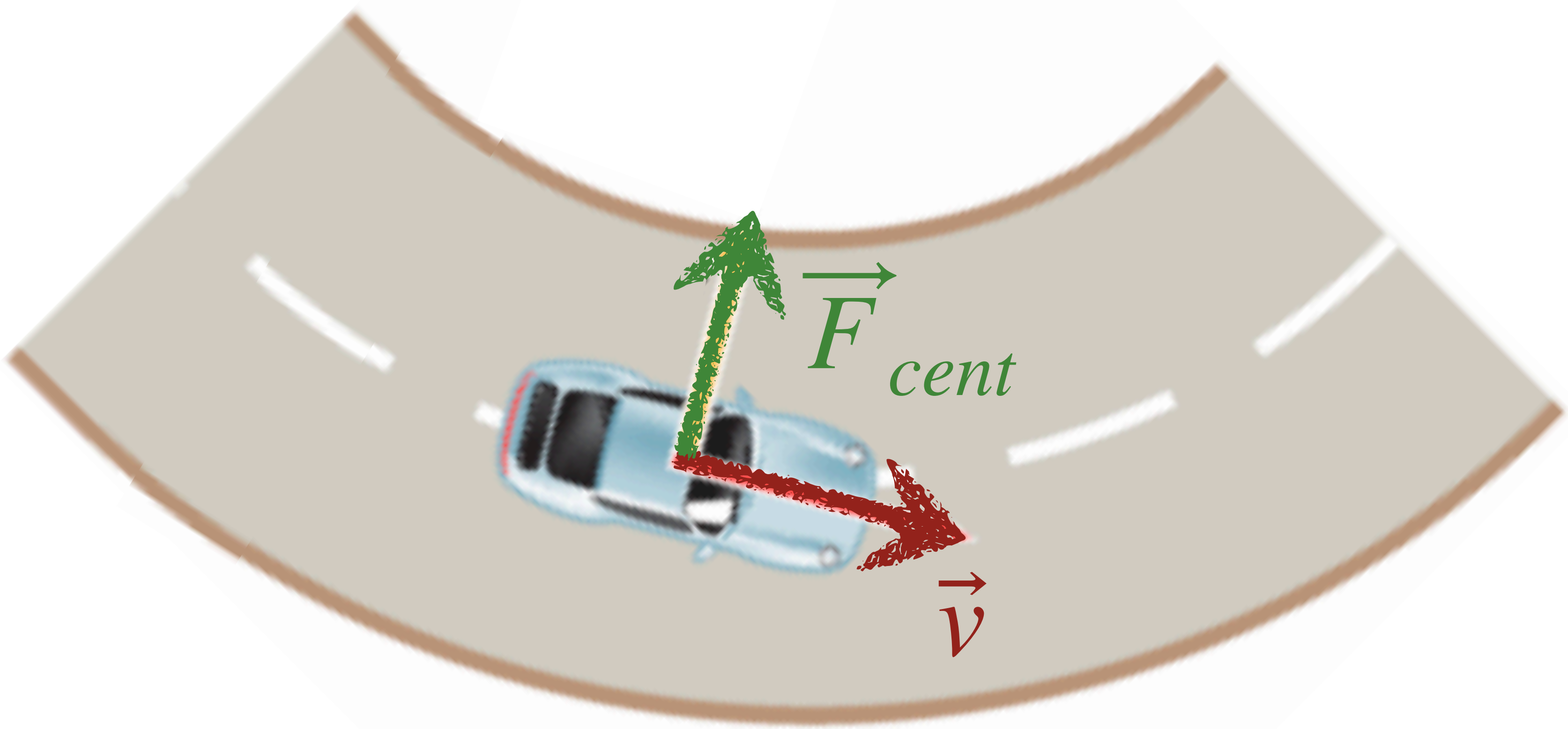
Circular motion in auto racing

- When a car goes around a curve, there must be a net force towards the center of the curve



Circular motion in auto racing

- When a car goes around a curve, there must be a net force towards the center of the curve
- If the road is flat, this force is supplied by friction



What if the frictional force is insufficient?

DEMO (681)

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Rotating pen

Circular motion in auto racing

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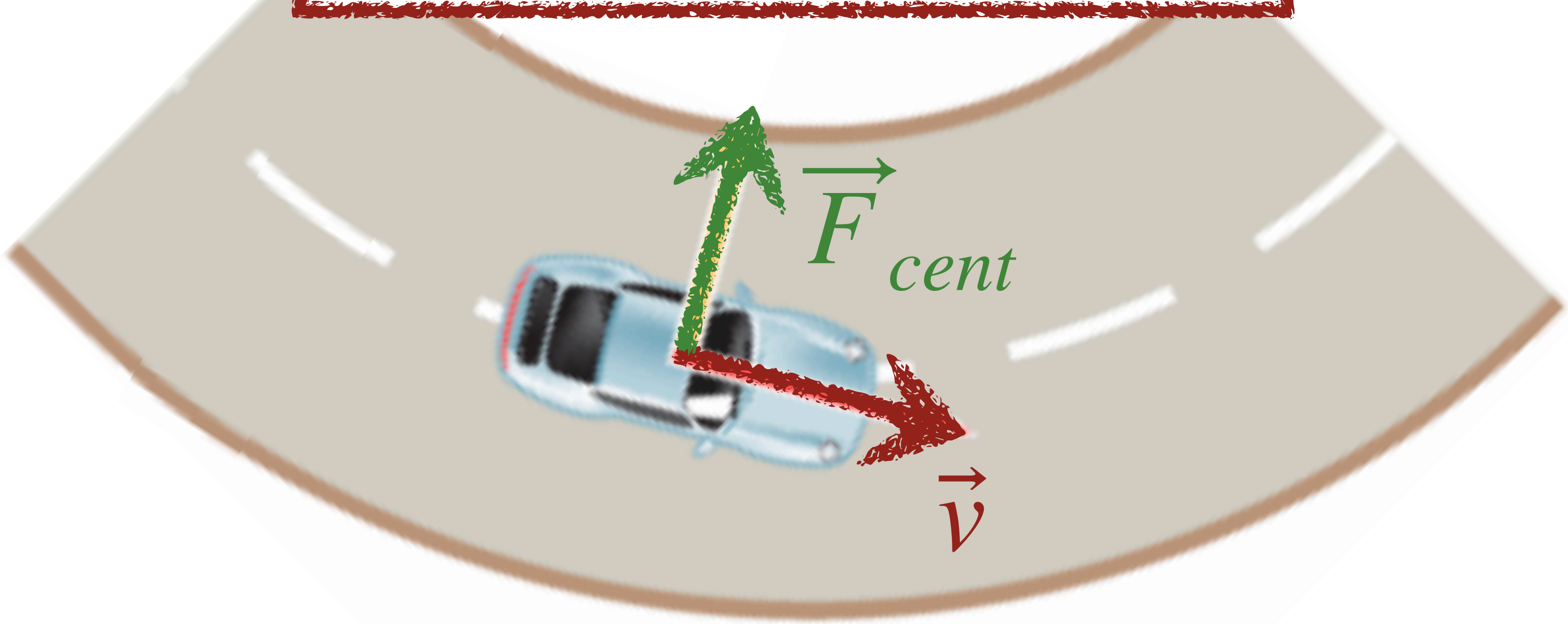
Newton's 1st law



What if the frictional force is insufficient?

Circular motion in auto racing

- If friction is insufficient, the car will tend to move in a straight line (see skid marks)

Circular motion in auto racing

- If friction is insufficient, the car will tend to move in a straight line (see skid marks)
- If tires roll without slipping, the friction is static

Circular motion in auto racing

- If friction is insufficient, the car will tend to move in a straight line (see skid marks)
- If tires roll without slipping, the friction is static
- If they slip, it is bad:
    1. Kinetic friction is smaller than static
    2. Static friction can point inwards (i.e. opposing the <u>impending</u> motion), while kinetic friction only opposes the direction of motion

Conceptual question

A particle moves with constant speed along the circular path shown on the right. Its velocity vector at two different times is also shown.

What is the direction of the acceleration when the particle is at point x?

A. ←

B. →

C. ↑

D. ↓

E. ⊙ (out of the page)

F. ⊗ (into of the page)

Conceptual question

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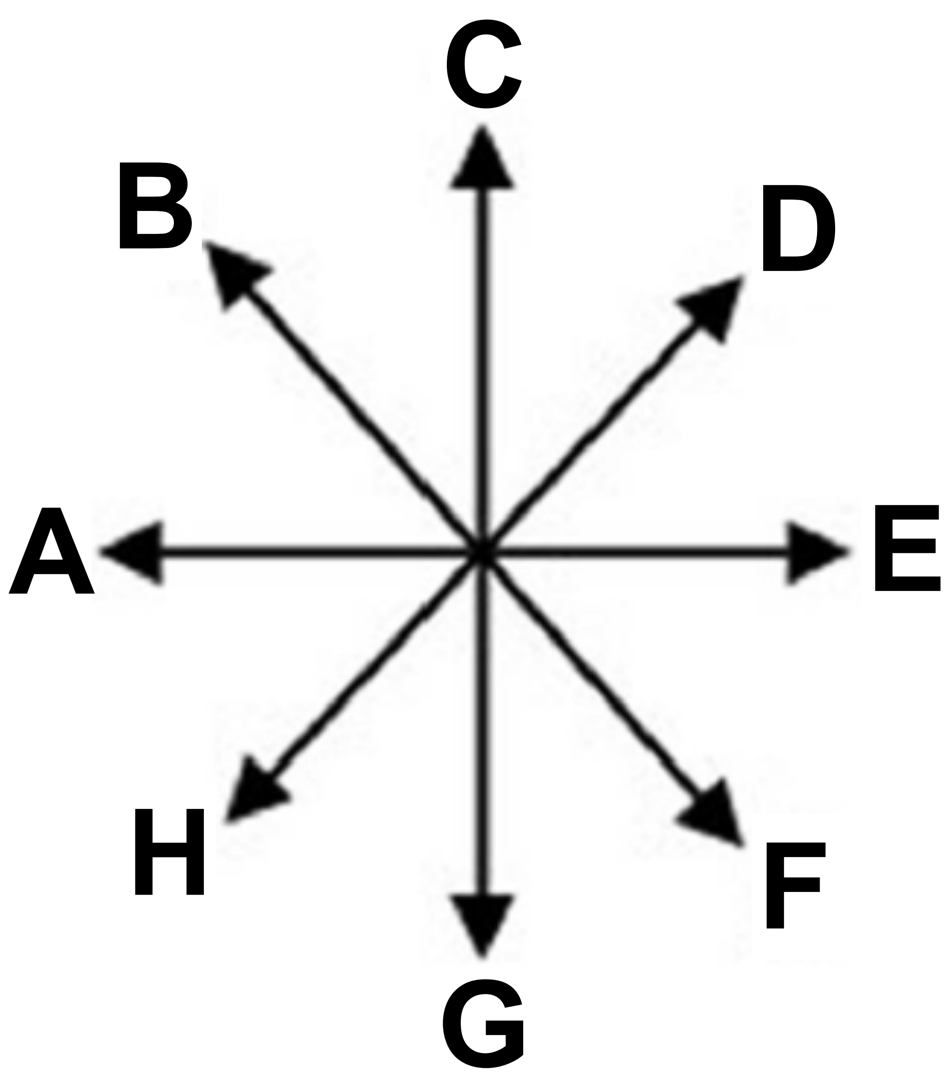
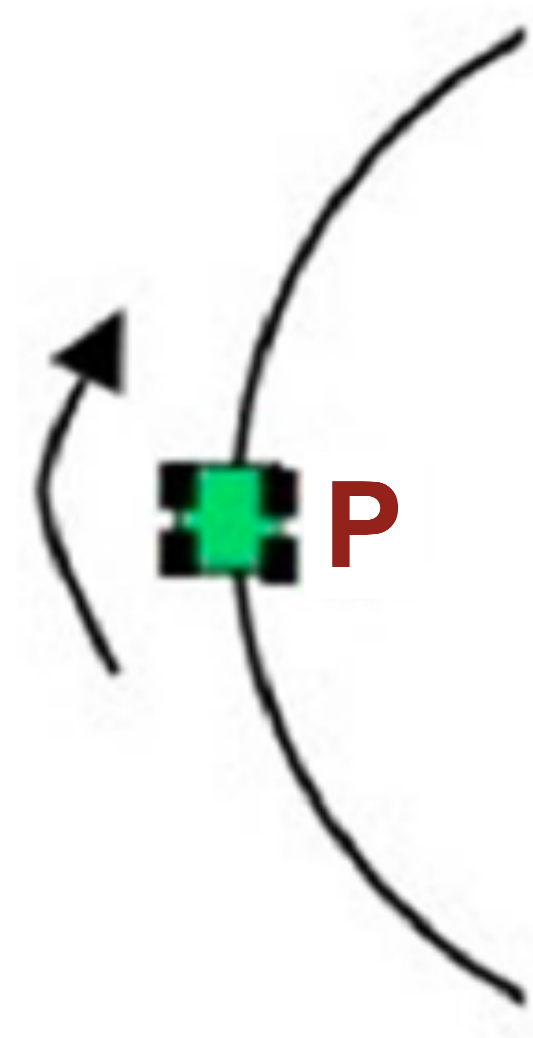
A poorly drawn golf cart moves around a circular path on a level surface with *decreasing* speed.

Which arrow could indicate the direction of the car's acceleration while passing the point **P**?





Polar coordinates

- Even when the origins are identical, there are *many* ways to specify the location of a point

- Some can be very useful and save you **much algebra**!

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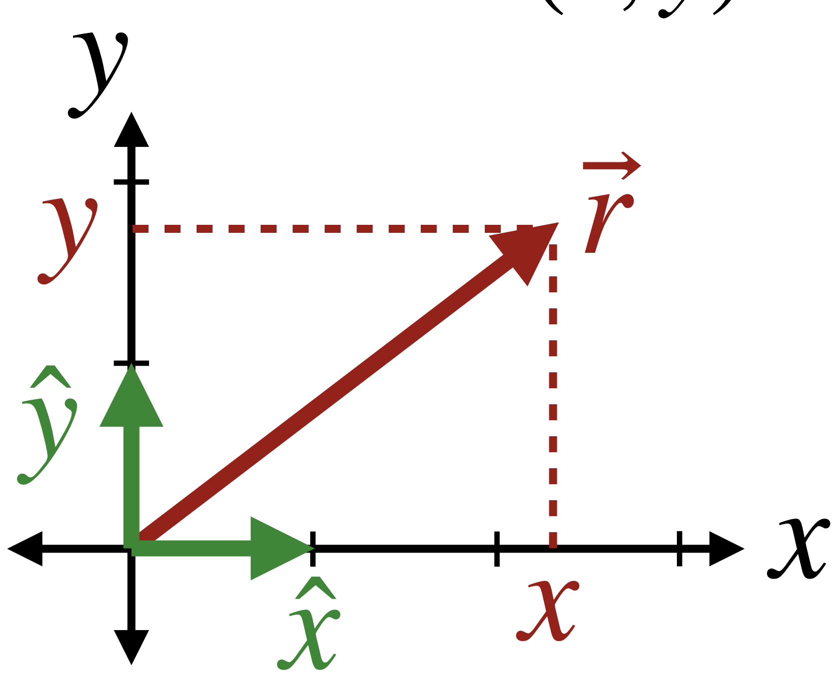
Cartesian (x, y)



$$\vec{r} = x\hat{x} + y\hat{y}$$

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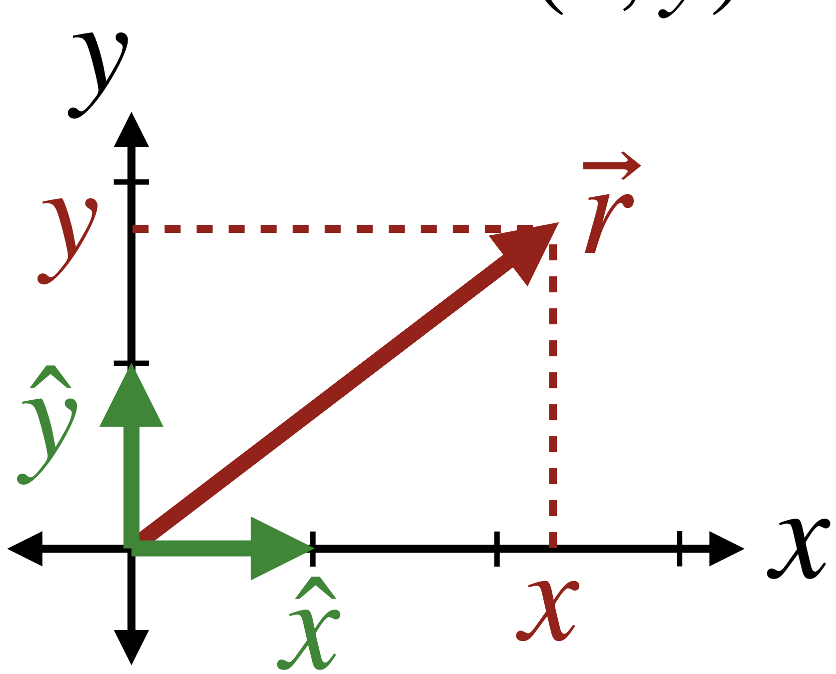

