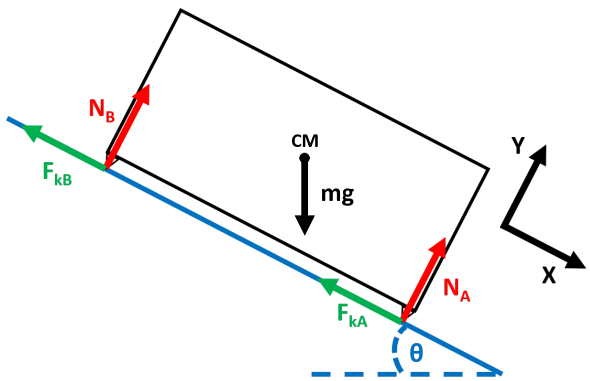
Solution to the Final Exam

17 January 2025

PHYS-101(en)

1. Block on an inclined surface

a. (**2.0 points**) The diagram may look as follows, where CM indicates the center of mass of the block:



b. (**4.0 points**) First, apply Newton's second law to solve for the acceleration of the block:

$$\sum F_y : N_A + N_B - mg\cos(\theta) = 0 \Rightarrow N_A + N_B = mg\cos(\theta) \tag{1}$$

$$\sum F_x : mg\sin(\theta) - F_{kA} - F_{kB} = ma_x \tag{2}$$

$$ma_x = mg\sin(\theta) - \mu_k N_A - \mu_k N_B = mg\sin(\theta) - \mu_k(N_A + N_B) \tag{3}$$

Substituting in the result from equation 1 to equation 3, we find:

$$ma_x = mg\sin(\theta) - \mu_k\, mg\cos(\theta) \tag{4}$$

$$a_x = g\left[\sin(\theta) - \mu_k \cos(\theta)\right] \tag{5}$$

The block therefore experiences constant acceleration in the x-direction, so the following equations apply:

$$v_x(t) = v_0 + a_x\, t \tag{6}$$

$$x(t) = x_0 + v_0\, t + \frac{1}{2} a_x\, t^2 \tag{7}$$

After traveling some distance d, we can express equations 6 and 7 as:

$$d = x(t_d) - x_0 = v_0\, t_d + \frac{1}{2} a_x\, t_d^2 \tag{8}$$

$$v_f = v_x(t_d) = v_0 + a_x\, t_d \quad \Rightarrow \quad t_d = \frac{1}{a_x}(v_f - v_0) \tag{9}$$

Substituting the result for t_d into equation 8, we find:

$$d = v_0 \frac{1}{a_x}(v_f - v_0) + \frac{1}{2} a_x \frac{1}{a_x^2}(v_f - v_0)^2 \tag{10}$$

$$d = \frac{v_0}{a_x}(v_f - v_0) + \frac{1}{2a_x}(v_f^2 - v_0^2) = \frac{1}{2a_x}(v_f^2 - v_0^2) \tag{11}$$

Rearranging and introducing the value for a_x found in equation 5, we arrive at an expression for v_f:

$$v_f^2 = v_0^2 + 2d\, a_x \tag{12}$$

$$|\vec{v_f}| = \sqrt{v_0^2 + 2dg\,[\sin(\theta) - \mu_k \cos(\theta)]} \tag{13}$$

c. (**3.0 points**) For $|\vec{v_f}|$ to be less than $|\vec{v_0}|$, the second term inside the radical $(2dg\,[\sin(\theta) - \mu_k \cos(\theta)])$ must be negative, as d and g are both positive. This gives us the condition:

$$2dg\,[\sin(\theta) - \mu_k \cos(\theta)] < 0 \quad \Rightarrow \quad \sin(\theta) - \mu_k \cos(\theta) < 0 \tag{14}$$

$$\mu_k > \frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta) \tag{15}$$

A full stop occurs when the terms inside the radical cancel out. To avoid that, it must be the case that:

$$v_0^2 + 2dg\,[\sin(\theta) - \mu_k \cos(\theta)] > 0 \quad \Rightarrow \quad v_0^2 > -2dg\,[\sin(\theta) - \mu_k \cos(\theta)] \tag{16}$$

$$\frac{v_0^2}{2dg} > \mu_k \cos(\theta) - \sin(\theta) = \cos(\theta)\,[\mu_k - \tan(\theta)] \tag{17}$$

