Last Name .... First Name..................

Computational Neuroscience: Neuronal Dynamics
Exam
June 2023

- Write your name in legible letters on top of this page.
- Check that your exam has 12 pages (numbered 1-12).
- The exam lasts 180 min.
- Write **all** your answers in a legible way on the exam (no extra sheets).
- No documentation is allowed apart from 1 sheet A5 of **handwritten** notes.
- No calculator is allowed.
- Have your student card displayed before you on your desk.
- Keep your bag next to your chair, but do not touch or open it during the exam

Evaluation:

1. .... / 7 pts (Separation of time scales. Estimated time 20 min)
2. .... / 15 pts (Phase plane analysis. Estimated time 75 min)
3. .... / 7 pts (Hopfield Model. Estimated time 35 min)
4. .... / 10 pts (Stochastic spiking/Fokker-Planck. Estimated time 50 min)

Total: ..../ 39 pts

1 Separation of time scales (7 points). Estimated time: 20 minutes

A homogeneous group of excitatory neurons (E) interacts with a homogeneous group of inhibitory neurons (I). In a population rate model, this is sometimes described by two equations:

$$\tau_E \frac{dh_E}{dt} = -h_E + w_{EE}\, \phi_E(h_E) - w_{EI}\, \phi_I(h_I) \quad (1)$$

$$\tau_I \frac{dh_I}{dt} = -h_i + w_{IE}\, \phi_E(h_E) \quad (2)$$

where τ_E and τ_I are time constants and have units of time.

a) *Understanding the equation in biological terms.* Sometimes different interpreations are possible. If so, choose only one, but stay consistent throughout.

(i) What is your interpretation of h_E? What are the units of h_E according to your interpretation?

....

(ii) Which parameter describes the coupling strength from excitation to inhibition? What are the units of this parameter according to your interpretation?

....

(iii) How do you interpret the quantity $\phi_I(h_I)$? What are the units of $\phi_I(h_I)$ according to your interpretation?

....

(iv) We look at the inputs to an excitatory neuron. What is the condition that the total amount of inhibition equals the total amount of excitation?

....

number of points: ....../[2]

b) *Separation of time scales* Assume that $\tau_I \ll \tau_E$. Simplify the dynamics to a single differential equation:

....

number of points: ....../[1]

Space for your calcualations. Do not write answers in this space.

c) We now set

$w_{EE} = 2\alpha;$

$w_{EI} = 4\alpha + \epsilon;$

$w_{IE} = 1/2;$

$\phi_E(h_E) = \exp[h_E];$

$\phi_I(h_I) = [h_I]_+ = h_I$ for $h_I \geq 0$ and 0 otherwise

(i) Write down the resulting differential equation. You may assume that $h_I > 0$.

(*) ....

(ii) Make a graph in the space below. To do so, draw the left-hand-side (y-axis) as a function of the righ-hand-side of your equation (*). The x-axis should be plotted in the range $-1 < x < 4$. Consider three cases: $\epsilon = 0$; $0 < \epsilon \ll 1$; $-1 \ll \epsilon < 0$.

(iii) In the above graph, indicate the stability of the fixed point(s)

number of points: ....../[3]

d) Your friend Alice runs simulations of a model with interacting populations of excitatory and inhibitory neurons similar to the system of Eqs. (1) and (2). She reports that IF (A) the total amount of inhibitory inputs into an excitatory neuron is at least as strong as that of excitatory inputs AND IF (B) inhibition is fast compared to excitation, her simulations never 'explode'. Do you believe that her observations are correct? Give a reason for your answer.

Hints: To understand her condition (A) reconsider your answer to the last question in part a). What is the role of ϵ? Include the case $|\epsilon| > 1$ in your considerations. What can be the reason of 'exploding simulations'?

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....

....

number of points: ....../[1]

2 Phase Plane Analysis for a network with Excitation-Inhibition balance (15 points). Estimated time: 75 minutes

Excitatory neurons interact with inhibitory neurons. In a population rate model, this is sometimes described by two equations:

$$\tau_E \frac{dh_E}{dt} = -h_E + 2\alpha \exp[h_E] - 4\alpha\, h_I + S(t) \tag{3}$$

$$\tau_I \frac{dh_I}{dt} = -h_i + (1/2) \exp[h_E] \tag{4}$$

where τ_E and τ_I are time constants and have units of time, $S(t)$ is external input, and α is an important paramter that we will manipulate in a later question. You can assume that h_E is unit-free.