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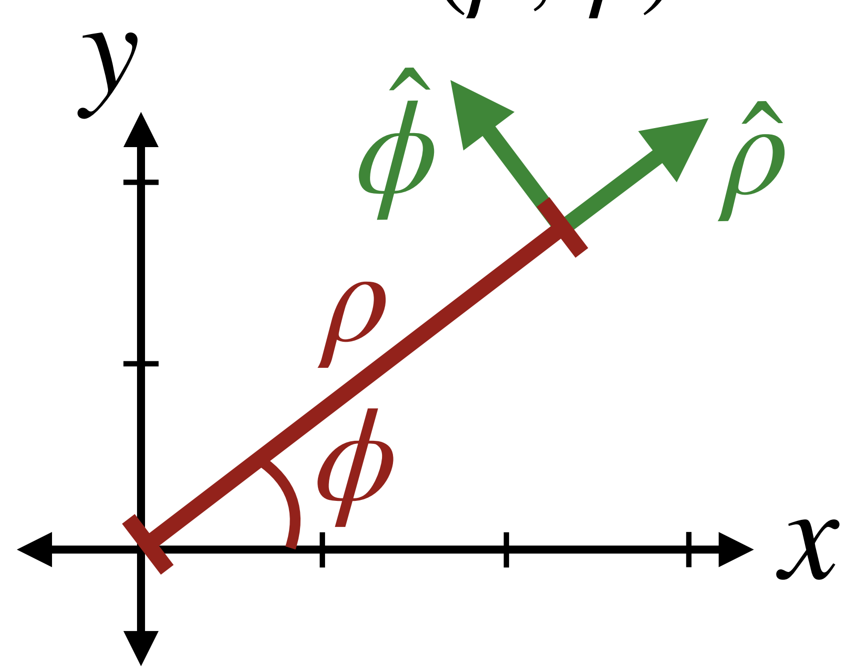

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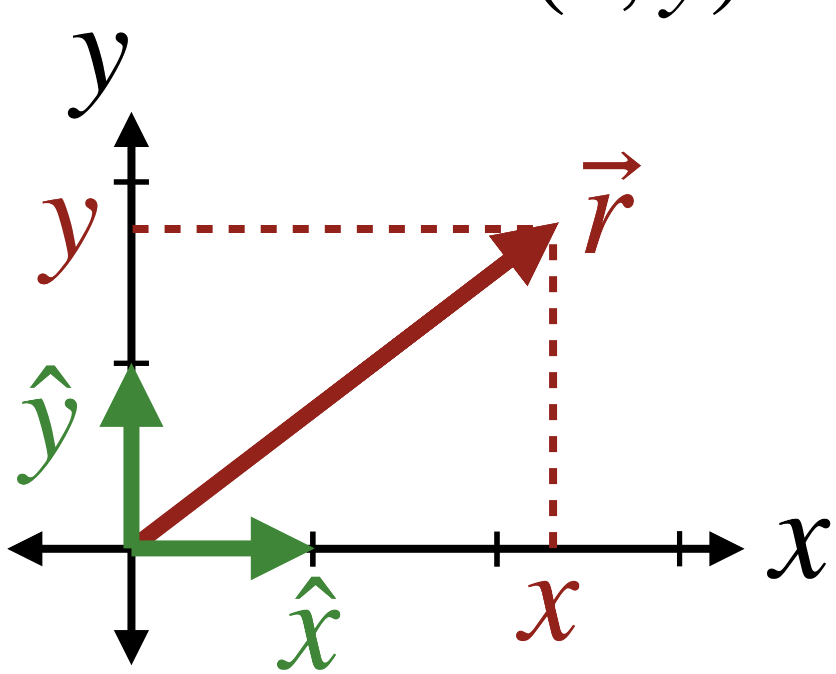

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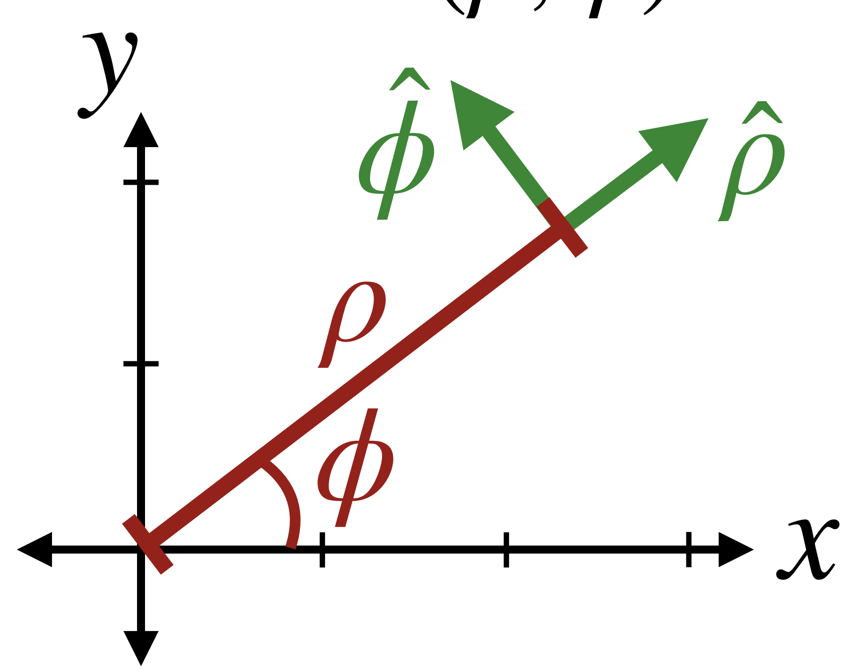

$0 \leq \rho < \infty$

$0 \leq \phi < 2\pi$ radians

or

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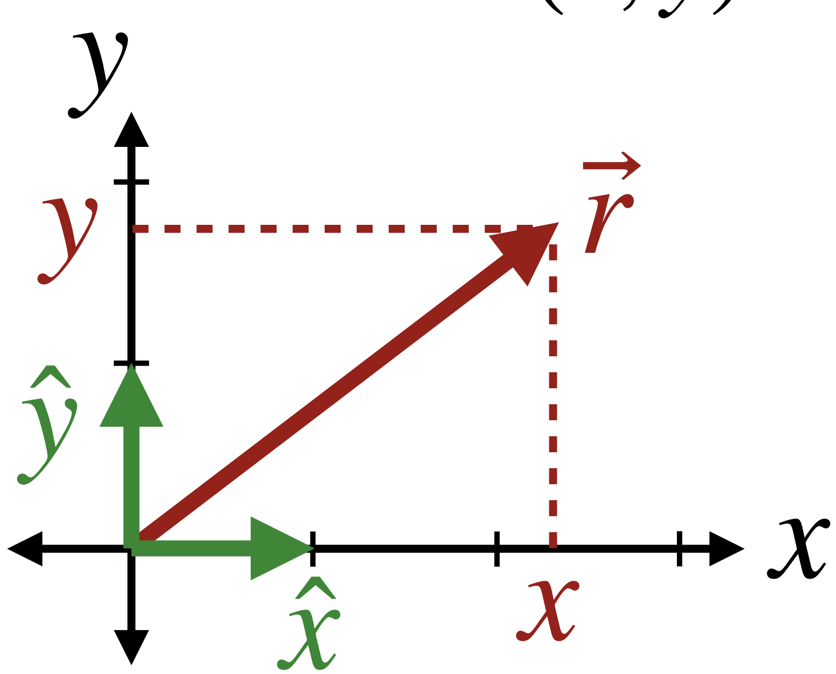

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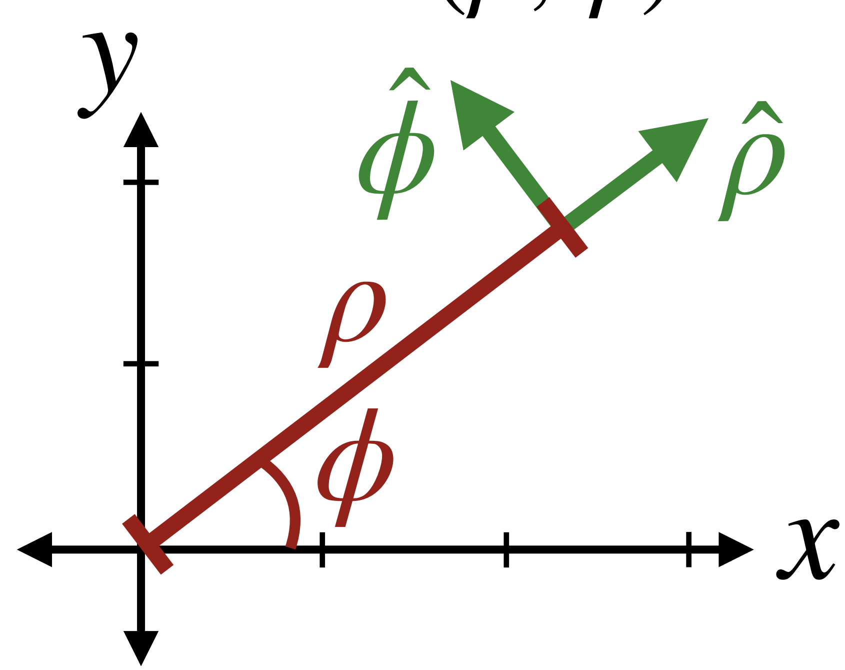

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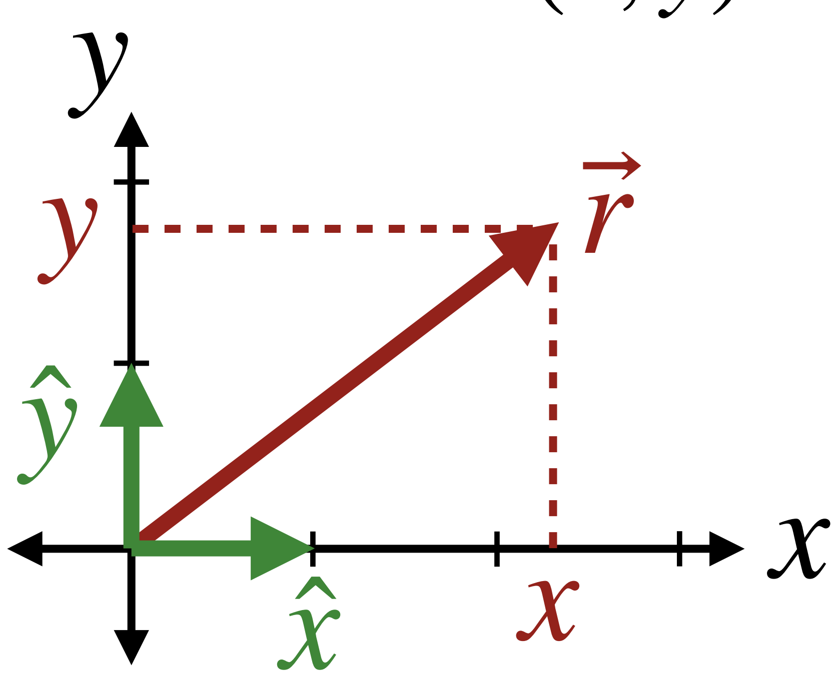

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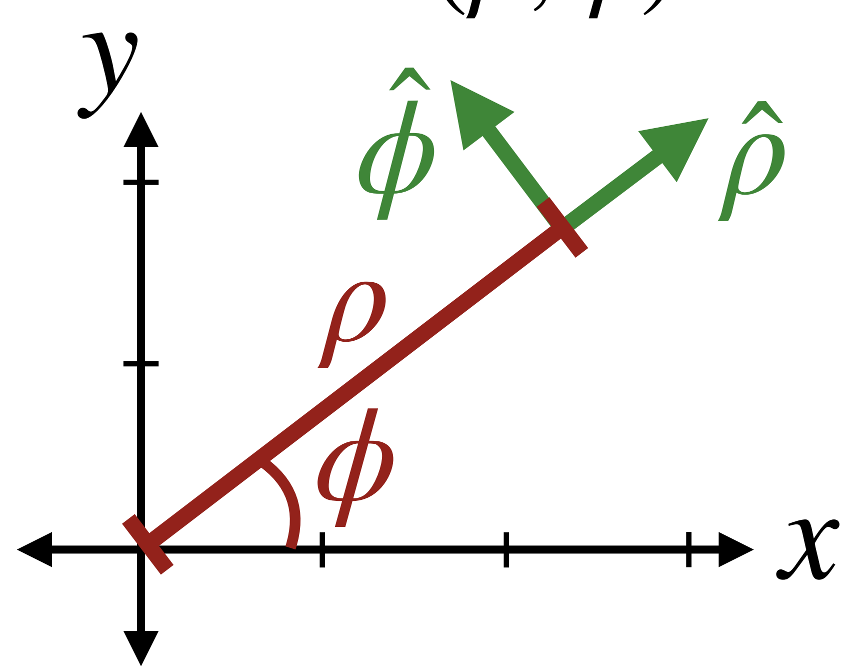

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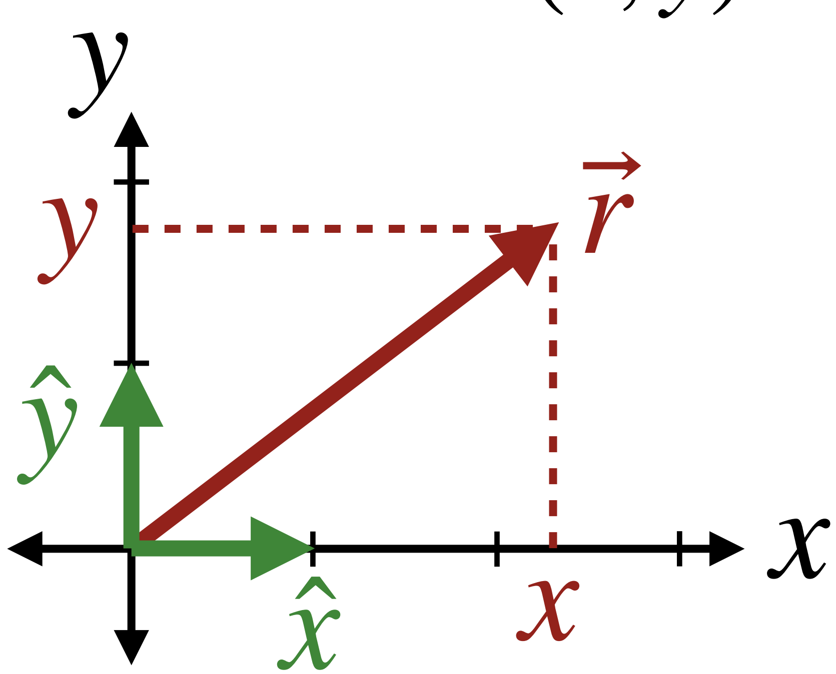