This gives us the following condition for μ_k to ensure that the block does not come to a stop:

$$\mu_k < \tan(\theta) + \frac{v_0^2}{2dg\cos(\theta)} \tag{18}$$

The two conditions for μ_k from equations 15 and 18 can then be written as a single expression in the form provided in the problem statement:

$$\tan(\theta) < \mu_k < \tan(\theta) + \frac{v_0^2}{2dg\cos(\theta)} \tag{19}$$

d. (**4.0 points**) To solve for the normal force acting on point B ($\vec{N}_B$), you can use the fact that the rotational state of the block does not change around any choice of pivot point, i.e., $\vec{\tau}_{\text{net}} = 0$. By choosing the CM as the pivot point, $\tau \neq 0$ for frictional forces. Note that, due to its uniform density, the CM of the block is at the center of the rectangle. We can then write the following, where the subscripts FkA and FkB correspond to the kinetic friction at points A and B:

$$\vec{\tau}_{\text{net,CM}} = \vec{\tau}_{N_A} + \vec{\tau}_{FkA} + \vec{\tau}_{N_B} + \vec{\tau}_{FkB} + \vec{\tau}_{mg} \tag{20}$$

We can then write expressions for the components of $\vec{\tau}_{\text{net,CM}}$. As the weight acts on the pivot, $\vec{\tau}_{mg} = 0$. For the rest, we have:

$$\vec{\tau}_{N_A} = \vec{r}_{CM,A} \times \vec{N}_A = (l\,\hat{x} - h\,\hat{y}) \times (N_A\,\hat{y}) = l\,N_A\,\hat{z} \tag{21}$$

$$\vec{\tau}_{FkA} = \vec{r}_{CM,A} \times \vec{F}_{kA} = (l\,\hat{x} - h\,\hat{y}) \times (-F_{kA}\,\hat{x}) = h\,F_{kA}\,(\hat{y} \times \hat{x}) = -h\,\mu_k\,N_A\,\hat{z} \tag{22}$$

$$\vec{\tau}_{N_B} = \vec{r}_{CM,B} \times \vec{N}_B = (-l\,\hat{x} - h\,\hat{y}) \times (N_B\,\hat{y}) = -l\,N_B\,\hat{z} \tag{23}$$

$$\vec{\tau}_{FkB} = \vec{r}_{CM,B} \times \vec{F}_{kB} = (-l\,\hat{x} - h\,\hat{y}) \times (-F_{kB}\,\hat{x}) = h\,F_{kB}\,(\hat{y} \times \hat{x}) = -h\,\mu_k\,N_B\,\hat{z} \tag{24}$$

Substituting into $\vec{\tau}_{\text{net,CM}}$:

$$\vec{\tau}_{\text{net,CM}} = l\,N_A\,\hat{z} - h\,\mu_k\,N_A\,\hat{z} - l\,N_B\,\hat{z} - h\,\mu_k\,N_B\,\hat{z} = [N_A\,(l - h\,\mu_k) - N_B\,(l + h\,\mu_k)]\,\hat{z} \tag{25}$$

Since $\vec{\tau}_{\text{net,CM}} = 0$, we can set equation 25 equal to zero:

$$0 = N_A\,(l - h\,\mu_k) - N_B\,(l + h\,\mu_k) \tag{26}$$

We can then use equation 1 to find that $N_A = mg\cos(\theta) - N_B$. Plugging this into equation 26 and solving for N_B:

$$0 = (l - h\,\mu_k)\,[mg\cos(\theta) - N_B] - (l + h\,\mu_k)\,N_B = (l - h\,\mu_k)\,mg\cos(\theta) - 2\,l\,N_B \tag{27}$$

$$N_B = \frac{1}{2}(1 - \frac{h}{l}\,\mu_k)\,mg\cos(\theta) \quad \Rightarrow \quad |\vec{N}_B| = \frac{1}{2}\,mg\cos(\theta)\,|1 - \frac{h}{l}\,\mu_k| \tag{28}$$