Your friend Alice informs you that, based on her simulations with paramters $\tau_E = 10ms$ and $\tau_I = 2ms$ and $\alpha = 1$, the system has a stable fixed point.

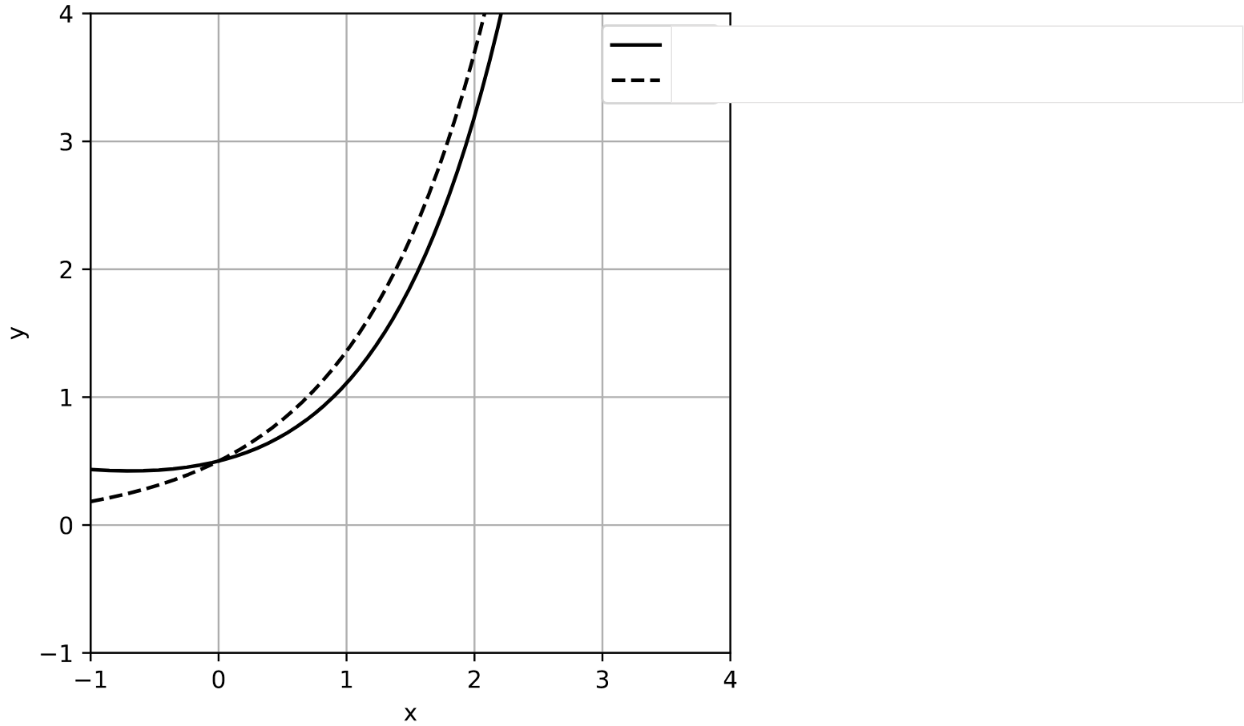
Hint. For numerical evaluations of the Euler number you can use $e = 2.7$ and $e^2 = 7.4$.

a) Calculate the nullclines of the two differential equations (3) and (4) for $S(t) = 0$.

Nullcline1: ....

Nullcline2: ....

number of points: ....../[2]



Phase plane with paramters $\tau_E = 10ms$ and $\tau_I = 2ms$ and $\alpha = 1$.

b) Label the two nullclines and the x-axis in the graph on the left page (bottom).

solid line = ...

dashed line = ...

x-axis: x= ....

number of points: ....../[1]

c) Evaluate at the point $(x, y) = (0, 0)$ the derivatives using the set of parameters $\tau_E = 10ms$ and $\tau_I = 2ms$ and $\alpha = 1$ (either a fraction or 1 digit after the comma is sufficient).

$dh_E/dt =$ ....

$dh_I/dt =$ ....

number of points: ....../[1]

d) Evaluate at the point $(x, y) = (1, 0.5)$ the derivatives using the set of parameters $\tau_E = 10ms$ and $\tau_I = 2ms$ and $\alpha = 1$ (either a fraction or up to 1 digit after the comma is sufficient).

$dh_E/dt =$ ....

$dh_I/dt =$ ....

number of points: ....../[1]

e) Add in your graph flow arrows on the nullclines (at least four on each nullcline).

number of points: ....../[2]

f) Construct a trajectory, starting at $(0, 0)$, and draw it in the graph on the left.

number of points: ....../[1]

g) Assume that the dynamical system is exactly at the fixed point. At time t_{pulse} you apply a pulse stimulus $S(t) = \tau_E \delta(t - t_{\text{pulse}})$ where δ denotes the Dirac-δ-function.