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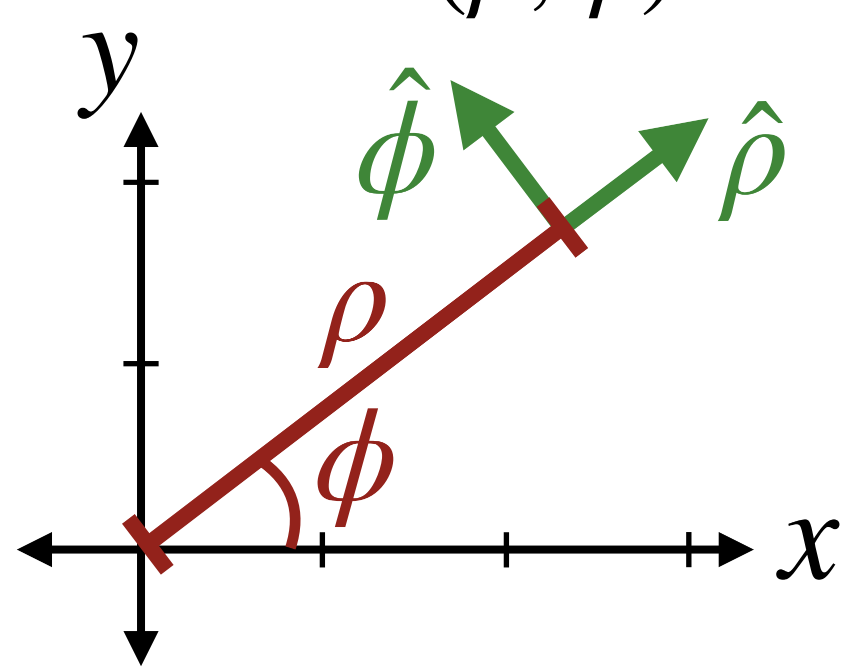

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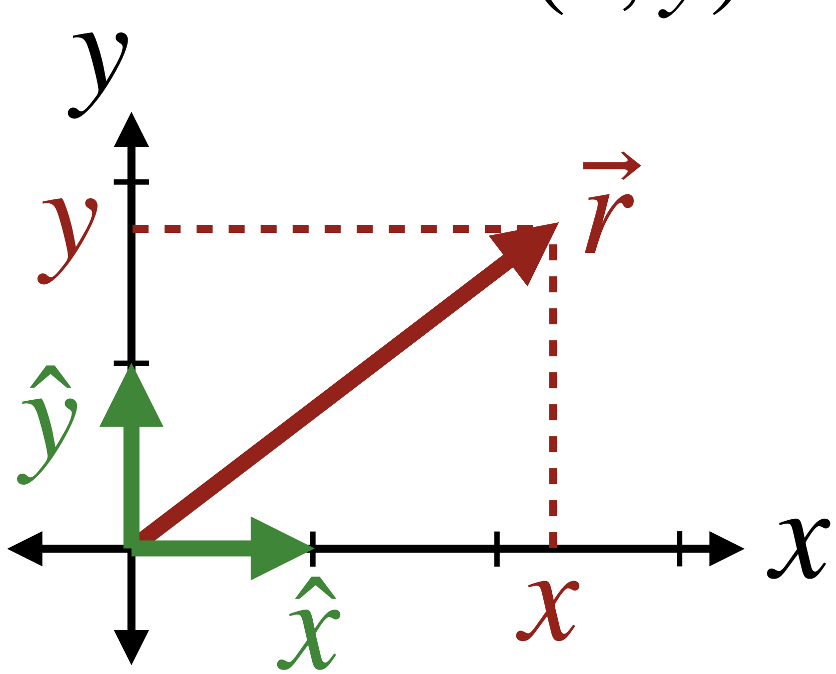

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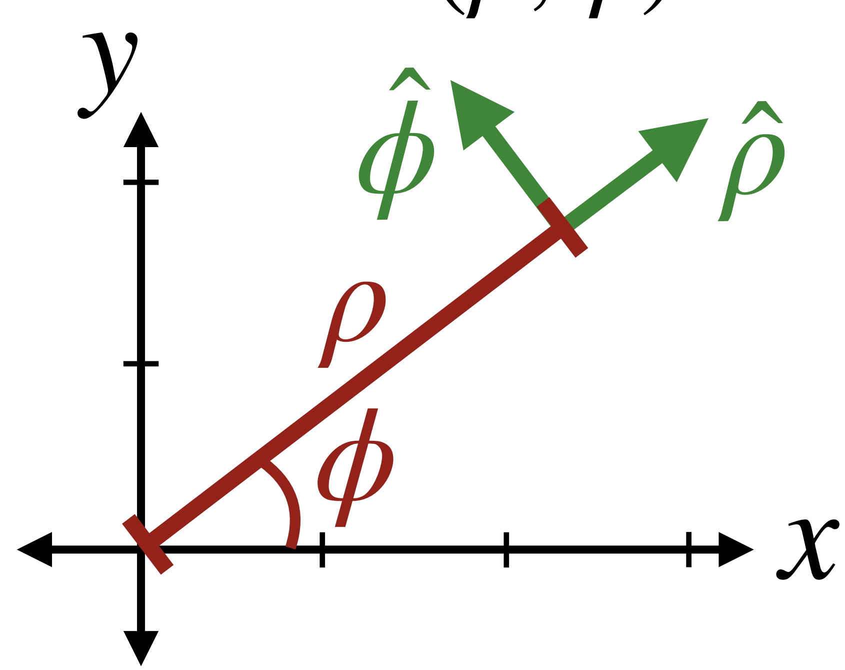

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$$\vec{r} = \rho\hat{\rho}$$

Cylindrical coordinates

- Just like polar, but adding the Cartesian *axial* direction $\hat{z}$

Cylindrical coordinates

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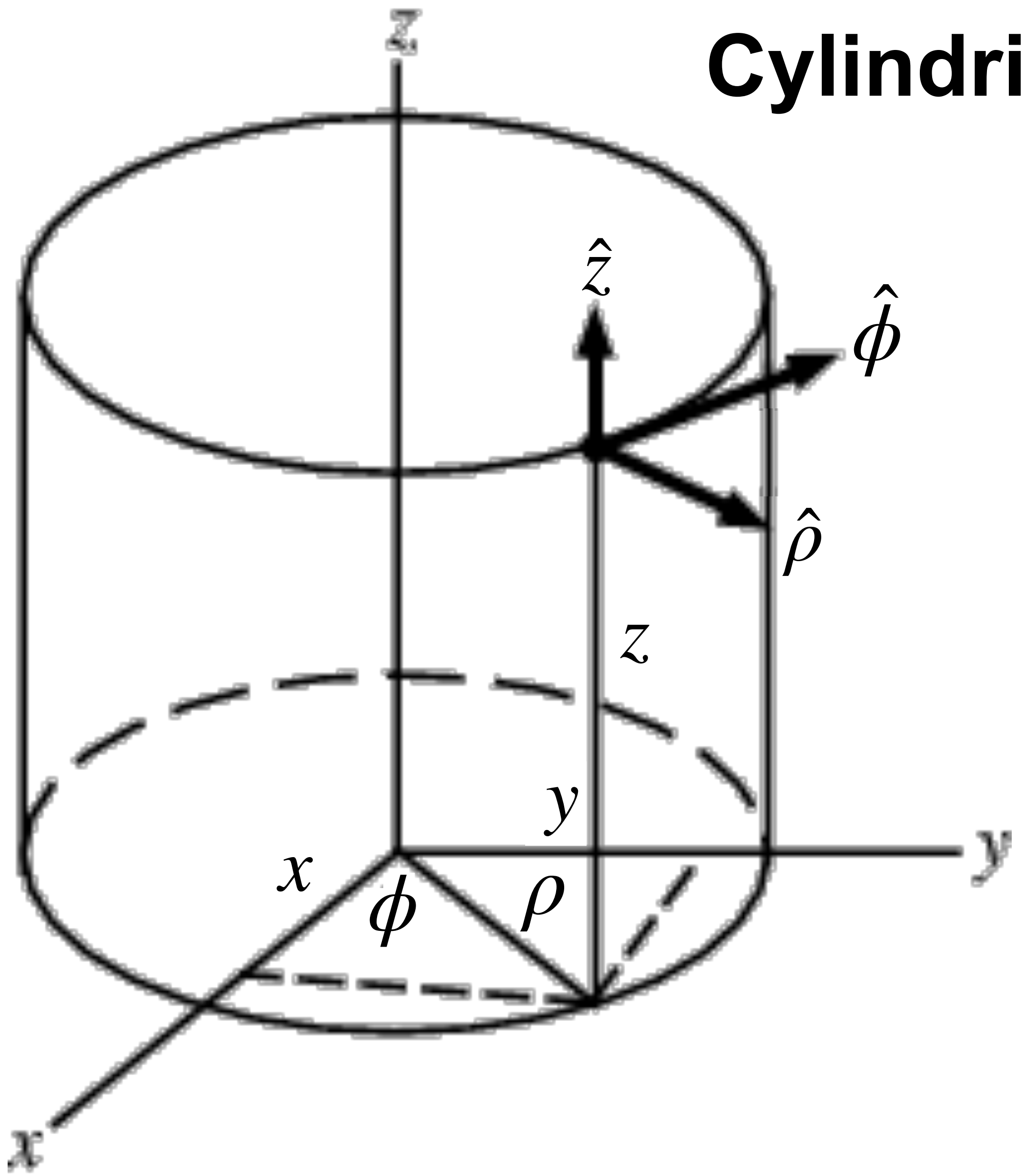
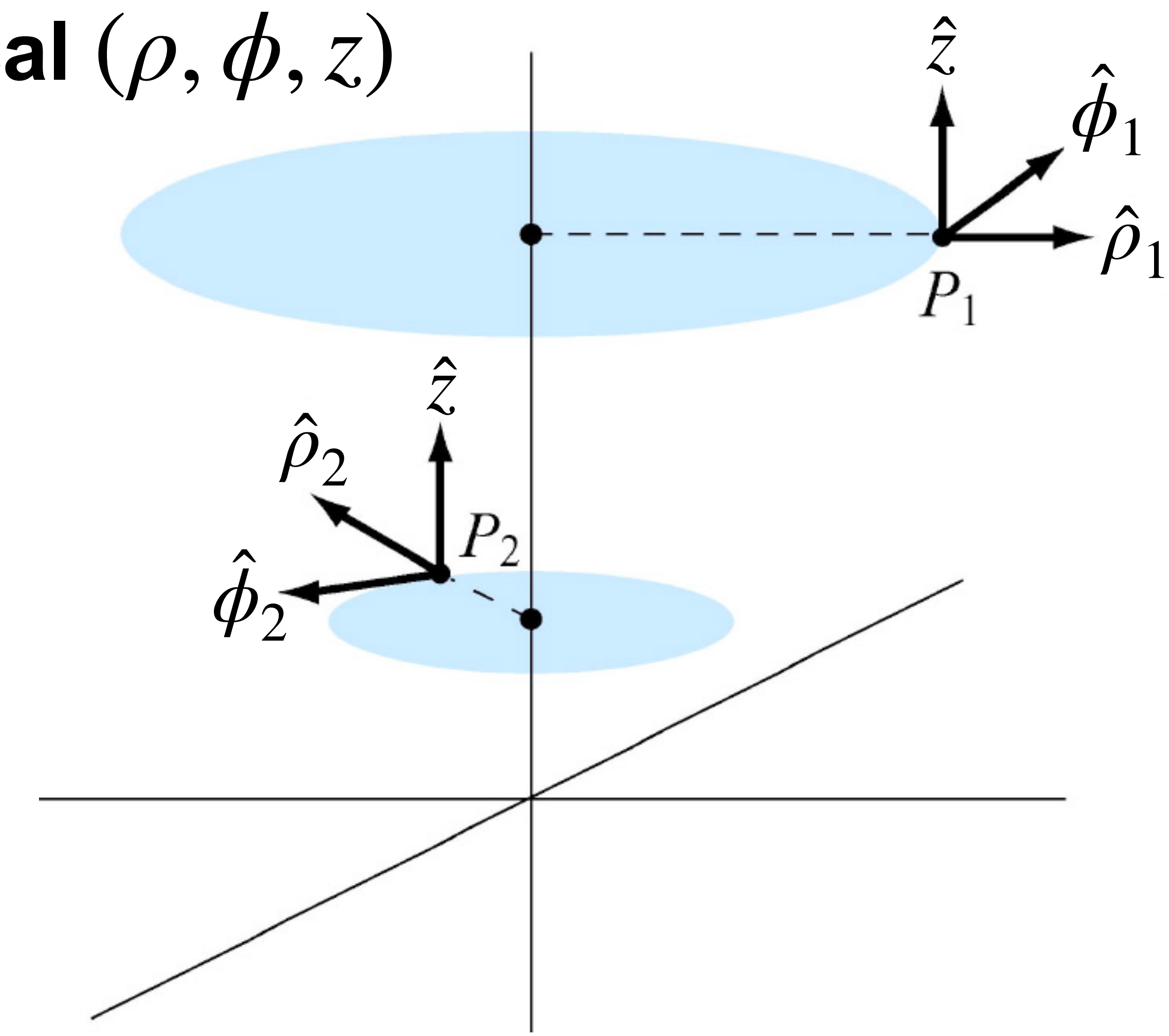
- At different locations, the *radial* $\hat{\rho}$ and *azimuthal* $\hat{\phi}$ directions change

Cylindrical (ρ, ϕ, z)