e. (**2.0 points**) For the block to always remain in contact with the surface, $N_B > 0$. Since $\cos(\theta) \geq 0$ for $0 \leq \theta \leq \frac{\pi}{2}$, the requirement is satisfied if $1 - \frac{h}{l}\mu_k > 0$. Solving for μ_k, the block remains in contact with the surface if:

$$\mu_k < \frac{l}{h} \tag{29}$$

f. (**2.0 points**) If the block does not slide, friction is static. The forces acting on the block are then as follows, where F_{sA} and F_{sB} are the forces of static friction acting on the block at points A and B, respectively:

$$\sum F_y : N_A + N_B - mg\cos(\theta) = 0 \Rightarrow N_A + N_B = mg\cos(\theta) \tag{30}$$

$$\sum F_x : mg\sin(\theta) - F_{sA} - F_{sB} = ma_x \tag{31}$$

We know that $F_{sA} + F_{sB} \leq \mu_s\,N_A + \mu_s\,N_B$, and from equation 30 that $\mu_s\,(N_A + N_B) = \mu_s\,mg\cos(\theta)$. The maximum value of the sum of friction forces is then:

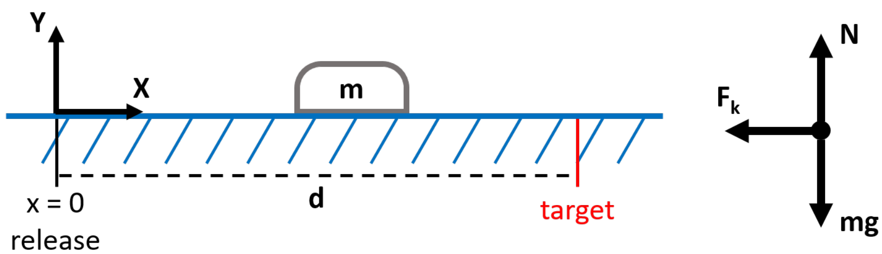
$$(F_{sA} + F_{sB})_{\max} = \mu_s\,mg\cos(\theta) \tag{32}$$

According to equation 30, the block will start sliding down the inclined surface (i.e., $a_x > 0$), only when:

$$mg\sin(\theta) - (F_{sA} + F_{sB})_{\max} > 0 \quad \Rightarrow \quad mg\sin(\theta) > \mu_s\,mg\cos(\theta) \quad \Rightarrow \quad \tan(\theta) > \mu_s \tag{33}$$

2. Curling

a. (**1.0 points**) The diagram may look something like what is shown below. The forces acting on the stone after it is released are also indicated at right:



b. (**3.0 points**) Applying Newton's second law to solve for the acceleration of the stone:

$$\sum F_y : N - mg = 0 \Rightarrow N = mg \tag{1}$$
$$\sum F_x : -F_k = ma_x \Rightarrow ma_x = -\mu_k N = -\mu_k mg \Rightarrow a_x = -\mu_k g \tag{2}$$

This tells us that the stone has a constant acceleration up to the moment it stops, t_s. We can then describe the velocity of the stone before it stops as:

$$v_x(t) = v_0 + a_x t = v_0 - \mu_k g t \tag{3}$$
$$\vec{v}(t) = (v_0 - \mu_k g t)\,\hat{x} \tag{4}$$

Note that, if $t > t_s$, then $\vec{v}(t) = 0$. To find t_s, we solve equation 4 for $t = t_s$:

$$v_x(t_s) = 0 = v_0 - \mu_k g t_s \quad \Rightarrow \quad t_s = \frac{v_0}{\mu_k g} \tag{5}$$

c. (**3.0 points**) Since acceleration is constant for $0 \le t \le t_s$, we can describe the position of the stone in the x-direction as a function of time using:

$$x(t) = x_0 + v_0 t - \frac{1}{2}\mu_k g t^2 \tag{6}$$

In our given choice of reference frame, $x_0 = 0$. To stop on the target, we need $x(t_s) = d$. Substituting this condition into equation 6:

$$d = x(t_s) = v_0 t_s - \frac{1}{2}\mu_k g t^2 = v_0 \left(\frac{v_0}{\mu_k g}\right) - \frac{1}{2}\mu_k g \left(\frac{v_0}{\mu_k g}\right)^2 \tag{7}$$
$$d = \frac{1}{\mu_k g}(v_0^2 - \frac{1}{2}v_0^2) = \frac{1}{2\mu_k g} v_0^2 \tag{8}$$

