Neural Networks and Biological Modeling

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Answers to question set 11

Exercise 1: Stochastic input

1.1 The expression for $u(t)$ was given by

$$u(t) = u_{rest} + \frac{R}{\tau_m} \int_0^t e^{-s/\tau_m} I(t-s) ds. \tag{1}$$

Taking the average of Eq.(1) over multiple repetitions or over a population of neurons, we obtain

$$\begin{aligned}\langle u(t) \rangle &= u_{\text{rest}} + \frac{R}{\tau_m} \int_0^t e^{-s/\tau_m} \underbrace{\langle I(t) \rangle}_{=I_0} ds \\ &= u_{\text{rest}} + R I_0 [1 - e^{-t/\tau_m}] \,.\end{aligned}$$

1.2 Using $(RI(t) - R\langle I(t) \rangle) = \xi(t)$ and $\langle \xi(t)\xi(t') \rangle = \tau_m a^2 \delta(t-t')$.

$$\begin{aligned}\langle [u(t) - \langle u(t) \rangle]^2 \rangle &= \left\langle \left[\frac{1}{\tau_m} \int_0^t e^{-s/\tau_m} \xi(s) ds \right]^2 \right\rangle \\ &= \frac{1}{\tau_m^2} \int_0^t \int_0^t e^{-(s+s')/\tau_m} \langle \xi(s)\xi(s') \rangle ds ds' \\ &= \frac{a^2}{\tau_m} \int_0^t \int_0^t e^{-(s+s')/\tau_m} \delta(s-s') ds ds' \\ &= \frac{a^2}{\tau_m} \int_0^t e^{-2s/\tau_m} ds \\ &= \frac{a^2}{2} \left(1 - e^{-2t/\tau_m}\right).\end{aligned}$$

Exercise 2: Diffusive noise (stochastic spike arrival)

2.1 The expression for $u(t)$ was given by

$$u(t) = u_{rest} + \frac{qR}{\tau} \int_0^t e^{-s/\tau} S(t-s) ds. \tag{2}$$

Taking the average of Eq.(2) over multiple repetitions or over a population of neurons, we obtain

$$\langle u(t) \rangle = u_{\text{rest}} + \frac{q}{C} \int_0^t e^{-s/\tau} \underbrace{\langle S(t-s) \rangle}_{=\nu} ds \tag{3}$$

$$= u_{\text{rest}} + \frac{q\tau\nu}{C} [1 - e^{-t/\tau}] \,. \tag{4}$$

2.2 Using $\langle S(t)\rangle = \nu$, $\tau = RC$ and $\langle S(t)S(t')\rangle = \nu\delta(t-t') + \nu^2$.

$$\begin{aligned}
\langle u(t)u(t)\rangle &= u_{\rm rest}^2 + \frac{2qu_{\rm rest}}{C}\int_0^t e^{-s/\tau}\langle S(t-s)\rangle\, ds \;+\frac{q^2}{C^2}\int_0^t\!\!\int_0^t e^{-s/\tau}e^{-s'/\tau}\langle S(t-s)S(t-s')\rangle\, ds\, ds' \\
&= u_{\rm rest}^2 + \frac{2u_{\rm rest}q\tau\nu}{C}[1-e^{-t/\tau}] + \frac{q^2\tau\nu}{2C^2}[1-e^{-2t/\tau}] + \frac{q^2\tau^2\nu^2}{C^2}[1-e^{-t/\tau}]^2. \\
&= \left(u_{rest} + \frac{q\tau\nu}{C}\left[1-e^{-t/\tau}\right]\right)^2 + \frac{q^2\tau\nu}{2C^2}\left[1-e^{-2t/\tau}\right]
\end{aligned}$$

note that the units are consistent since q/C have units of voltage and $\tau\nu$ has no units.

2.3 It is easy to obtain the variance from the previous two questions:

$$\langle u(t)^2\rangle - \langle u(t)\rangle^2 = \frac{\nu q^2R^2}{2\tau}[1-e^{-2t/\tau}].$$

2.4 Same technique as previous problem, but we replace the upper bound of the integral by ∞.

$$\begin{aligned}
\langle u(t)u(t')\rangle &= u_{\rm rest}^2 + \frac{2qu_{\rm rest}}{C}\int_0^\infty e^{-s/\tau}\langle S(t-s)\rangle\, ds \;+\frac{q^2}{C^2}\int_0^\infty\!\!\int_0^\infty e^{-s/\tau}e^{-s'/\tau}\langle S(t-s)S(t'-s')\rangle\, ds\, ds' \\
&= u_{\rm rest}^2 + \frac{2u_{\rm rest}q\tau\nu}{C} + \frac{q^2\tau\nu}{2C^2}e^{(t-t')/\tau} + \frac{q^2\tau^2\nu^2}{C^2}. \\
&= \left(u_{rest} + \frac{q\tau\nu}{C}\right)^2 + \frac{\tau\nu q^2}{2C^2}e^{(t-t')/\tau} \qquad \text{, for } t' > t\ .
\end{aligned}$$

