Neural Networks and Biological Modeling

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Question set 13

Exercise 1: Low-dimensional dynamics in a field model

We study a recurrent network composed of N rate neurons, where the membrane potential $h_i(t)$ of each neuron $i = 1, \dots, N$ follows the dynamics:

$$\frac{d}{dt} h_i(t) = -\frac{1}{\tau} h_i(t) + \frac{J}{N} \sum_{j=1}^{N} W_{ij} \phi(h_j(t)) \tag{1}$$

where $\phi(h)$ is a non-linear voltage-to-rate function. We consider that each neuron i has a spatial position $\boldsymbol{z}_i$ in a space V, and that the density of neurons at position $\boldsymbol{z}$ is given by the distribution $\rho(\boldsymbol{z})$. These positions determine the recurrent connections, via the expression:

$$W_{ij} = \sum_{\mu=1}^{D} f_\mu(\boldsymbol{z}_i) g_\mu(\boldsymbol{z}_j) \tag{2}$$

where f_μ, g_μ ($\mu = 1, \dots, D$) are continuous functions of the neurons' positions.

In the limit of $N \to \infty$, the model corresponds to a neural field model, where the membrane potential of each neuron depends on its position: we have $h_i(t) = h(t, \boldsymbol{z}_i)$. The sum over all neurons j in Eq.(1) can be replaced by an integral over all the positions $\boldsymbol{z}'$, so that we obtain the neural field equation:

$$\frac{d}{dt} h(t, \boldsymbol{z}) = -\frac{1}{\tau} h(t, \boldsymbol{z}) + J \int_V \sum_{\mu=1}^{D} f_\mu(\boldsymbol{z}) g_\mu(\boldsymbol{z}') \phi(h(t, \boldsymbol{z}')) \rho(\boldsymbol{z}') d\boldsymbol{z}' \tag{3}$$

As a field model, this is an infinite-dimensional dynamical system. Yet, due to the particular form of the connectivity in Eq.(2), it can be reduced to a D-dimensional description. The goal of this exercise is to derive the hidden D-dimensional dynamics.

In this exercise, we assume that the functions f_μ are orthonormal:

$$\int_V f_\mu(\boldsymbol{z}) f_\nu(\boldsymbol{z}) \rho(\boldsymbol{z}) d\boldsymbol{z} = \delta_{\mu\nu} = \{1 \text{ if } \mu = \nu, \text{ and } 0 \text{ otherwise}\} \tag{4}$$

1.1 Assume that the field $h(t, \boldsymbol{z})$ is given by a linear combination of the functions f_μ, with time-dependent coefficients $\kappa_\mu(t)$ ($\mu = 1, \dots, D$); that is:

$$h(t, \boldsymbol{z}) = \sum_{\mu=1}^{D} f_\mu(\boldsymbol{z}) \kappa_\mu(t) \tag{5}$$

What is the expression of each coefficient κ_μ in terms of the field $h(t, \boldsymbol{z})$?

Hint: compute the projection of the field on the function f_μ: $\int_V f_\mu(\boldsymbol{z}) h(t, \boldsymbol{z}) \rho(\boldsymbol{z}) d\boldsymbol{z}$.

1.2 We are now interested in the fixed points of Eq.(3). Find a closed-form expression solved by the coefficients κ_μ in the steady-state.

Hint: use Eq.(5) to replace the field with the variables κ_μ.

1.3 Starting from the field dynamics of Eq.(3), derive a closed-form expression for the dynamics of the coefficients κ_μ.

Hint: use Eq.(5) to compute the time derivative of the variables $\kappa_\mu(t)$.

1.4 Consider that, at initial time $t = 0$, the field is a linear combination of the functions f_μ, *plus* an additional term:

$$h(t=0,\boldsymbol{z}) = \sum_\mu f_\mu(\boldsymbol{z})\kappa_\mu(0) + \Delta h(\boldsymbol{z})$$

where $\int_V \Delta h(\boldsymbol{z}) f_\mu(\boldsymbol{z})\rho(\boldsymbol{z})d\boldsymbol{z} = 0$, for all μ. What are the dynamics of Δh? Why is Eq.(5) a good assumption?

1.5 Consider now that the network receives an external input, given by an additional term in Eq.(3):

$$I^{\text{ext}}(\boldsymbol{z}) = \sum_{\mu=1}^{D} f_\mu(\boldsymbol{z})I_\mu(t)$$

What are the dynamics of the coefficients κ_μ now? Can the external input affect the fixed points?

Exercise 2: Application to the ring model

We now apply the results of the first exercise to the ring model, in which neurons have angular positions $z_i \in [0, 2\pi[$ uniformly distributed on a ring, and the membrane potential of each neuron i follows the dynamics:

$$\frac{d}{dt}h_i(t) = -\frac{1}{\tau}h_i(t) + \frac{J}{N}\sum_{j=1}^{N} W_{ij}\phi(h_j(t)) \tag{6}$$

$$W_{ij} = 2\cos(z_i - z_j) \tag{7}$$

The interactions can be written in the form of Eq.(2), by using the sum expansion of the cosine: we have

$$W_{ij} = 2\cos(z_i - z_j) = 2\cos(z_i)\cos(z_j) + 2\sin(z_i)\sin(z_j) = \sum_{\mu=1}^{2} f_\mu(z_i)g_\mu(z_j) \tag{8}$$

where $f_1(z) = g_1(z) = \sqrt{2}\cos(z)$, and $f_2(z) = g_2(z) = \sqrt{2}\sin(z)$.

We consider the limit of $N \to \infty$, where the membrane potentials are described by a neural field over the ring: $h_i(t) = h(t, z_i)$.

2.1 What is the density of neurons, $\rho(z)$? Check that the orthonormality condition of Eq.(4) is satisfied.

*H*int: the distribution of the neurons integrates to 1, i.e. $\int_0^{2\pi} \rho(z)dz = 1$.

In the following, consider the step transfer function given by: $\phi(x) = RH(x)$, where $H(x) = \mathbf{1}\{x > 0\}$ is the Heaviside function, and R some active-state firing rate.

2.2 Find the value of $A > 0$ such that $h_i(t) = A\cos(z_i)$ is a steady-state of the dynamics.

To find the solution, use the field formulation of Eq.(5). The steady state corresponds to: $h(z) = A/\sqrt{2}f_1(z)$, so that $\kappa_1 = A/\sqrt{2}$, and $\kappa_2 = 0$.

2.3 Given the initial condition $h_i(t=0) = A_0\cos(z_i)$, find the solution to the dynamics. Use your result to express the solution for the initial condition $h_i(t=0) = A_0\sin(z_i)$.

*H*int: look for a solution of the form: $h(t, z) = A(t)\cos(z)$.

2.4 Show that there is a continuous set of attractors of the dynamics, that forms a circle in the space of (κ_1, κ_2), with radius $\sqrt{2}\tau JR/\pi$. What is the flow of the dynamics in the phase plane of (κ_1, κ_2)?

*H*int: express the dynamics of the vector $\boldsymbol{\kappa}(t) = (\kappa_1(t), \kappa_2(t))$ in polar coordinates.