Finally, solving for the required initial velocity v_0 of the stone:

$$v_0 = \sqrt{2\mu_k g d} \tag{9}$$

d. (**3.0 points**) To find the work done on the stone by friction, we first recognize that the friction force ($\vec{F}_k$) is constant over the entire trajectory of the stone. Therefore, we can apply the expression of work done by a constant force:

$$W_f = \vec{F}_k \cdot \vec{l} = (-F_k\,\hat{x}) \cdot (d\,\hat{x}) = (-F_k\,d)\,(\hat{x} \cdot \hat{x}) = -F_k\,d = -\mu_k\,m\,g\,d \tag{10}$$

Here, it is important to recognize that $W_f < 0$.

To now solve for the change in kinetic energy, we use:

$$\Delta K = K_f - K_i = \frac{1}{2}m\,v_f^2 - \frac{1}{2}m\,v_0^2 \tag{11}$$

Since $v_f = 0$, we find:

$$\Delta K = -\frac{1}{2}m\,v_0^2 \tag{12}$$

Substituting in the expression found for v_0 in equation 9:

$$\Delta K = -\frac{1}{2}m(\sqrt{2\mu_k\,g\,d})^2 = -m\,\mu_k\,g\,d \tag{13}$$

This is the same expression found for W_f in equation 10, so $\Delta K = W_f$. This result should be expected from the "Work-Kinetic energy" theorem, as friction is the only force doing work on the stone.

e. (**3.0 points**) The work done by friction can be separated into the part of the stone's trajectory with sweeping and the part without, denoted by subscripts s and ns, respectively:

$$W_f = \int_d \vec{F}_k \cdot d\vec{l} = \int_s \vec{F}_k \cdot d\vec{l} + \int_{ns} \vec{F}_k \cdot d\vec{l} \tag{14}$$

The force is constant in each part, so:

$$\int_s \vec{F}_k \cdot d\vec{l} = \vec{F}_{k,s} \cdot \int_s d\vec{l} = \vec{F}_{k,s} \cdot \vec{l}_s \tag{15}$$

$$\int_{ns} \vec{F}_k \cdot d\vec{l} = \vec{F}_{k,ns} \cdot \int_{ns} d\vec{l} = \vec{F}_{k,ns} \cdot \vec{l}_{ns} \tag{16}$$

From the problem statement, we know that:

$$\vec{F}_{k,ns} = -\mu_k\,mg\,\hat{x} \qquad \vec{F}_{k,s} = -\frac{1}{2}\mu_k\,mg\,\hat{x} \tag{17}$$

$$\vec{l}_{ns} = l_{ns}\,\hat{x} \qquad \vec{l}_s = \frac{d}{3}\,\hat{x} \tag{18}$$

where l_{ns} is the total distance without sweeping. We can then find an expression for the work done by friction:

$$W_f = \vec{F}_{k,s} \cdot \vec{l}_s + \vec{F}_{k,ns} \cdot \vec{l}_{ns} = -(\frac{1}{2}\mu_k\,mg)\left(\frac{d}{3}\right) - (\mu_k\,mg)\,(l_{ns}) \tag{19}$$

$$W_f = -\left(l_{ns} + \frac{d}{6}\right)\,\mu_k\,mg \tag{20}$$

From the "Work-Kinetic energy" theorem, we then have:

$$\Delta K = \frac{1}{2}m\,v_f^2 - \frac{1}{2}m\,v_0^2 = -\frac{1}{2}m\,v_0^2 = -m\,\mu_k\,g\,d = W_f \tag{21}$$

Equating the result from equation 20 and solving for l_{ns}:

$$\Delta K = W_f \qquad \Rightarrow \qquad -\,m\,\mu_k\,g\,d = -\left(l_{ns} + \frac{d}{6}\right)\mu_k\,mg \tag{22}$$

$$d = l_{ns} + \frac{d}{6} \qquad \Rightarrow \qquad l_{ns} = d - \frac{d}{6} = \frac{5d}{6} \tag{23}$$

The total distance then traveled by the stone is:

$$l_{tot} = l_s + l_{ns} = \frac{d}{3} + \frac{5d}{6} = \frac{7d}{6} \tag{24}$$

f. (**3.0 points**) Using the impulse approximation, we can say that the total momentum is conserved between right before and right after the collision. The use of the impulse approximation is necessary because the presence of an external force during the collision (friction) tells us that momentum is not conserved during the collision. The impulse approximation, however, allows us to simplify the calculations and proceed as if momentum is conserved.