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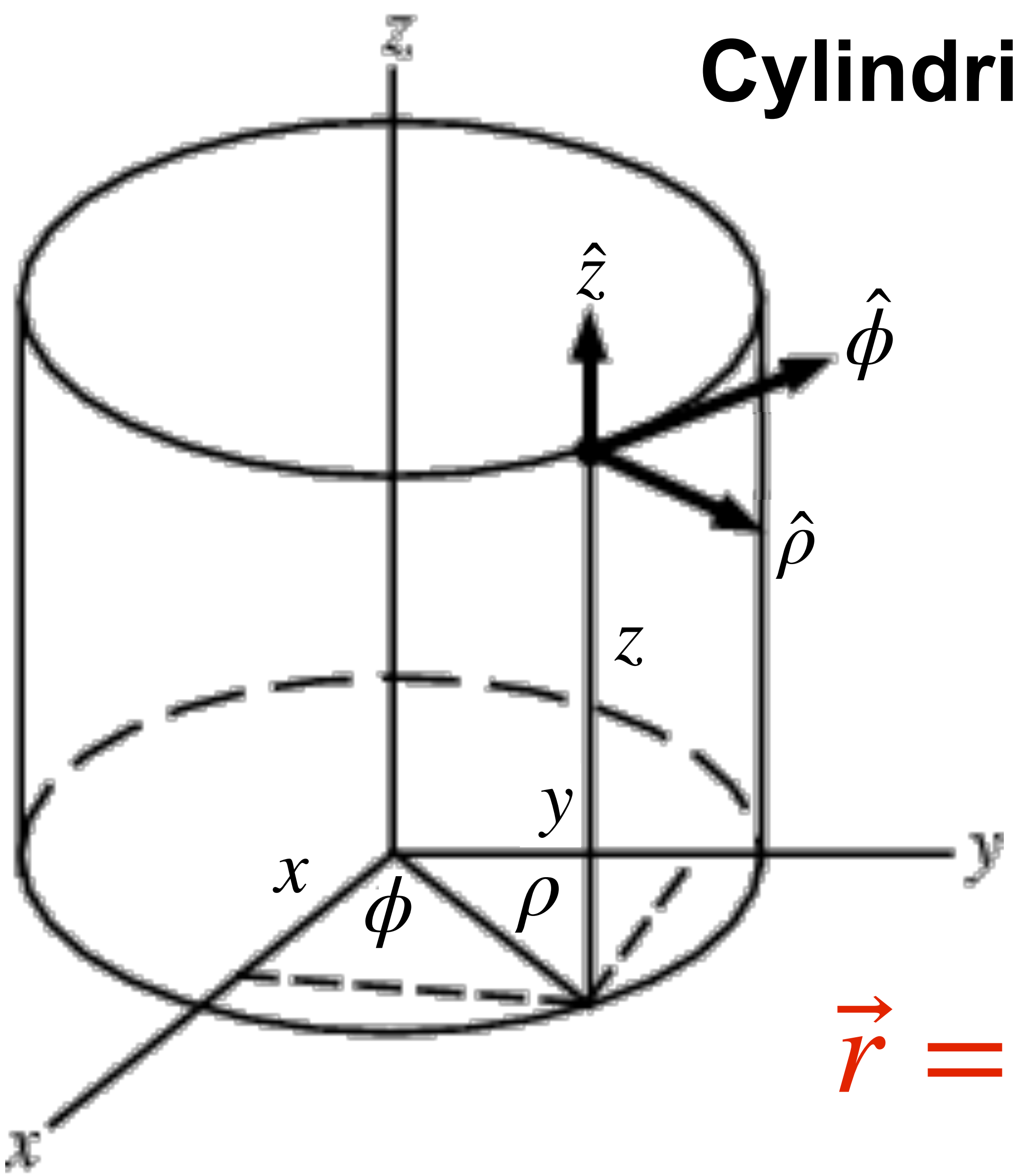
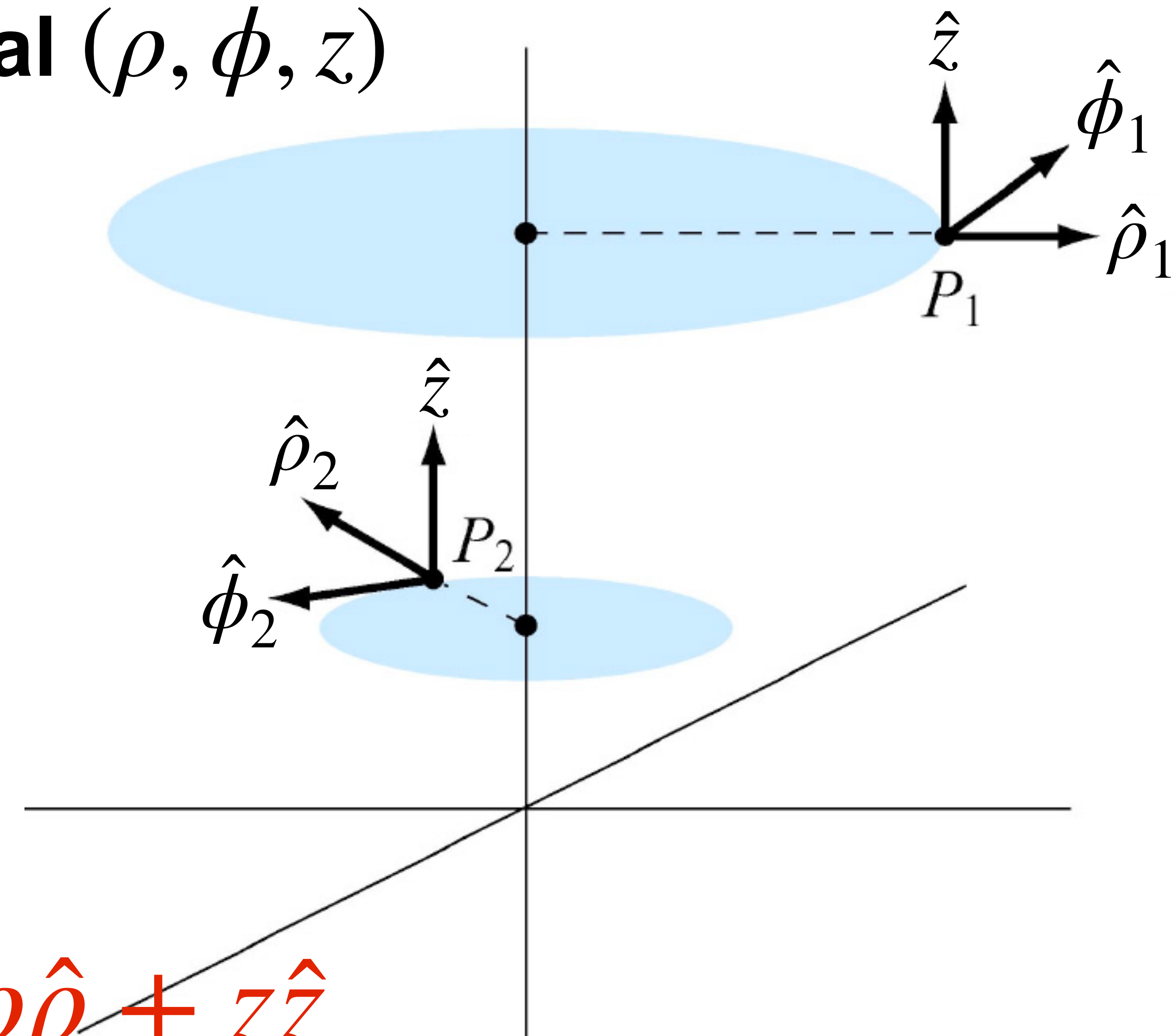


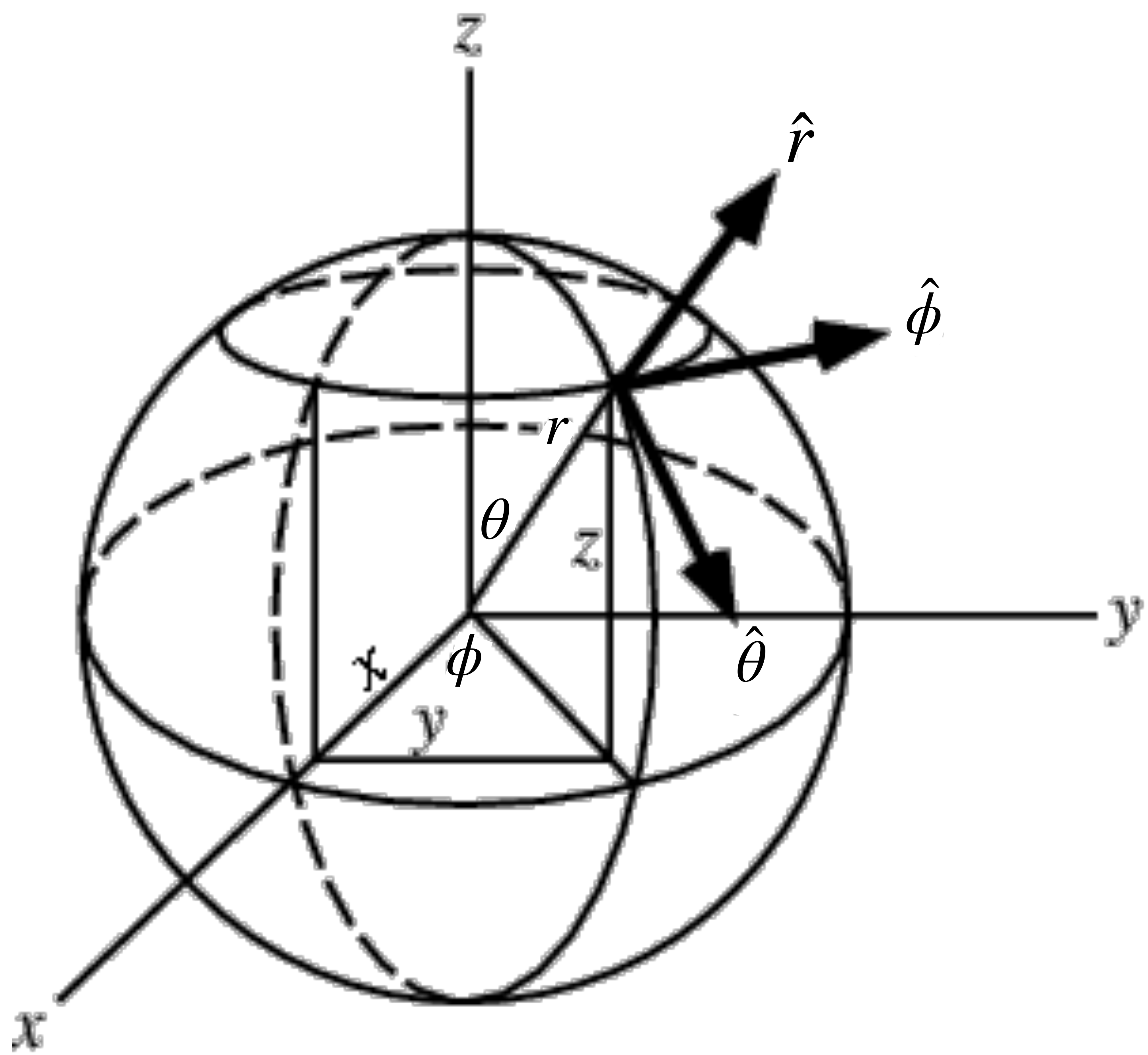


$$\vec{r} = \rho\hat{\rho} + z\hat{z}$$

Spherical coordinates

- One radial coordinate and two angles

Spherical (r, θ, ϕ)



Conversions between coordinates!

Transformation	Coordinate variables	Unit vectors	Vector components
Cartesian to cylindrical	$\rho = \sqrt{x^2 + y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\rho} = \cos\phi\hat{x} + \sin\phi\hat{y}$ $\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$ $\hat{z} = \hat{z}$	$A_\rho = A_x\cos\phi + A_y\sin\phi$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = \rho\cos\phi$ $y = \rho\sin\phi$ $z = z$	$\hat{x} = \cos\phi\hat{\rho} - \sin\phi\hat{\phi}$ $\hat{y} = \sin\phi\hat{\rho} + \cos\phi\hat{\phi}$ $\hat{z} = \hat{z}$	$A_x = A_\rho\cos\phi - A_\phi\sin\phi$ $A_y = A_\rho\sin\phi + A_\phi\cos\phi$ $A_z = A_z$
Cartesian to spherical	$r = \sqrt{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}\left(\sqrt{x^2 + y^2}/z\right)$ $\phi = \tan^{-1}(y/x)$	$\hat{r} = \sin\theta\cos\phi\hat{x}$ $+\sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$ $\hat{\theta} = \cos\theta\cos\phi\hat{x}$ $+\cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$ $\hat{\phi} = -\sin\phi\hat{y} + \cos\phi\hat{z}$	$A_r = A_x\sin\theta\cos\phi$ $+A_y\sin\theta\sin\phi + A_z\cos\theta$ $A_\theta = A_x\cos\theta\cos\phi$ $+A_y\cos\theta\sin\phi - A_z\sin\theta$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$
Spherical to Cartesian	$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$	$\hat{x} = \sin\theta\cos\phi\hat{r}$ $+\cos\theta\cos\phi\hat{\theta} - \sin\phi\hat{\phi}$ $\hat{y} = \sin\theta\sin\phi\hat{r}$ $+\cos\theta\sin\phi\hat{\theta} + \cos\phi\hat{\phi}$ $\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}$	$A_x = A_r\sin\theta\cos\phi$ $+A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ $A_y = A_r\sin\theta\sin\phi$ $+A_\theta\cos\theta\sin\phi + A_\phi\cos\phi$ $A_z = A_r\cos\theta - A_\theta\sin\theta$
Cylindrical to spherical	$r = \sqrt{\rho^2 + z^2}$ $\theta = \tan^{-1}(\rho/z)$ $\phi = \phi$	$\hat{r} = \sin\theta\hat{\rho} + \cos\theta\hat{z}$ $\hat{\theta} = \cos\theta\hat{\rho} - \sin\theta\hat{z}$ $\hat{\phi} = \hat{\phi}$	$A_r = A_\rho\sin\theta + A_z\cos\theta$ $A_\theta = A_\rho\cos\theta - A_z\sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$\rho = r\sin\theta$ $\phi = \phi$ $z = r\cos\theta$	$\hat{\rho} = \sin\theta\hat{r} + \cos\theta\hat{\theta}$ $\hat{\phi} = \hat{\phi}$ $\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}$	$A_\rho = A_r\sin\theta + A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_r\cos\theta - A_\theta\sin\theta$

Example: Cylindrical coordinates

Write the following in Cartesian and cylindrical coordinates:

A. The equation of a sphere of radius r_0 centered at the origin.

Example: Cylindrical coordinates

Write the following in Cartesian and cylindrical coordinates:

A. The equation of a sphere of radius r_0 centered at the origin.

B. The equation of a cylinder parallel to the z axis with a radius ρ_0 and length L, whose center is at the origin.

Motion in cylindrical coordinates

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- Derive the expressions for the velocity and acceleration of an object in cylindrical coordinates

Motion in cylindrical coordinates

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Motion in spherical coordinates

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Motion in spherical coordinates

- Derive the expressions for the velocity and acceleration of an object in spherical coordinates
- Just kidding, it's quite horrible:

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi}$$

$$\begin{aligned}\vec{a} = &\left(\ddot{r} - r\left(\dot{\theta}\right)^2 - r\left(\dot{\phi}\right)^2\sin^2\theta\right)\hat{r} \\ &+\left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\left(\dot{\phi}\right)^2\sin\theta\cos\theta\right)\hat{\theta} \\ &+\left(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta\right)\hat{\phi}\end{aligned}$$

DEMO (52)

Flexible spinning rings

Quantifying speed in **uniform** circular motion

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- The *period* T is the time the object takes to complete one full revolution; units of [s]

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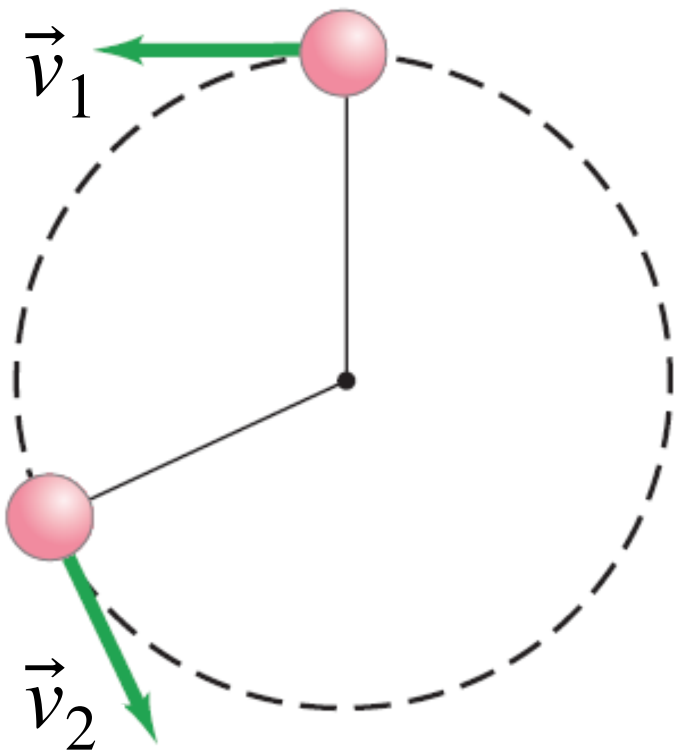
- Since the distance traveled per revolution is $2\pi\rho_0$, we can calculate the speed

$$v = \frac{2\pi\rho_0}{T} = 2\pi\rho_0 f = \rho_0\overline{\omega}$$

Velocity in circular motion

- Find the velocity of an object moving in a circle with a radius ρ_0.