For the general case the autocorrelation is:

$$\langle u(t)u(t')\rangle = \left(u_{rest} + \frac{q\tau\nu}{C}\right)^2 + \frac{\tau\nu q^2}{2C^2}e^{|t-t'|/\tau}.$$

2.5 We have $S(t) = w_1S_1(t) + w_2S_2(t)$, $\langle S_1\rangle = \nu_1$, $\langle S_2\rangle = \nu_2$, $\langle S_1(t)S_2(t')\rangle = \nu_1\nu_2$,$\langle S_1(t)S_1(t')\rangle = \nu_1\delta(t-t') + \nu_1^2$ and $\langle S_2(t)S_2(t')\rangle = \nu_2\delta(t-t') + \nu_2^2$.
The mean and the autocorrelation in the steady state regime ($t\to\infty$) will respectively be

$$\begin{aligned}
\langle u(t)\rangle &= u_{\rm rest} + \frac{q}{C}\int_0^\infty e^{-s/\tau}\underbrace{\langle w_1S_1(t-s) + w_2S_2(t-s)\rangle}_{=\nu_1+\nu_2} ds \\
&= u_{\rm rest} + \frac{q\tau}{C}(w_1\nu_1 + w_2\nu_2)\,,
\end{aligned}$$

and

$$\langle u(t)u(t')\rangle = u_{\rm rest}^2 + 2u_{\rm rest}\frac{q\tau}{C}(w_1\nu_1 + w_2\nu_2) + \frac{q^2\tau^2}{C^2}(w_1\nu_1 + w_2\nu_2)^2 + \frac{q^2\tau}{2C^2}(w_1^2\nu_1 + w_2^2\nu_2)e^{-|(t-t')|/\tau}.$$

2.6 In this case, we have $\nu_1 = \nu_2 = \nu$ and $\langle S_1(t)S_1(t')\rangle = \langle S_2(t)S_2(t')\rangle = \langle S_1(t)S_2(t')\rangle = \nu\delta(t-t') + \nu^2$.
We find

$$\langle u(t)\rangle = u_{\rm rest} + \frac{q\tau}{C}(w_1 + w_2)\nu\,,$$

and

$$\langle\, u(t)u(t')\,\rangle = u_{\text{rest}}^2 + 2u_{\text{rest}}\frac{q\tau}{C}(w_1+w_2)\nu + \frac{q^2\tau^2}{C^2}(w_1+w_2)^2\nu^2 + \frac{q^2\tau}{2C^2}(w_1+w_2)^2\nu e^{-|(t-t')|/\tau}.$$

Note that the correlations between the two spike trains increase the autocorrelation of u since $(w_1+w_2)^2 > w_1^2+w_2^2$.

Exercise 3: Renewal process

Given an output spike at $t=\hat{t}$, the survivor function $S(t-\hat{t})$ is given by

$$S(t-\hat{t}) = \exp\Big[-\int_{\hat{t}}^{t}\rho(t'|\hat{t})dt'\Big] = \exp\Big[-\int_{\hat{t}}^{t}\rho(t'-\hat{t})dt'\Big] = \exp\Big[-\int_{0}^{t-\hat{t}}\rho(s)ds\Big]\,.$$

where we made the variable change $s=t'-\hat{t}$.
The interspike interval distribution is $P(t-\hat{t}) = \rho(t-\hat{t})S(t-\hat{t})$. Thus we only need to calculate the integral of the hazard function $\rho(t-\hat{t})$. This gives

$$\int_0^{t-\hat{t}}\rho(s)ds = \begin{cases} \int_0^{t_{\text{abs}}}\rho(s)ds = 0 & \text{for } s<t_{\text{abs}} \\ \int_0^{t_{\text{abs}}}\rho(s)ds + \int_{t_{\text{abs}}}^{t-\hat{t}}\rho(s)ds = \frac{\rho_0}{4}\left(t-\hat{t}-t_{\text{abs}}\right)^2 & \text{for } t_{\text{abs}}<s<t_{\text{abs}}+2 \\ \int_0^{t_{\text{abs}}}\rho(s)ds + \int_{t_{\text{abs}}}^{t_{\text{abs}}+2}\rho(s)ds + \int_{t_{\text{abs}}+2}^{t-\hat{t}}\rho(s)ds = \rho_0\left(-1+t-\hat{t}-t_{\text{abs}}\right) & \text{for } t_{\text{abs}}+2<s\,. \end{cases}$$