Plot qualitatively the trajectory for $t > t_{\text{pulse}}$ in the graph on the left.

number of points: ....../[1]

Space for your calcualations. Do not write answers in this space.

h) Introduce a parameter $\beta = \tau_E/\tau_I$. The aim is to analyze the stability of the fixed point for **arbitrary** α **and** β **and** τ_E. You will find two Eigenvalues:

$\lambda_1 =$ ....

$\lambda_2 =$ ....

number of points: ....../[2]

i) Evaluate the stability for $\alpha = 1$ and $\beta = 5$ and $\tau_E = 10ms$.

The fixed point is stable/unstable because ....

....

number of points: ....../[1]

k) Evaluate the stability for $\alpha = 4$ and $\beta = 5$ and $\tau_E = 10ms$.

The fixed point is stable/unstable because ....

....

number of points: ....../[1]

m) For the set of parameters $\alpha = 4$ and $\beta = 5$ and $\tau_E = 10ms$, redo qualitatively the plot from part g), i.e., the trajectory after the same pulse stimulus as in g). **Plot the trajectory in the new graph top right.**

Hint: How is the initial slope of the trajectory different between g) and m)?

number of points: ....../[1]

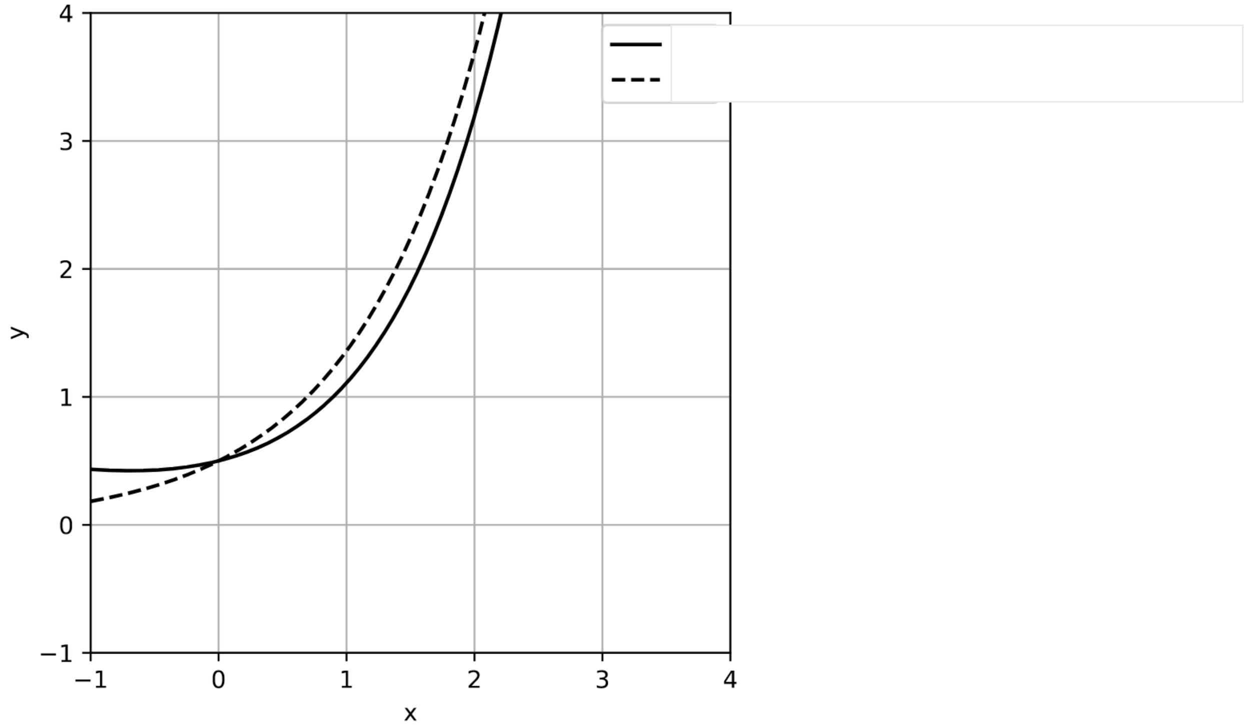
n) For the set of parameters $\alpha = 4$ and $\tau_E = 10ms$ and $\tau_I = 0.1ms$ (i.e. $\beta = 100$), redo qualitatively the plot from part g), i.e., the trajectory after the same pulse stimulus as in g). **Plot the trajectory in the new graph bottom right.**