Quantifying velocity in circular motion

- The angular *speed* is $\omega = \dot{\phi} = \dfrac{d\phi}{dt}$

- Like velocity, the angular velocity is a vector $\overrightarrow{\omega}$, but defined given an axis of rotation

- To find the direction of $\overrightarrow{\omega}$ we need to introduce *cross (vector) products*

Review: Cross (or vector) product

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- Two vectors are multiplied in a cross product to produce another vector

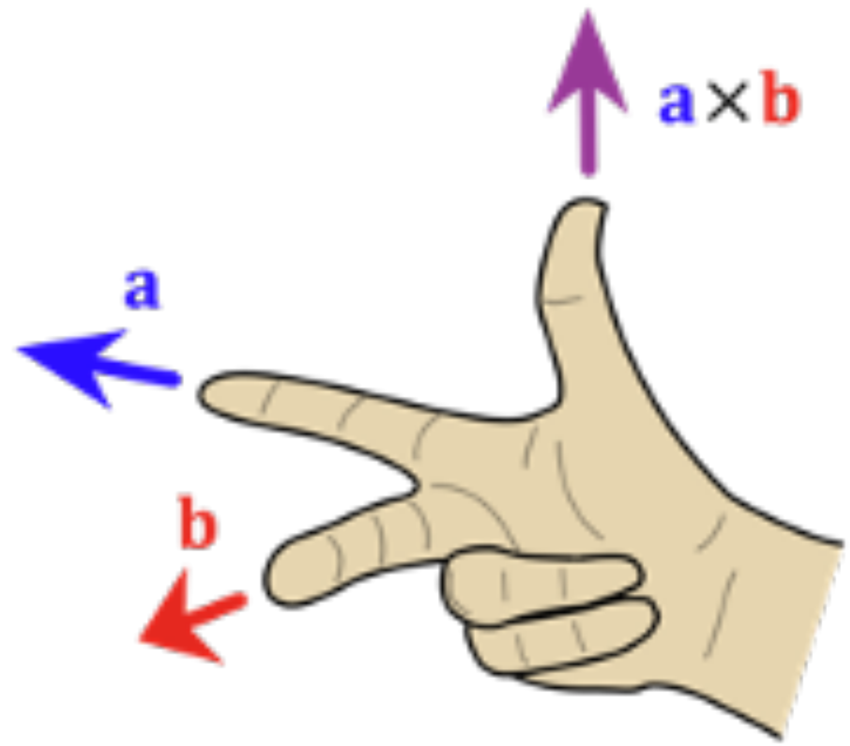
$$\vec{a} \times \vec{b} = \vec{c}$$

Review: Cross (or vector) product

- Two vectors are multiplied in a cross product to produce another vector

$$\vec{a} \times \vec{b} = \vec{c}$$

- Magnitude: $|\vec{c}| = c = ab\sin\theta$
- Direction: Use right hand rule



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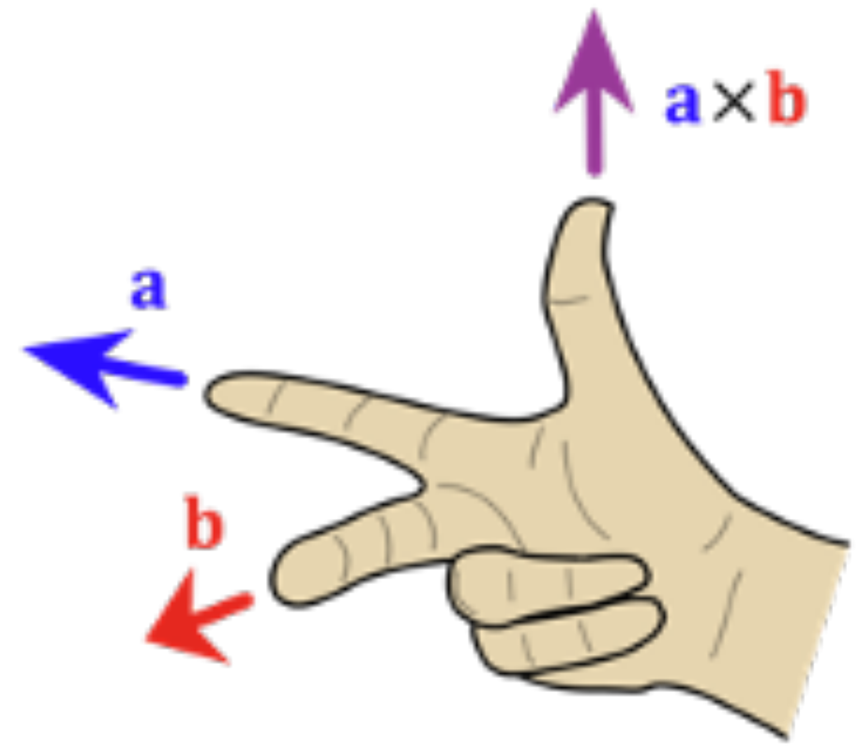

- If $\vec{a} \,||\, \vec{b}$, then $\vec{a} \times \vec{b} = 0$ or if $\vec{a} \perp \vec{b}$, then $|\vec{a} \times \vec{b}| = ab$
- Not commutative: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- Distributive: $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- Derivative product rule: $\frac{d}{dt}(\vec{a} \times \vec{b}) = \frac{d\vec{a}}{dt} \times \vec{b} + \vec{a} \times \frac{d\vec{b}}{dt}$

Review: Cross (or vector) product

- How to compute $\vec{a} \times \vec{b}$ component-by-component

$$\vec{a} \times \vec{b} = (a_y b_z - a_z b_y)\hat{x} + (a_z b_x - a_x b_z)\hat{y} + (a_x b_y - a_y b_x)\hat{z}$$

where $\vec{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$

$$\vec{b} = b_x\hat{x} + b_y\hat{y} + b_z\hat{z}$$

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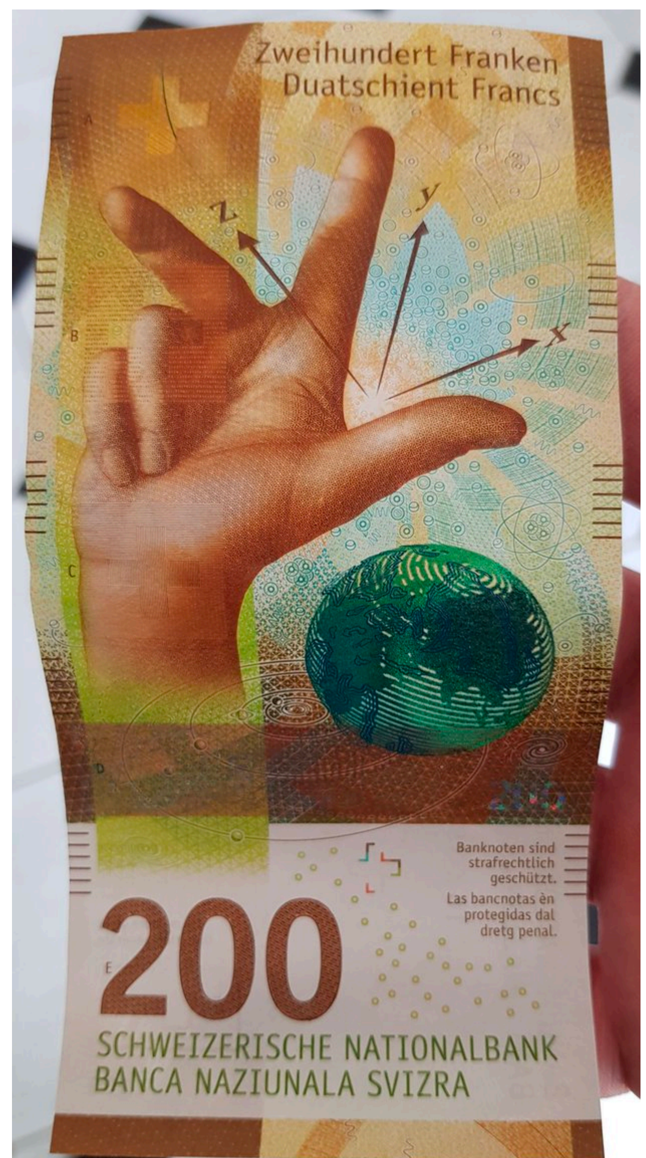

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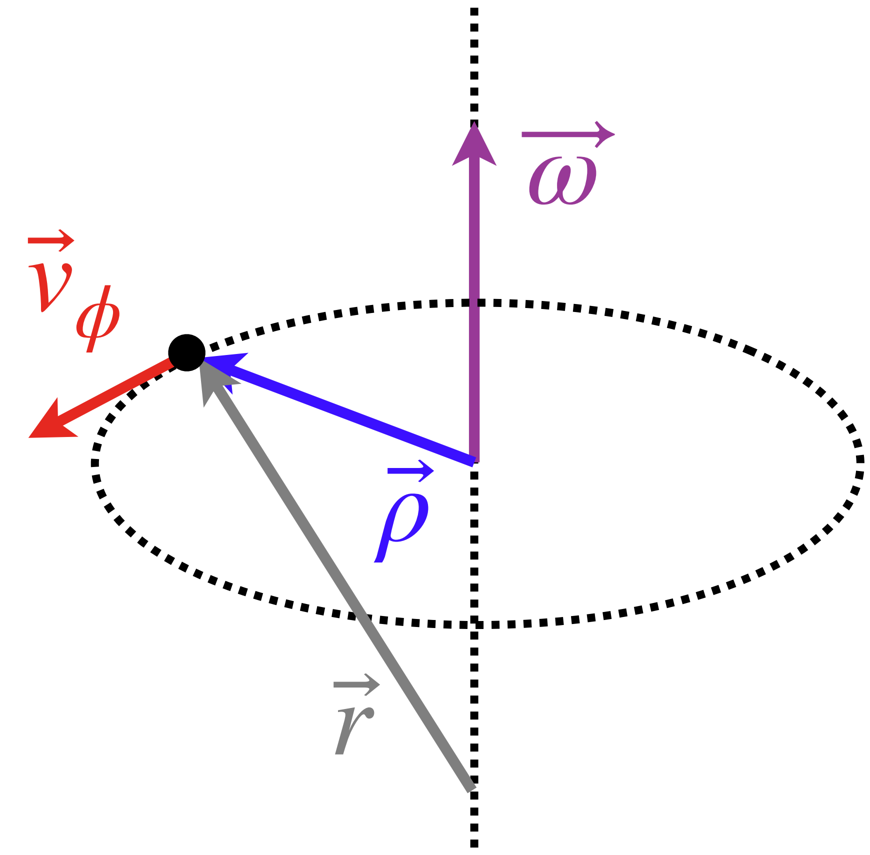
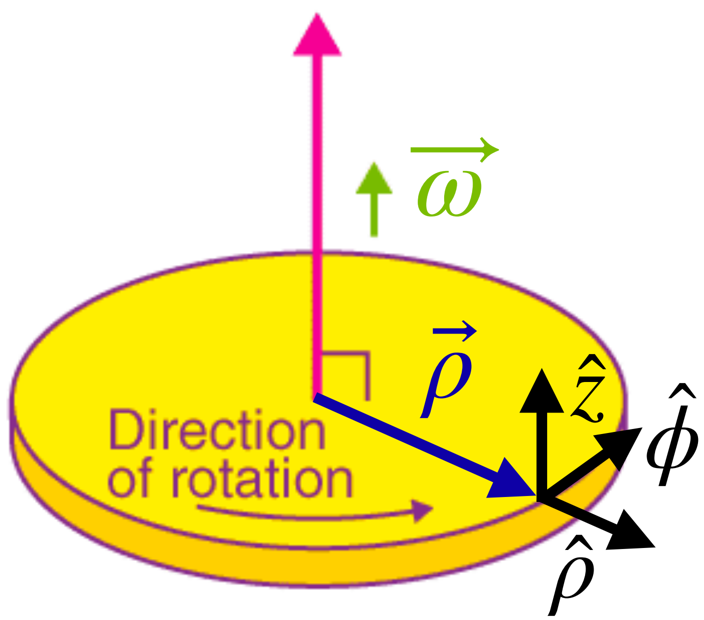
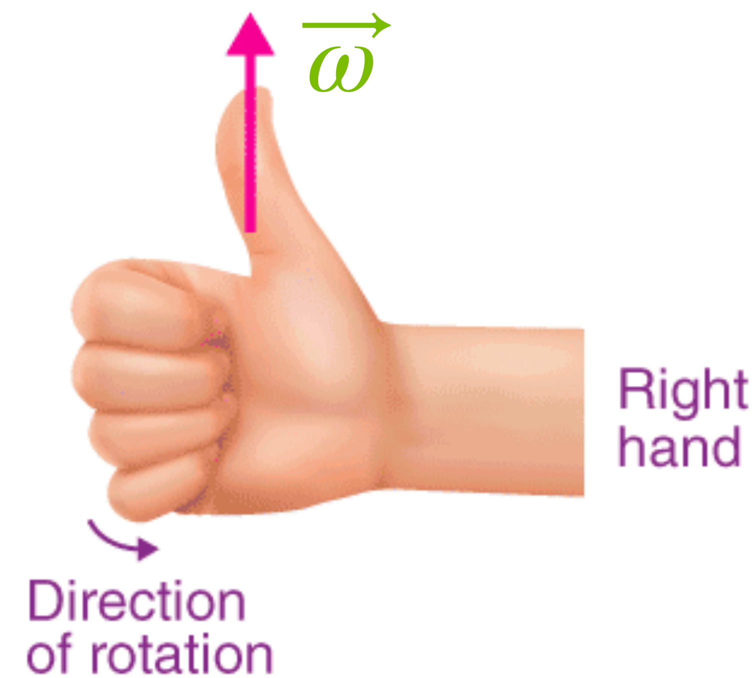
- Points along the axis of rotation according to the right-hand rule



$$\overrightarrow{\omega} = \frac{\vec{\rho} \times \vec{v}_\phi}{\rho^2} \qquad \text{so} \qquad \vec{v}_\phi = \overrightarrow{\omega} \times \vec{\rho}$$

Quantifying velocity in circular motion





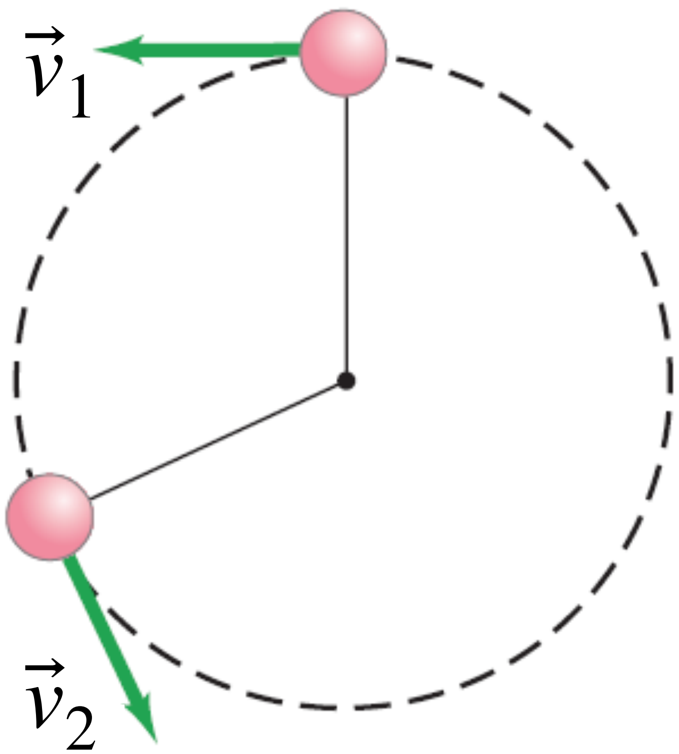
- Alternatively, there's a different right hand rule that shows the direction of $\vec{\omega}$
- Often (but not always!) $\vec{\omega}$ is in the $\pm\hat{z}$ direction, due to the way we often define our cylindrical coordinate systems
- We can also use this to reinterpret some past results:

$$d\hat{\rho}/dt = \omega\hat{\phi} = \vec{\omega} \times \hat{\rho} \quad \text{and} \quad d\hat{\phi}/dt = -\omega\hat{\rho} = \vec{\omega} \times \hat{\phi}$$

Acceleration in circular motion

EPFL
Swiss Plasma Center

- Find the acceleration of an object moving in a circle with a radius ρ_0.