To show that the velocity of the sticking stones after the collision is in the same direction and has half the speed as your stone's velocity right before the collision, we use conservation of momentum, as implied by the impulse approximation. The subscripts 1 and 2 correspond to the moving stone (your stone) and the target stone (your opponent's stone), respectively. The subscript c denotes the combined stones immediately following the collision.

$$\vec{p}_i = m\,\vec{v}_{1,i} + m\,\vec{v}_{2,i} \tag{25}$$

$$\vec{p}_f = 2m\,\vec{v}_{c,f} \tag{26}$$

Since $\vec{v}_{2,i} = 0$, we have:

$$\vec{p}_i = \vec{p}_f \quad \Rightarrow \quad m\,\vec{v}_{1,i} = 2m\,\vec{v}_c \quad \Rightarrow \quad \vec{v}_c = \frac{1}{2}\vec{v}_{1,i} \tag{27}$$

g. (**2.0 points**) To stop at the target, which is $\frac{1}{10}d$ away from the location of the collision, the object of mass $2m$ must undergo a change in kinetic energy of:

$$\Delta K = 0 - \frac{1}{2}(2m)\,v_c^2 \tag{28}$$

As friction is the only force doing work on the stones, this change in kinetic energy must be equal to the work done by friction. This allows us to solve for v_c:

$$W_f = \vec{F}_k \cdot \vec{l} = -\mu_k\,(2m)\,g\left(\frac{1}{10}d\right) = \Delta K = -\frac{1}{2}(2m)\,v_c^2 \tag{29}$$

$$v_c^2 = \frac{1}{5}\mu_k\,g\,d \quad \Rightarrow \quad v_c = \sqrt{\frac{1}{5}\mu_k\,g\,d} \tag{30}$$

h. (**2.0 points**) From part (f), the speed of your stone immediately before the collision must be twice v_c. To find v_n, we can then use the change in kinetic energy from the moment you release the stone to right before the collision:

$$\Delta K = \frac{1}{2}m\,(2\,v_c^2) - \frac{1}{2}m\,v_n^2 = \frac{1}{2}m\left[\frac{4}{5}\mu_k\,g\,d - v_n^2\right] \tag{31}$$

As before, this change in kinetic energy must be equal to the work done by friction over the path of the stone:

$$W_f = \vec{F}_k \cdot \vec{l} = -\mu_k \, mg \left(\frac{9}{10} d \right) \tag{32}$$

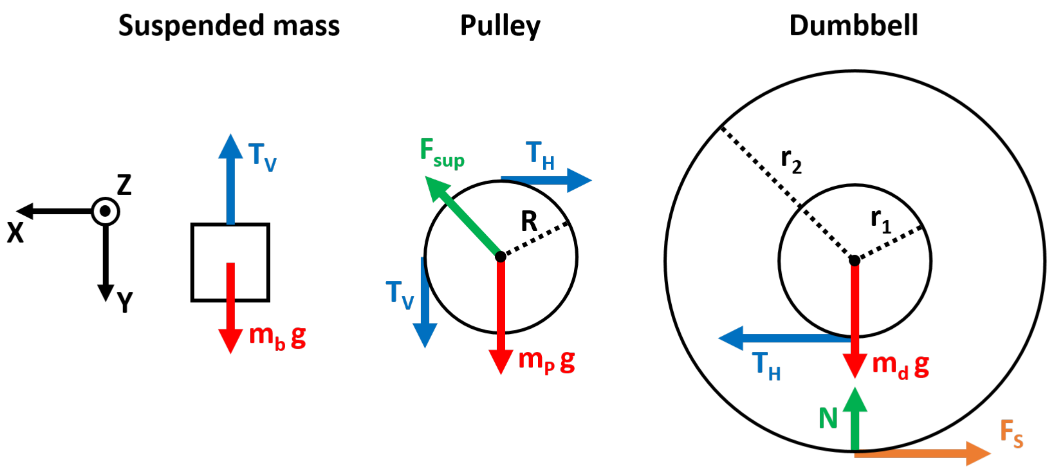
Equating W_f and ΔK, we can solve for v_n:

$$\Delta K = W_f \quad \Rightarrow \quad \frac{1}{2} m \left[\frac{4}{5} \mu_k \, g \, d - v_n^2 \right] = - \, \mu_k \, mg \left(\frac{9}{10} d \right) \tag{33}$$

$$v_n^2 - \frac{4}{5} \mu_k \, g \, d = \frac{g}{5} \mu_k \, g \, d \quad \Rightarrow \quad v_n = \sqrt{\frac{13}{5} \mu_k \, g \, d} \tag{34}$$

3. Dumbbell, pulley and block

a. (**2.0 points**) The diagrams may look as follows for the suspended mass, pulley, and dumbbell:



The "no slipping" condition suggests that static friction (F_s) is acting on point C_2. This force acts in the negative x direction, opposing the tension force of the rope (T_H).

b. (**5.0 points**) To find the acceleration of the dumbbell's CM, first apply Newton's second law to the dumbbell;

$$\sum F_y : m_d\, g - N = 0 \Rightarrow N = m_d\, g \tag{1}$$

$$\sum F_x : T_H - F_s = m_d\, a_{CM} \Rightarrow F_s = T_H - m_d\, a_{CM} \tag{2}$$

We also need to consider the torques about the axis of the dumbbell, as the dumbbell can rotate about its axis. The net torque can be described as:

$$\vec{\tau}_{\text{net}} = \vec{\tau}_{m_d\, g} + \vec{\tau}_N + \vec{\tau}_{F_s} + \vec{\tau}_{T_H} \tag{3}$$

Since the weight is applied at the pivot, $\vec{\tau}_{m_d\, g} = 0$. Breaking out the other individual components of $\vec{\tau}_{\text{net}}$, we have:

$$\vec{\tau}_N = (r_2\, \hat{y}) \times (N\, \hat{y}) = r_2\, N\, (\hat{y} \times \hat{y}) = 0 \tag{4}$$

$$\vec{T}_{F_s} = (r_2\, \hat{y}) \times (-F_s\, \hat{x}) = -r_2\, F_s\, (\hat{y} \times \hat{x}) = r_2\, F_s\, \hat{z} \tag{5}$$

$$\vec{T}_{T_H} = (r_1\, \hat{y}) \times (T_H\, \hat{x}) = r_1\, T_H\, (\hat{y} \times \hat{x}) = -r_1\, T_H\, \hat{z} \tag{6}$$

Substituting these values into equation 3, we find:

$$\vec{\tau}_{\text{net}} = (r_2\, F_s - r_1\, T_H)\, \hat{z} \tag{7}$$

A net torque will cause an angular acceleration $\vec{\alpha}_d$ in the $\hat{z}$ direction:

$$\vec{\tau}_{\text{net}} = I_d\, \vec{\alpha}_d \quad \Rightarrow \quad r_2\, F_s - r_1\, T_H = I_d\, \alpha_d \tag{8}$$

The "no-slipping" condition at C_2 implies that $\alpha_d = \frac{a_{CM}}{r_2}$. Replacing this and F_s (from equation 2) in equation 8, gives:

$$r_2\,(T_H - m_d\,a_{CM}) - r_1\,T_H = I_d\,\frac{a_{CM}}{r_2} \Rightarrow (r_2 - r_1)\,T_H = \frac{1}{r_2} I_d\,a_{CM} + m_d\,r_2\,a_{CM} \tag{9}$$

$$a_{CM} = T_H\,(r_2 - r_1)\,\frac{r_2}{I_d + m_d\,r_2^2} \tag{10}$$

Since $T_H > 0$ and $r_2 > r_1$, $a_{CM} > 0$. This tells us that the dumbbell will accelerate to the left (in the positive x direction).