Hint: How is the initial slope of the trajectory different between g) and n)?

number of points: ....../[1]

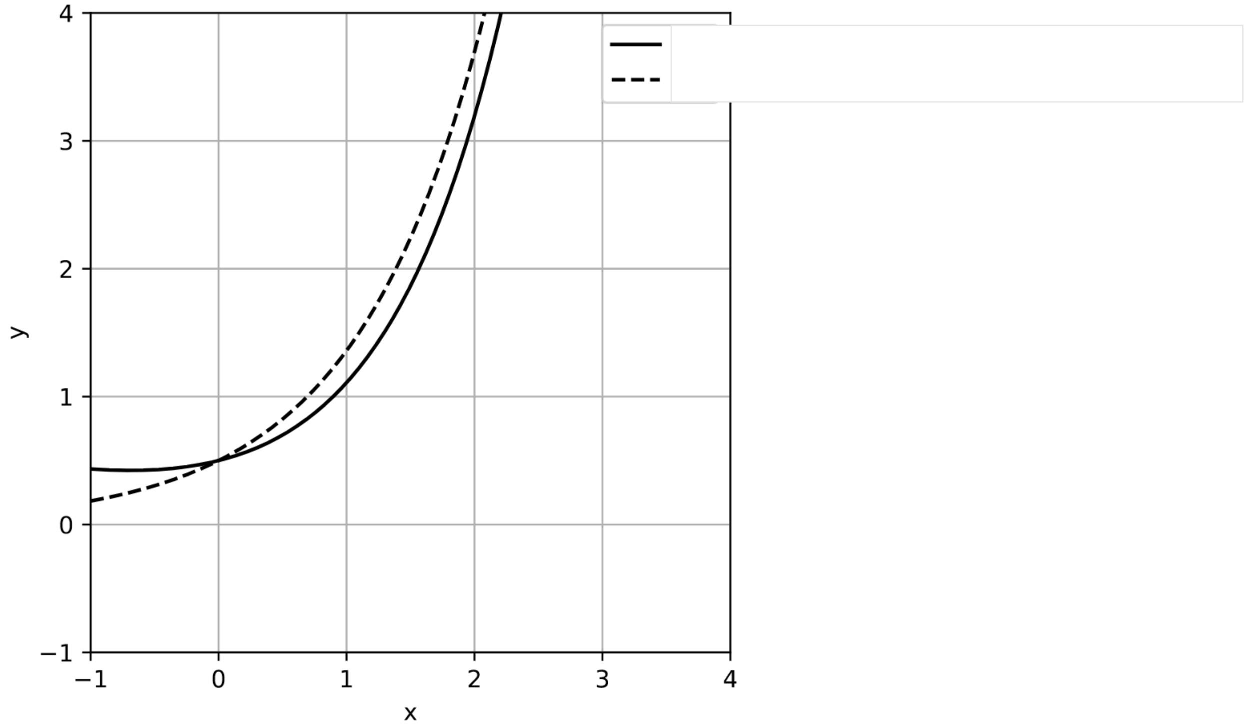
Space for your calcualations. Do not write answers in this space.

Graph for part m)



Phase plane with paramters $\tau_E = 10ms$ **and** $\beta = 5$ **and** $\alpha = 4$.

Graph for part n)



Phase plane with paramters $\tau_E = 10ms$ **and** $\tau_I = 0.1ms$ **and** $\alpha = 4$.

3 Hopfield model (7 points). Estimated time 35 minutes.

Consider a network of $N = 20000$ neurons that has stored 4 patterns

$\xi^1 = \{\xi^1_1, \ldots \xi^1_N\}$
$\xi^2 = \{\xi^2_1, \ldots \xi^2_N\}$
$\xi^3 = \{\xi^3_1 \ldots \xi^3_N\}$
$\xi^4 = \{\xi^4_1 \ldots \xi^4_N\}$

using the synaptic update rule $w_{ij} = (J/N) \sum_\mu \xi^\mu_i \xi^\mu_j$ where $J > 0$ is a parameter. **Each pattern has values $\xi^\mu_i = \pm 1$ so that exactly 50 percent of neurons in a pattern have $\xi^\mu_i = +1$.**

Assume stochastic dynamics: neurons receive an input $h_i(t) = \sum_j w_{ij} S_j(t)$ where $S_j(t) = \pm 1$ is the state of neuron j. Neurons update their state

$$Prob\{S_i(t+1) = +1 | h_i(t)\} = 0.5[1 + g(h_i(t))] \quad (5)$$

where g is an **odd and monotonically increasing function**: $g(h) = 6h$ for $|h| < 1/6$ and $g(h) = 1$ for $h \geq 1/6$ and $g(h) = -1$ for $h \leq -1/6$.

a) Rewrite the righ-hand-side of equation (5) by introducting an overlap $m^\mu(t) = (1/N) \sum_j \xi^\mu_j S_j(t)$.

