Neural Networks and Biological Modeling

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Question set 3

Exercise 1: Separation of time scales

A. One-dimensional system

Consider the following differential equation

$$\tau \frac{dx}{dt} = -x + c\,. \tag{1}$$

1.1 Find the fixed point x_0 of this system. Hint: a fixed point is a stationary solution $\Rightarrow \frac{dx}{dt} = 0$.

1.2 Show that the fixed point is a stable one, and that the solution of (1) converges exponentially towards the fixed point with a time constant τ. Hint: write down the solution assuming an initial condition $x(t=0) \neq x_0$.

1.3 Consider the case where c is time-dependent, namely,

$$c \equiv c(t) = \begin{cases} 0 & \text{for } t < 0 \\ c_0 & \text{for } 0 \leq t < 1 \\ 0 & \text{for } t > 1\,. \end{cases}$$

Calculate the solution $x(t)$ with initial condition $x(t = -10) = 0$.

1.4 Take the expression $x(t)$ you have found in the previous question. Consider $\tau = 0.5$ and $\tau = 0.01$ and sketch the function graph.

B. Separation of time scales

Consider the following system of equations:

$$\begin{aligned} \frac{du}{dt} &= f(u) - m \\ \epsilon \frac{dm}{dt} &= -m + c(u) \end{aligned}$$

with $\epsilon = 0.01$.

1.5 Exploit the fact that $\epsilon \ll 1$ and reduce the system to one equation (note the similarity between the m-equation and Eq.(1)).

1.6 Set $f(u) = -au + b$ where $a > 0$, $b \in \mathbb{R}$ and $c(u) = \tanh(u)$. Discuss the stability of the fixed points with respect to a and b. Hint: use the graphical analysis for one dimensional equations from week 1: when plotting $f(u)$ and $c(u)$ against u, you can read off the fixed point from that graph.

Exercise 2: Phase plane stability analysis

2.1 Linear system

Consider the following linear system:

$$\left[\begin{array}{ll} \frac{du}{dt} & = \alpha u - w \\ \frac{dw}{dt} & = \beta u - w\,. \end{array}\right.$$

These equations can be written in matrix form as $\frac{d}{dt}x = Ax$ where $x = \begin{pmatrix} u \\ w \end{pmatrix}$ and $A = \begin{pmatrix} \alpha & -1 \\ \beta & -1 \end{pmatrix}$. Determine the conditions for stability of the point $(u = 0, w = 0)$ in the case $\beta > \alpha$ by studying the eigenvalues of the above matrix. (Hint: Distinguish the cases of real and complex eigenvalues.)

2.2 Piecewise linear Fitzhugh-Nagumo model

The Fitzhugh-Nagumo model is defined by the equations

$$\left[\begin{array}{ll} \frac{du}{dt} & = F(u,w) = f(u) - w + I \\ \frac{dw}{dt} & = G(u,w) = bu - w \end{array}\right.$$

Here, $u(t)$ is the membrane potential and $w(t)$ is a second, time-dependent variable. I stands for the injected current. A simplified model is obtained by considering a piecewise linear $f(u)$:

$$f(u) = \begin{cases} -u & \text{if } u < 1 \\ \frac{u-1}{a} - 1 & \text{if } 1 \leq u < 1 + 2a \\ 2(1+a) - u & \text{if } u > 1 + 2a \end{cases}$$

with $a < 1$, $b > 1/a$.

(i) Sketch the "nullclines" $du/dt = 0$ and $dw/dt = 0$ in a (u,w)-plot. Consider the case $I = 0$. How does the fixed point move as I is varied? Sketch the form of the flow (i.e., the vector $(du/dt, dw/dt)$) along the nullclines and deduce qualitatively the shape of the trajectories.

(ii) Calculate the Jacobian matrix evaluated at the fixed point,

$$J = \begin{pmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial w} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial w} \end{pmatrix}.$$

Determine, by studying the eigenvalues of J, the linear stability of the fixed point as a function of I. What happens when the fixed point destabilizes?