Quantifying acceleration in circular motion

Swiss Plasma Center

- The magnitude of the angular acceleration is $\alpha = \ddot{\phi} = \frac{d^2\phi}{dt^2}$

Quantifying acceleration in circular motion

- The magnitude of the angular acceleration is $\alpha = \ddot{\phi} = \dfrac{d^2\phi}{dt^2}$
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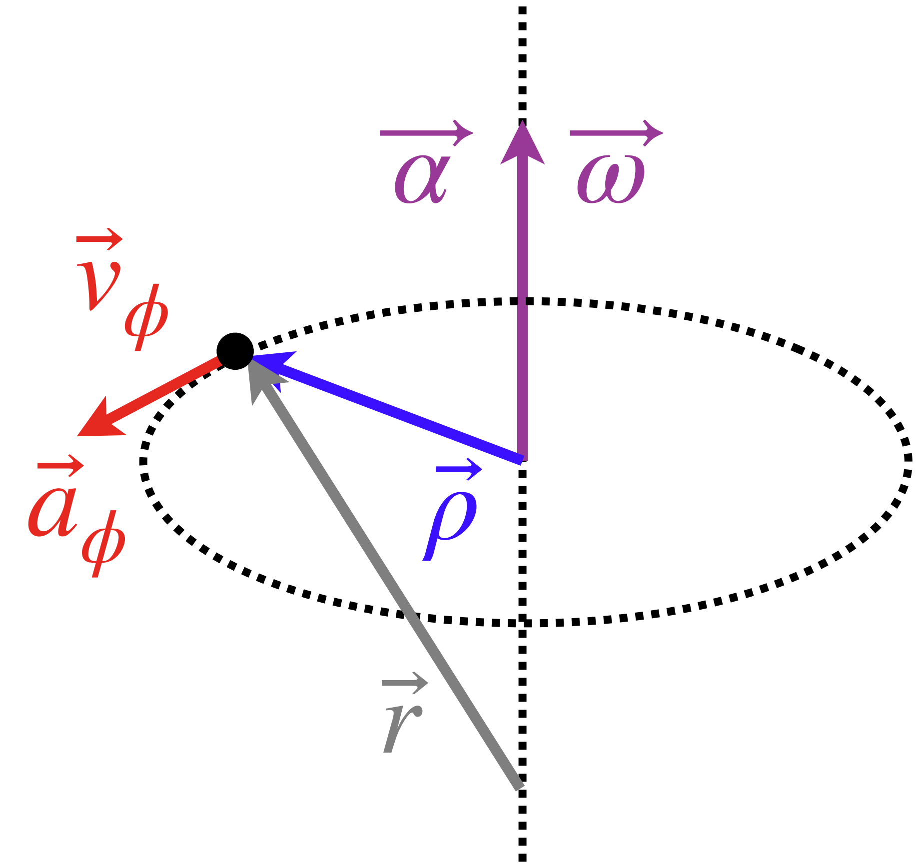
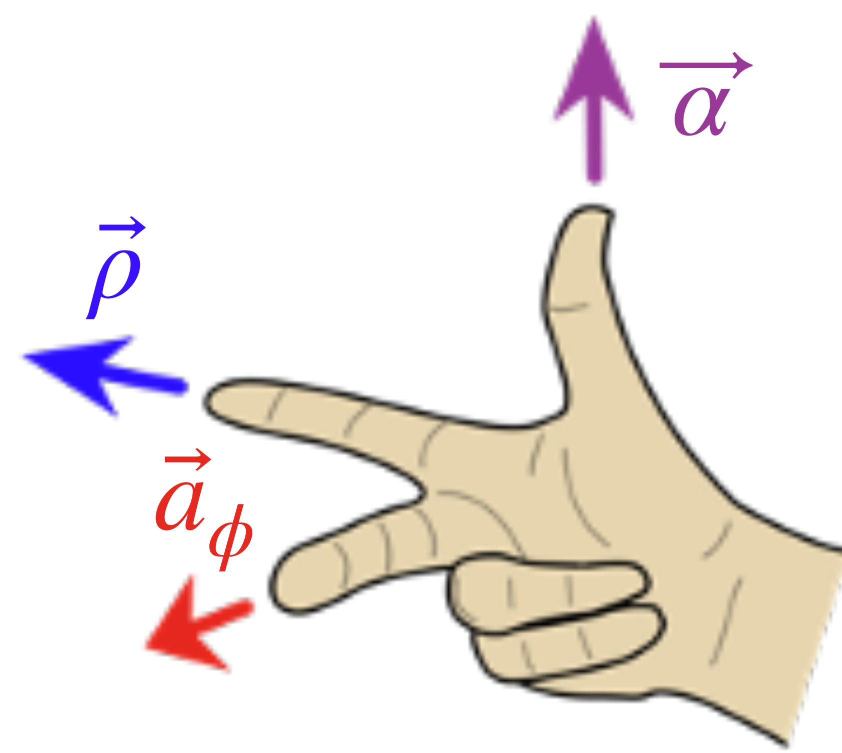
$$\vec{a}_\phi = \overrightarrow{\alpha} \times \vec{\rho} \quad \text{and} \quad \overrightarrow{\alpha} = \frac{\vec{\rho} \times \vec{a}_\phi}{\rho^2}$$





Quantifying acceleration in circular motion

- The magnitude of the angular acceleration is $\alpha = \ddot{\phi} = \dfrac{d^2\phi}{dt^2}$

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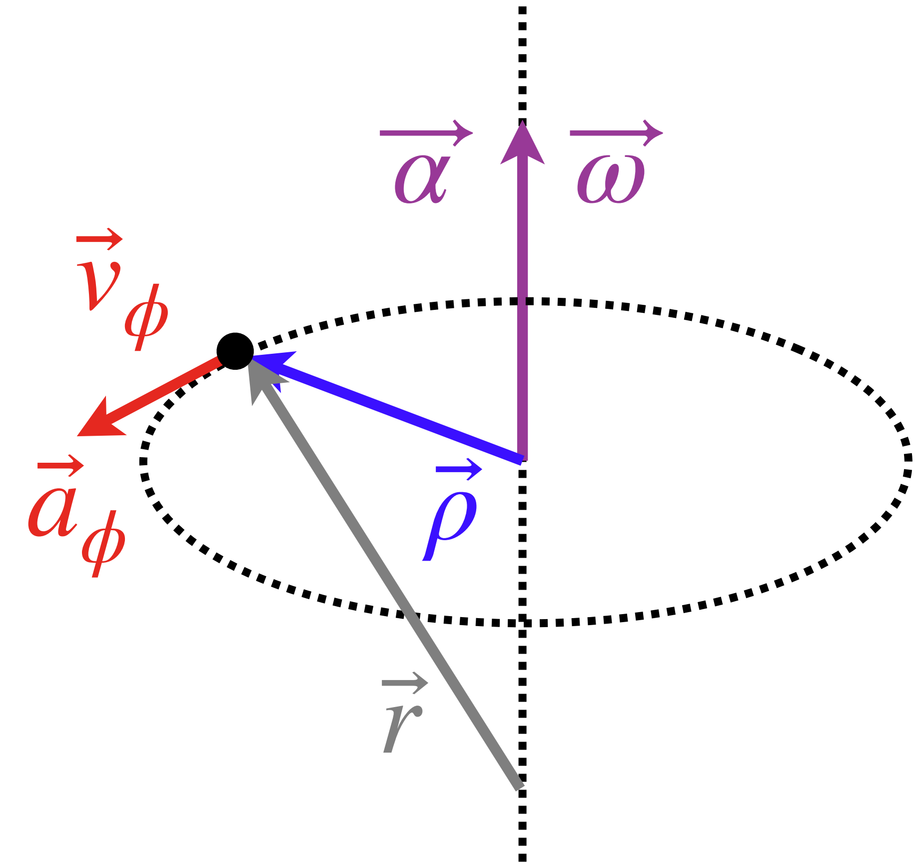
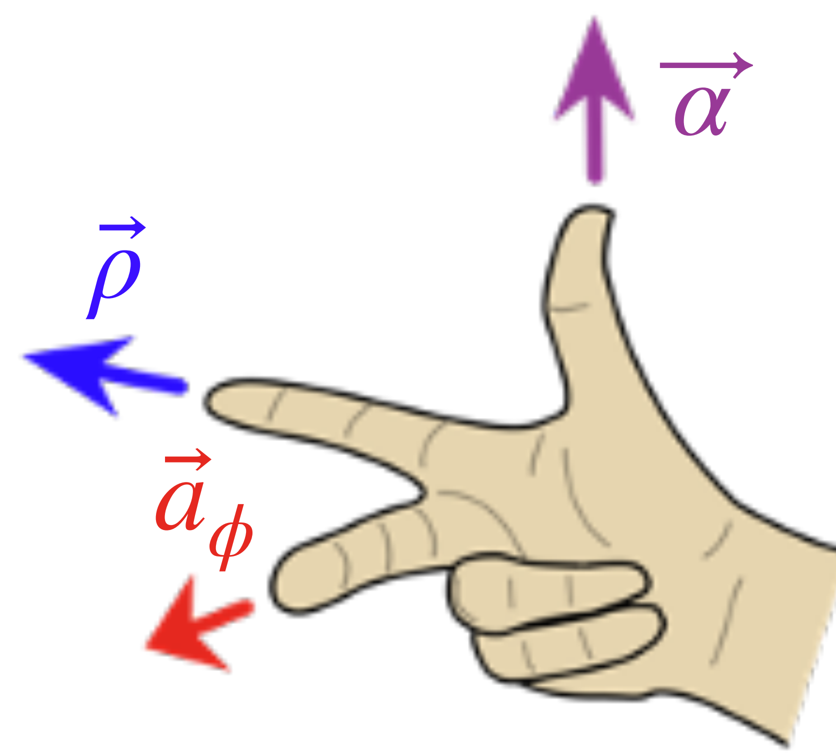
$$\vec{a}_\phi = \overrightarrow{\alpha} \times \vec{\rho} \quad \text{and} \quad \overrightarrow{\alpha} = \frac{\vec{\rho} \times \vec{a}_\phi}{\rho^2}$$

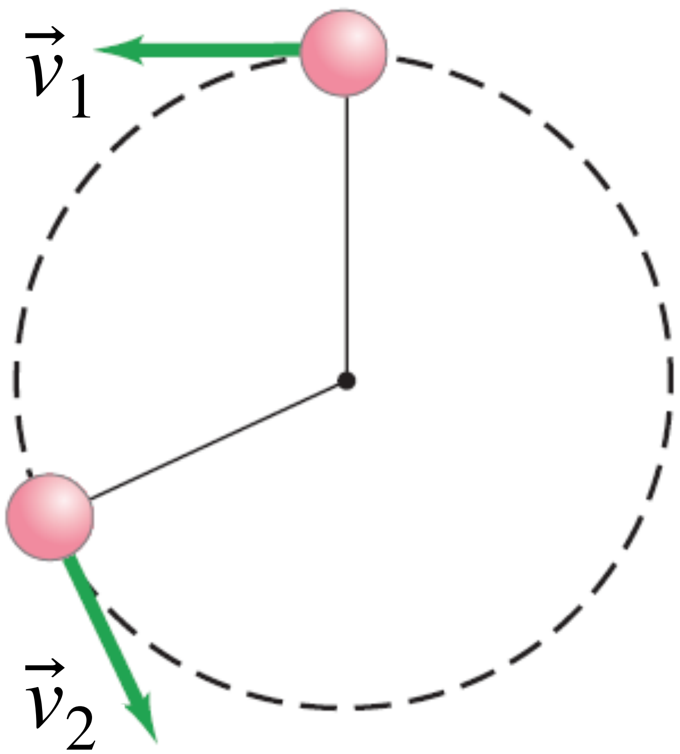
- If the direction of the rotation axis does not change, the angular acceleration vector points along it





Centripetal force in circular motion

- Find the force required to maintain an object moving in a circle with a radius ρ_0.



DEMO (32)

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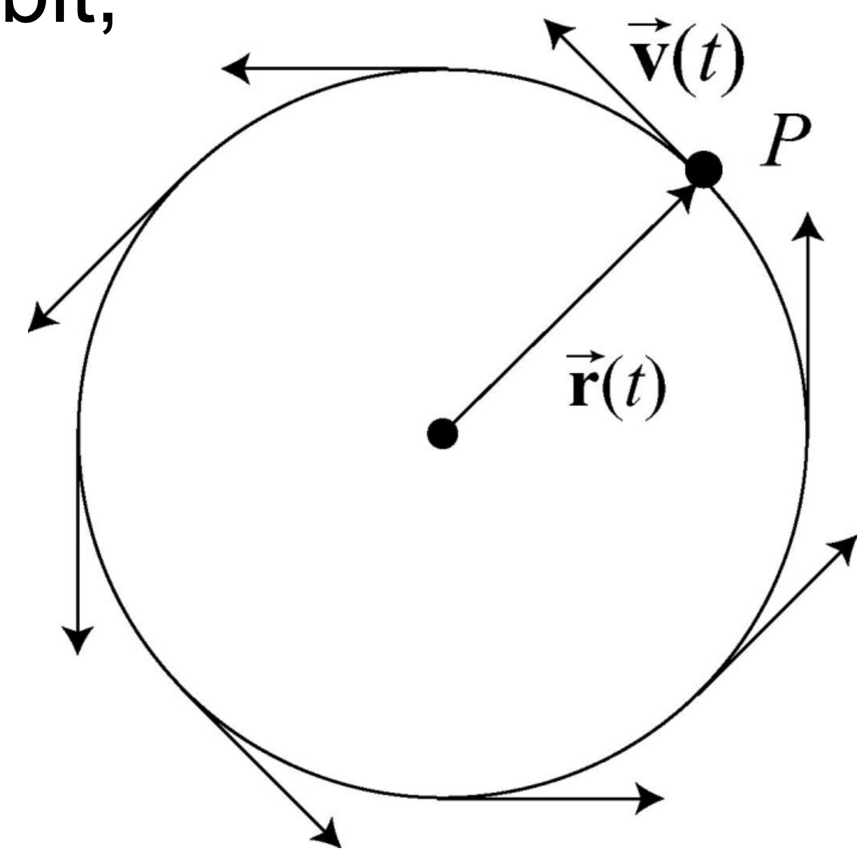
Spinning chain

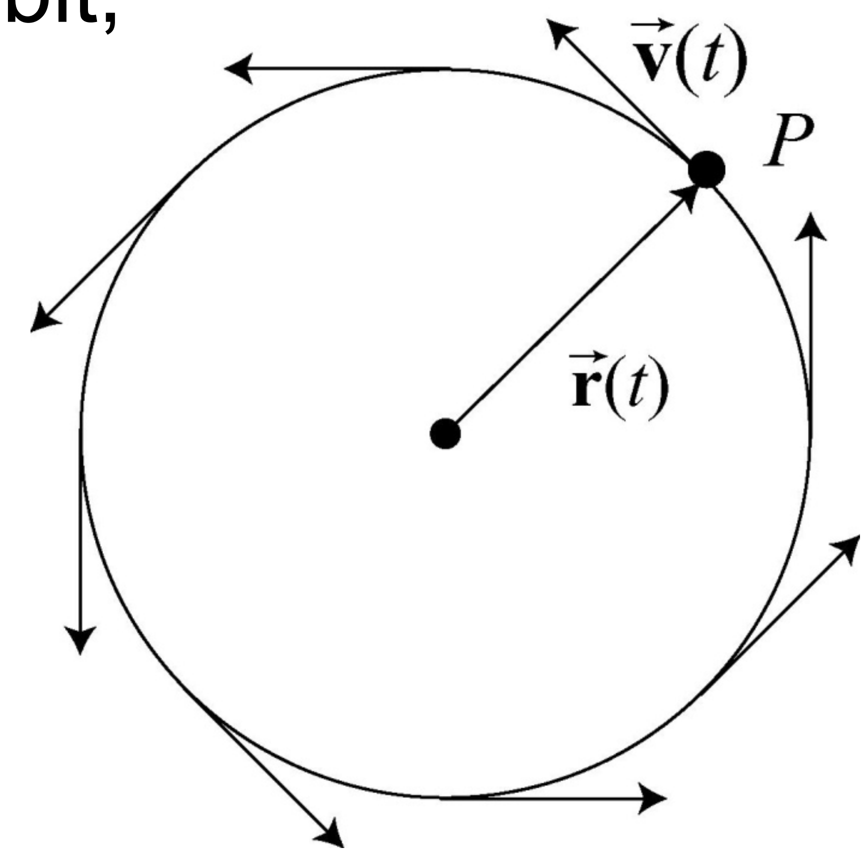
Circular motion summary

- When an object moves in a circular orbit, the direction of the velocity changes (and the speed *may* change as well)