c. (**4.0 points**) As seen from the dumbbell axis (i.e., from a reference frame whose origin coincides with the dumbbell axis but does not rotate), for the rope to not slip at C_1, it must have the same tangential velocity as the dumbbell at that point. The tangential speed (with respect to the axis) is simply $|v_{ax,C_1}| = r_1\,|\omega|$, where ω is the angular speed. Keep in mind that $|v_{ax,C_2}| = r_2\,|\omega| = |v_{CM}|$ and $|v_{ax,C_1}| = r_1 \left|\frac{v_{CM}}{r_2}\right| = \frac{r_1}{r_2}|v_{CM}|$ from the no-slip condition on the table.

To find the direction, we notice that since $a_{CM} > 0$ (the dumbbell moves to the left), the dumbbell must rotate counterclockwise. Therefore, because $v_{CM} > 0$:

$$\vec{v}_{ax,C_1} = \frac{r_1}{r_2}\,|v_{CM}|\,(-\hat{x}) = -\frac{r_1}{r_2}\,v_{CM}\,\hat{x} \tag{11}$$

In the original reference frame, the dumbbell axis has a translational motion with velocity $\vec{v}_{ax} = v_{CM}\,\hat{x}$ because the axis contains the CM. If $\vec{r}_{ax}$ is the position of the axis in the original reference frame and $\vec{r}_{ax,C_1}$ is the position of C_1 with respect to the axis, then the position of C_1 in the original reference frame is:

$$\vec{r}_{C_1} = \vec{r}_{ax} + \vec{r}_{ax,C_1} \tag{12}$$

After differentiating equation 12, we can express the velocity at C_1 in the original frame as:

$$\vec{v}_{C_1} = \vec{v}_{ax} + \vec{v}_{ax,C_1} = v_{CM}\,\hat{x} - \frac{r_1}{r_2}\,v_{CM}\,\hat{x} \tag{13}$$

$$\vec{v}_{C_1} = \left(1 - \frac{r_1}{r_2}\right)\,v_{CM}\,\hat{x} \tag{14}$$

The rope must then have this same velocity:

$$\vec{v}_{\text{rope}} = \vec{v}_{C_1} = \left(1 - \frac{r_1}{r_2}\right)\,v_{CM}\,\hat{x} \tag{15}$$

Since $v_{\text{rope}} < v_{CM}$, the axis of the dumbbell overtakes the rope. This implies that the rope winds up around the handle.

d. (**2.0 points**) Looking at our diagram in part (a) and applying Newton's second law to the block, we find:

$$\sum F_y : m_b\,g - T_V = m_b\,a_{by} \quad \Rightarrow \quad a_{by} = g - \frac{1}{m_b}\,T_V \tag{16}$$

e. (**4.0 points**) From part (a), we can compute the net torque around the axis of the pulley as follows:

$$\vec{\tau}_{\text{net}} = \vec{\tau}_{m_p\,g} + \vec{\tau}_{\text{sup}} + \vec{\tau}_{T_V} + \vec{\tau}_{T_H} \tag{17}$$

Here, both $\vec{\tau}_{m_p g} = 0$ and $\vec{\tau}_{\text{sup}} = 0$ because the forces act on the pivot point. The two remaining non-zero components of the net torque are then:

$$\vec{\tau}_{T_V} = (R\,\hat{x}) \times (T_V\,\hat{y}) = R\,T_V\,\hat{z} \tag{18}$$
$$\vec{\tau}_{T_H} = (-R\,\hat{y}) \times (-T_H\,\hat{x}) = -R\,T_H\,\hat{z} \tag{19}$$

Substituting these into equation 17, we find that the net torque is:

$$\vec{\tau}_{\text{net}} = R\,(T_V - T_H)\,\hat{z} \tag{20}$$

Since $\vec{\tau}_{\text{net}}$ can also be expressed in terms of the angular acceleration ($\vec{\alpha}_P$) and moment of inertia I_P of the pulley ($\vec{\tau}_{\text{net}} = I_P\,\vec{\alpha}_P$), we can then use the result from equation 20 to solve for the angular acceleration:

$$I_P\,\vec{\alpha}_P = R\,(T_V - T_H)\,\hat{z} \quad \Rightarrow \quad \vec{\alpha}_P = \frac{R}{I_P}\,(T_V - T_H)\,\hat{z} \tag{21}$$

f. (**3.0 points**) Since the rope does <u>not</u> slip on the pulley, it must accelerate with the tangential acceleration:

$$\alpha_P = \frac{a_{\text{rope}}}{R} \quad \Rightarrow \quad a_{\text{rope}} = R\,\alpha_P \tag{22}$$