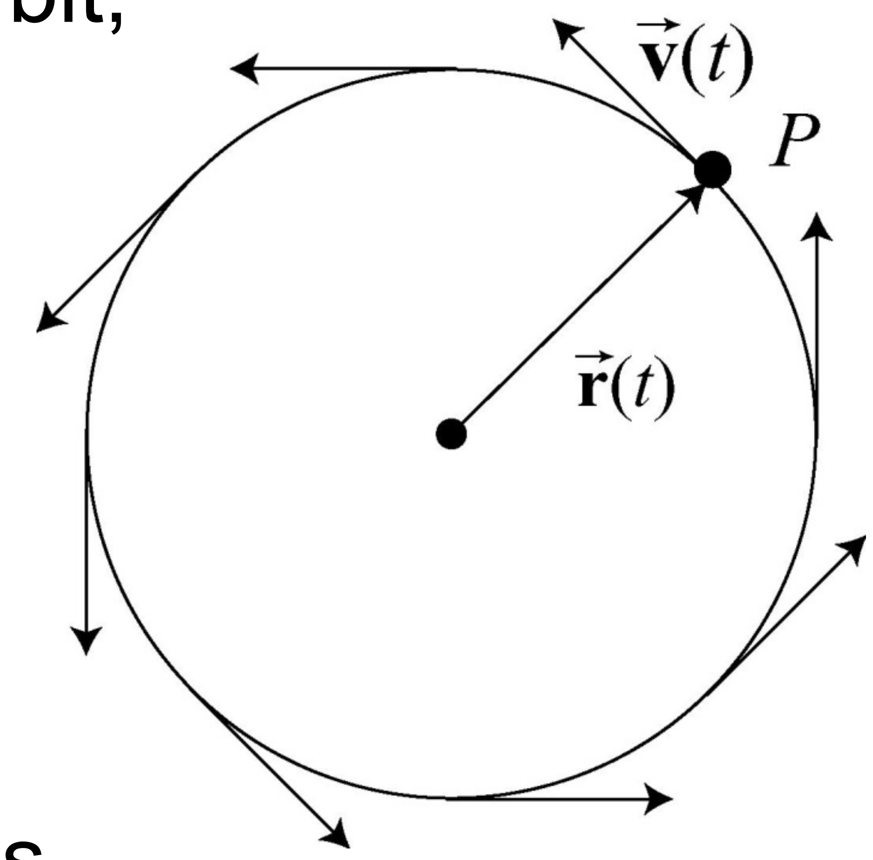
Circular motion summary

- When an object moves in a circular orbit, the direction of the velocity changes (and the speed *may* change as well)
- Instantaneous velocity is always tangent to the circle



Circular motion summary

- When an object moves in a circular orbit, the direction of the velocity changes (and the speed *may* change as well)
- Instantaneous velocity is always tangent to the circle
- The acceleration **will always have a radial component** (a_ρ) due to the change in direction of velocity, which is called the centripetal acceleration



Circular motion summary

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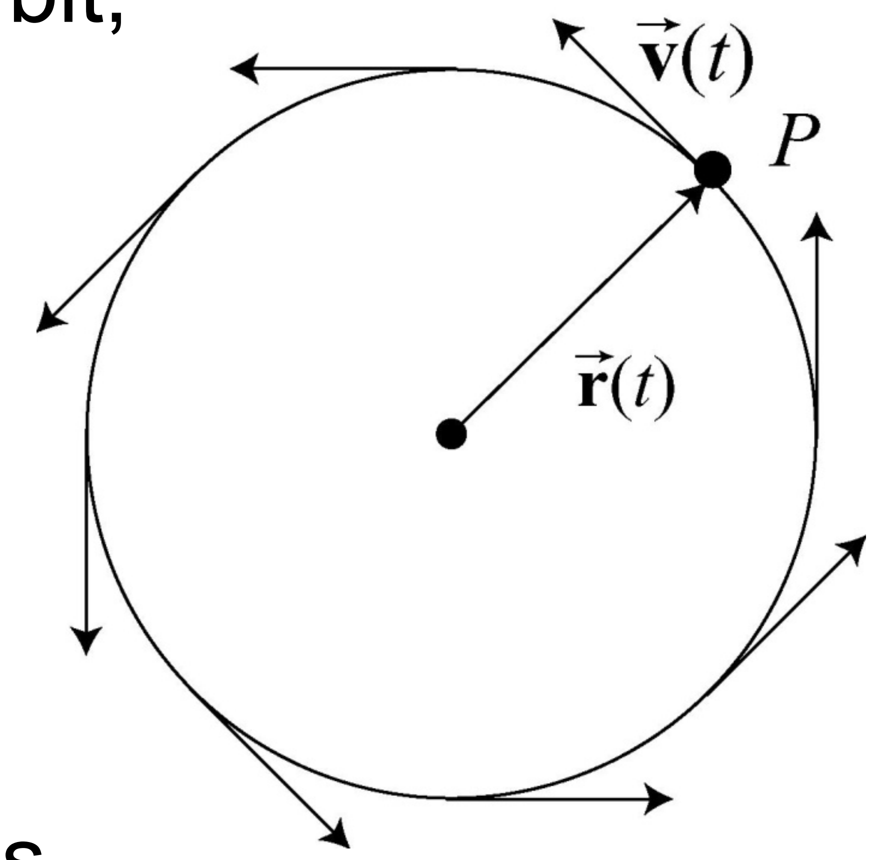
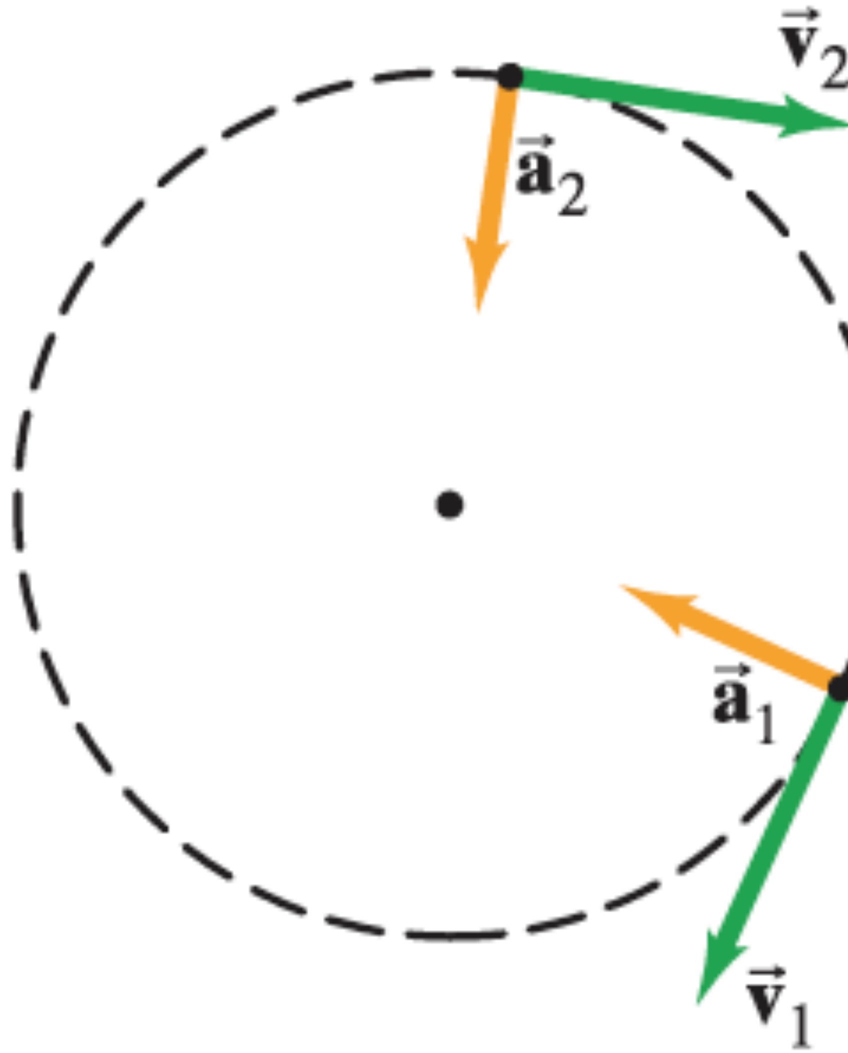


- When an object moves in a circular orbit, the direction of the velocity changes (and the speed *may* change as well)
- Instantaneous velocity is always tangent to the circle
- The acceleration **will always have a radial component** (a_ρ) due to the change in direction of velocity, which is called the centripetal acceleration
- The acceleration *may* have a tangential component (a_ϕ) if the speed changes
- When $a_\phi = 0$, the speed of the object remains constant

Uniform circular motion summary

- Motion in a circle of constant radius ρ_0 at constant angular velocity $\overrightarrow{\omega}$ (radians per second)
- Instantaneous velocity is still always tangent to the circle
- The acceleration will **only** have a radial component (a_ρ) due to the change in direction of velocity



Uniform circular motion summary

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- Motion in a circle of constant radius ρ_0 at constant angular velocity $\overrightarrow{\omega}$ (radians per second)
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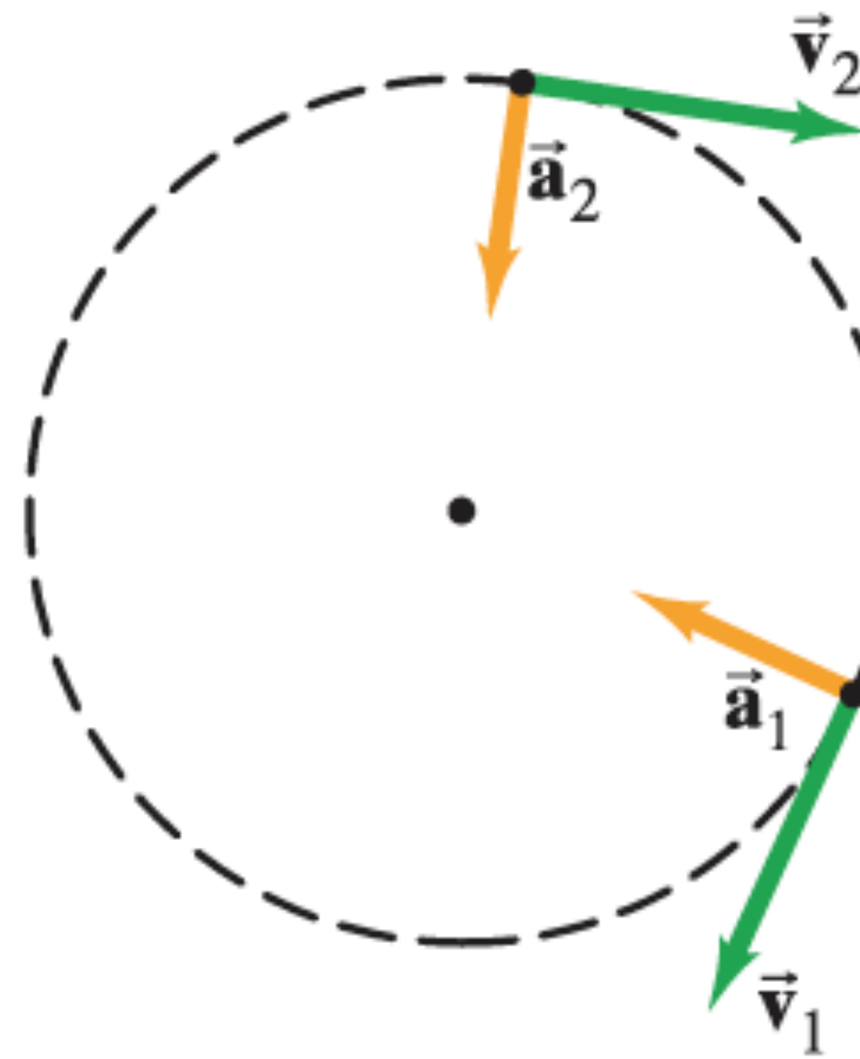
- The acceleration will **only** have a radial component (a_ρ) due to the change in direction of velocity



- Centripetal acceleration always points to the center of the circle

$$\vec{a}_{cent} = -\rho_0\omega^2\hat{\rho}$$

- Must be a centripetal force $\overrightarrow{F}_{cent} = m\vec{a}_{cent} = -m\rho_0\omega^2\hat{\rho}$

DEMO (457)

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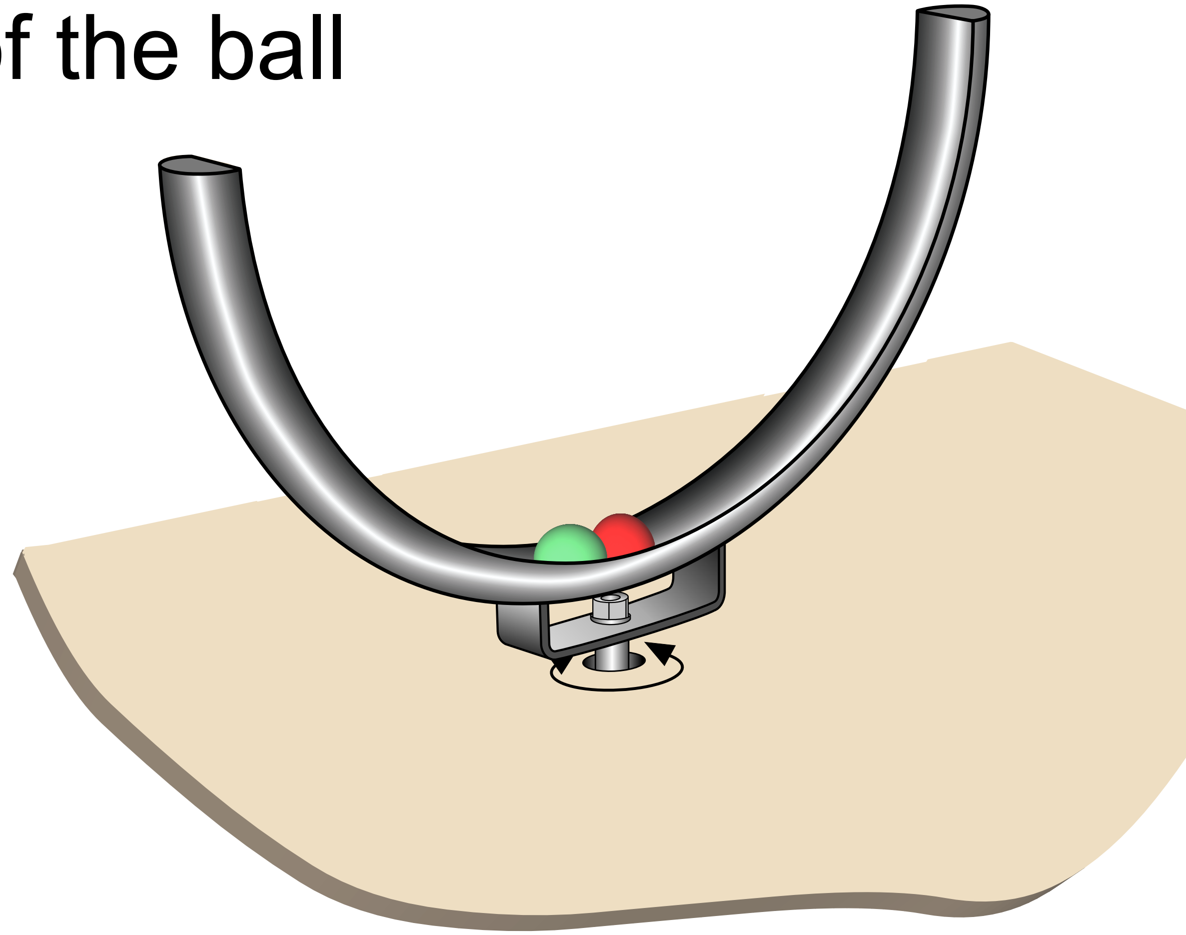
Ball on a rotating slide