Then, because the rope is attached to m_b and inextensible:

$$a_{by} = a_{\text{rope}} = R\,\alpha_P \tag{23}$$

Plugging the expression for α_P into equation 23, we can write an expression for T_H:

$$a_{by} = \frac{R^2}{I_P}\,(T_V - T_H) \quad \Rightarrow \quad T_H = T_V - \frac{I_P}{R^2}\,a_{by} \tag{24}$$

Then, rearranging the result from equation 16 to find an expression for T_V, we can solve for T_H with the requested terms:

$$T_H = m_b\,g - m_b\,a_{by} - \frac{I_P}{R^2}\,a_{by} \quad \Rightarrow \quad T_H = m_b\,g - a_{by}\,\left(m_b + \frac{I_P}{R^2}\right) \tag{25}$$

g. (**3.0 points**) To find the relation between the acceleration of the dumbbell's CM and the acceleration of the block, we can start with equation 15, which describes the relationship between the velocity of the dumbbell's CM and the velocity of the rope. We can then differentiate both sides of equation 15 with respect to time, to get an expression that relates the accelerations of the rope and dumbbell's CM:

$$\frac{d}{dt}(v_{\text{rope}}) = \left(1 - \frac{r_1}{r_2}\right)\,\frac{d}{dt}(v_{CM}) \quad \Rightarrow \quad a_{\text{rope}} = \left(1 - \frac{r_1}{r_2}\right)\,a_{CM} \tag{26}$$

Since $a_{\text{rope}} = a_{by}$ as argued in part (e) (see equation 23), we can relate the vertical acceleration of the block and the horizontal acceleration of the dumbbell's CM:

$$a_{by} = a_{CM}\left(1 - \frac{r_1}{r_2}\right) \tag{27}$$

Next, to solve for a_{CM} in terms of the parameters of the problem, we can first revisit equation 25. By substituting the result of equation 27, we can find an expression for T_H that includes a_{CM}:

$$T_H = m_b\, g - a_{by} \left(m_b + \frac{I_P}{R^2}\right) \quad \Rightarrow \quad T_H = m_b\, g - a_{CM} \left(1 - \frac{r_1}{r_2}\right) \left(m_b + \frac{I_P}{R^2}\right) \tag{28}$$

$$T_H = m_b\, g - a_{CM}\, \frac{(r_2 - r_1)}{r_2\, R^2}\, (I_P + m_b\, R^2) \tag{29}$$

We can then rewrite equation 10 as an expression for T_H that includes a_{CM}:

$$a_{CM} = T_H\, (r_2 - r_1)\, \frac{r_2}{I_d + m_d\, r_2^2} \quad \Rightarrow \quad T_H = a_{CM}\, \frac{I_d + m_d\, r_2^2}{r_2(r_2 - r_1)} \tag{30}$$

This allows us to equate equations 29 and 30 and find an expression for a_{CM} that includes only parameters given in the problem statement:

$$a_{CM}\, \frac{I_d + m_d\, r_2^2}{r_2(r_2 - r_1)} = m_b\, g - a_{CM}\, \frac{(r_2 - r_1)}{r_2\, R^2}\, (I_P + m_b\, R^2) \tag{31}$$

$$a_{CM} \left[\frac{I_d + m_d\, r_2^2}{r_2(r_2 - r_1)} + \frac{(r_2 - r_1)}{r_2\, R^2}\, (I_P + m_b\, R^2)\right] = m_b\, g \tag{32}$$

$$a_{CM} = m_b\, g \left[\frac{R^2\, r_2\, (r_2 - r_1)}{R^2\, (I_d + m_d\, r_2^2) + (r_2 - r_1)^2\, (I_P + m_b\, R^2)}\right] \tag{33}$$