Ball on a rotating slide

- Find the equilibrium position θ_0 of the ball



Conceptual question

responseware.eu
Session ID: epflphys101en

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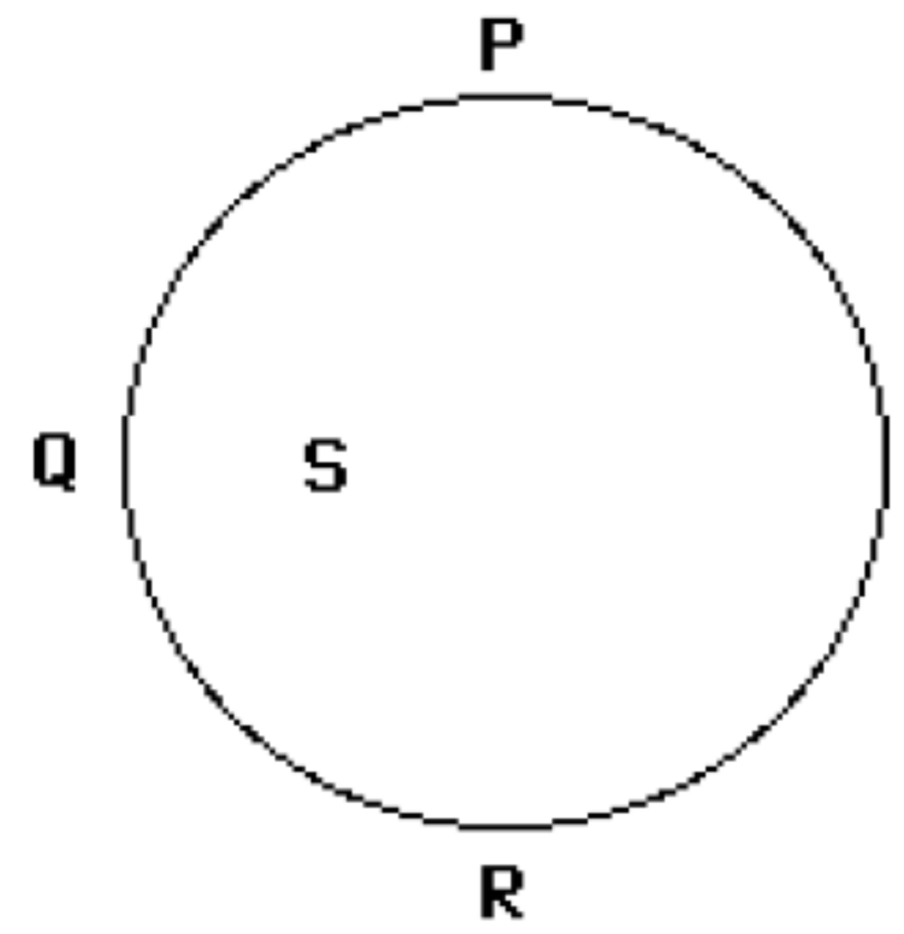
An object moves counter-clockwise along the circular path shown below. As it moves along the path, its acceleration vector continuously points towards S.



The object…

A. speeds up at P, Q, and R.
B. slows down at P, Q, and R.
C. speeds up at P and slows down at R.
D. slows down at P and speeds up at R.
E. No object can execute such motion.

Conceptual question

EPFL
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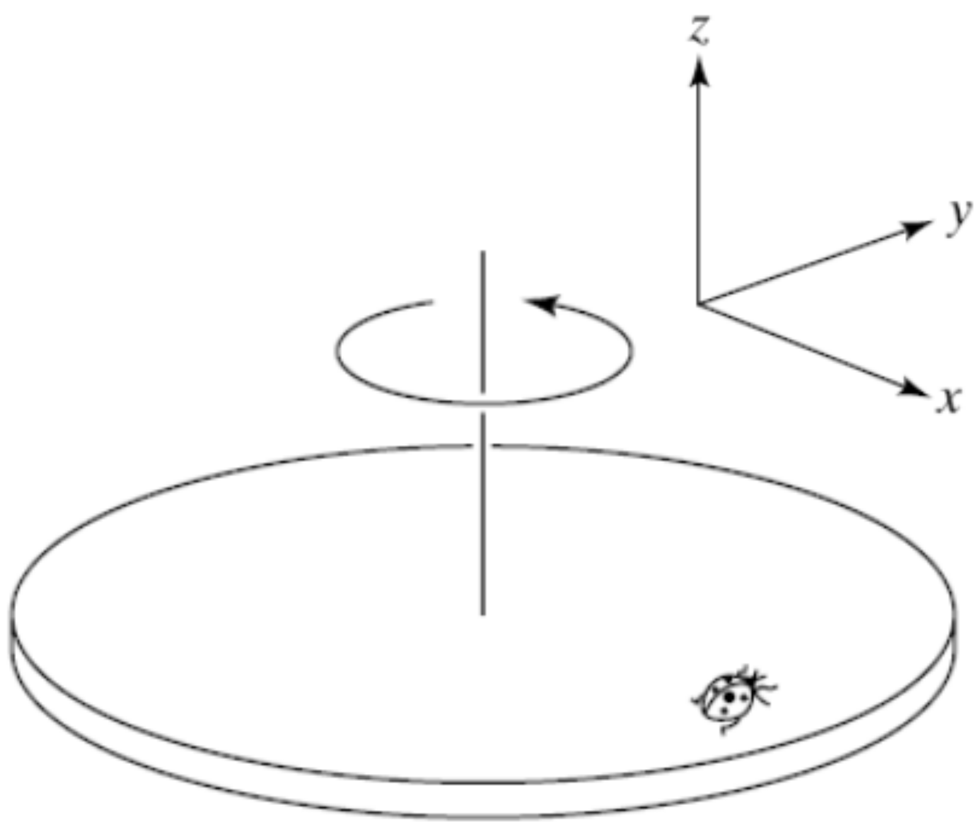
A ladybug sits at the outer edge of a merry-go-round that is turning and slowing down.



At the instant shown in the figure, the **<u>radial</u>** component of the ladybug's (Cartesian) acceleration is…

A. in the $+\hat{y}$ direction.
B. in the $-\hat{y}$ direction.
C. in the $-\hat{x}$ direction.
D. in the $+\hat{z}$ direction.
E. zero.

Conceptual question

responseware.eu
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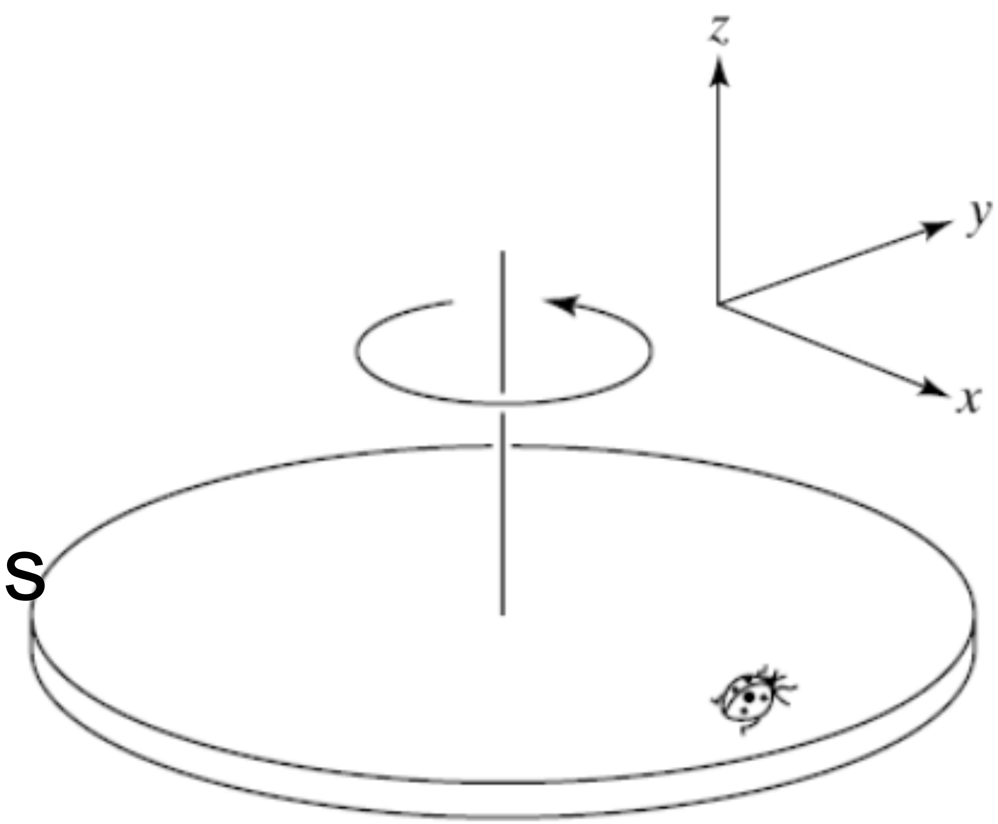
A ladybug sits at the outer edge of a merry-go-round that is turning and slowing down.

At the instant shown in the figure, the **tangential** component of the ladybug's (Cartesian) acceleration is…



A. in the $+\hat{y}$ direction.
B. in the $-\hat{y}$ direction.
C. in the $-\hat{x}$ direction.
D. in the $+\hat{z}$ direction.
E. zero.

Conceptual question

responseware.eu
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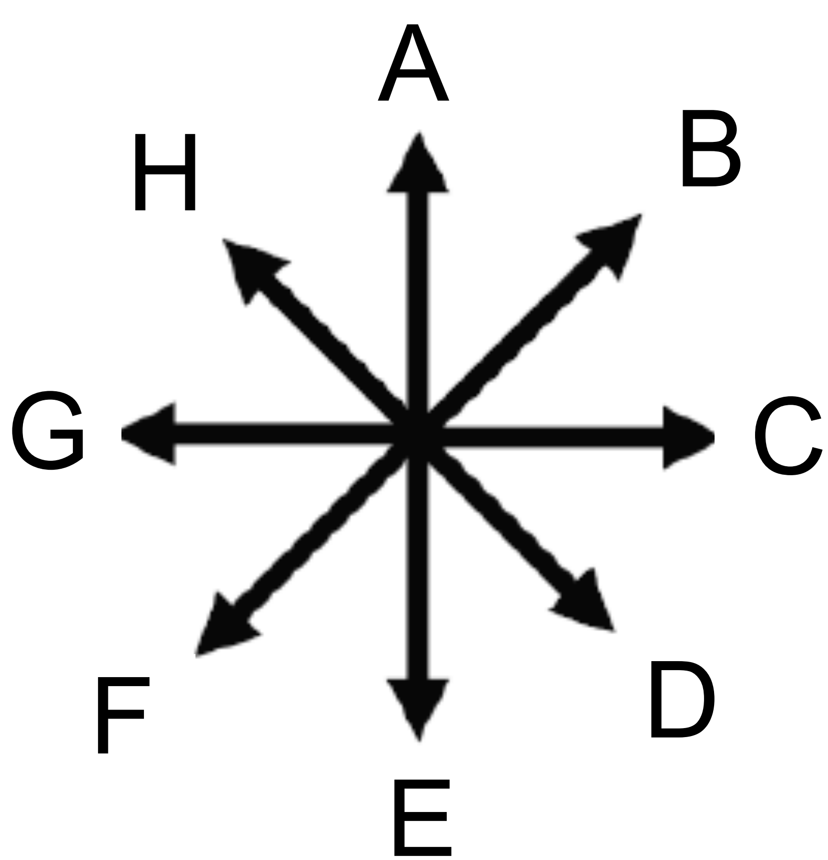
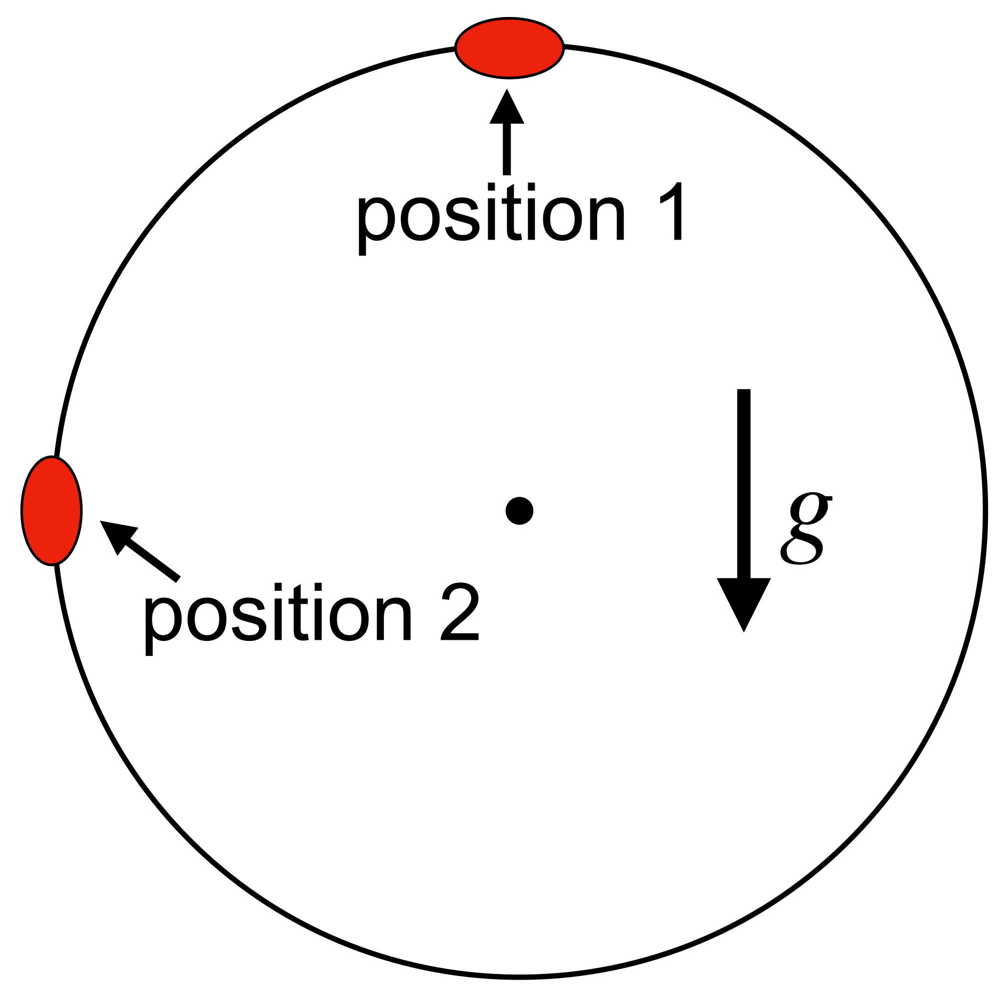
A bead is given a small push at the top of a hoop (position 1) and is constrained to slide around a frictionless circular wire (in a vertical plane). Which arrow best describes the direction of the acceleration when the bead is at the position 2?





Conceptual question

Consider a horse pulling a buggy. Is the following statement true?

The weight of the horse and the normal force exerted by the ground on the horse constitute an interaction pair that are always equal and opposite according to Newton’s third law.

A. Yes
B. No