....

number of points: ....../1

b) What is the significance of the overlap? Describe its meaning in one sentence. What is the percentage of neurons that have 'correct' activity, if the overlap is $m^\mu = 0.1$?

....

....

....

number of points: ....../1

c) Assume that the four patterns are orthogonal, i.e., $\sum_i \xi_i^\mu \xi_i^\nu = 0$ if $\mu \neq \nu$. **Assume that the overlap with pattern 4 at** $t = 0$ **has a value of 0.10 and** $m^\mu(0) = 0$ **for all other patterns.**

Suppose that neuron i is a neuron with $\xi_i^4 = -1$.

What is the probability that neuron i fires in time step 1? Give the formula for arbitrary J and evaluate then for $J = 1$.

....

....

What is the probability that another neuron k with $\xi_k^4 = +1$ fires in time step 1? Give the formula for arbitrary J and evaluate then for $J = 1$.

....

....

number of points: ....../2

d) For the same assumptions as in (c), what is the expected overlap for $< m^4(t) >$ after the first time step.

$\langle m^4(1) \rangle =$ ....

number of points: ....../2

e) In question (d) you calculated the EXPECTATION $\langle m^4(1) \rangle$. Can we drop the expectation sign in the limit of $N \to \infty$? Justify your answer in one sentence.

....

....

....

....

number of points: ....../1

4 Stochastic Spike arrival and Fokker-Planck equation (10 points). Estimated time 50 minutes

We consider three populations A,B,C of spiking neurons. Populaton A contains $N_E = 1000$ excitatory neurons and has all-to-all connections to neurons in population C. Populaton B contains $N_I = 250$ inhibitory neurons and has all-to-all connections to neurons in population C. There are no connections inside the populations.

Leaky Integrate-and-Fire neurons in C. We are interested in the firing activity of neurons in population C. The membrane time constant of neurons in population C is $\tau_m = 10ms$, the voltage scale is shifted so that the resting potential is at 0mV, the dynamics is that of leaky integrate-and-fire neurons, the threshold ϑ is 30mV above the resting potential, and after firing the membrane potential is reset to $u = 0$.

a) We consider an arbitrary neuron i in population C. Each excitatory input pulse causes a voltage jump by an amount of α (for example $\alpha = 0.5mV$ or $\alpha = 1mV$ are both biologically plausible choices). Each inhibitory input pulse causes a voltage jump by an amount of -2α.

Write down the differential equation for the voltage u_i of neuron i below ϑ. Denote spiking times by t_j^n, use the Dirac-δ-function and make sure that the jump size after spike arrivals are correct, that all input spikes are taken into account, and that all input spikes have the desired effect!

$\frac{du_i}{dt}$ = ...

number of points: ....../[1]

b) Each neuron in population A fires stochastically (Poisson) with firing rate $\nu_E = 10$Hz. Each neuron in population B fires stochastically (Poisson) with a firing rate $\nu_I = 20$Hz.

Assume that α is small enough so that it is unlikely that the membrane potential reaches ϑ.

Calculate the mean voltage of neuron i and write down the result:

$\langle u_i \rangle$ = ...

number of points: ....../[1]

Space for calculations. Do not use to write down answers!

c) Your friend Barbara works in a lab where they can routinely block the activity of inhibitory neurons. Under the assumption that neurons in population B do not send spikes, but neurons in population A are unaffected by the manipulation and fire at $\nu_E = 10$Hz as before, what is mean voltage as a function of α?

Assume that α is small enough so that it is unlikely that potential reaches ϑ.

$\langle u_i \rangle =$ ....

number of points: ....../[1]

d) We now undo the blocking so that as before neurons in population A fire at $\nu_E = 10$Hz and those in B at $\nu_I = 20$Hz.

Assume that α is small enough so that it is unlikely that the potential reaches ϑ.

Calculate the variance $\langle (\Delta u)^2 \rangle$ of the membrane potential. Give 2 lines of calculation and the final result

(1) ....

(2) ....

(3) ....

(4) $\langle (\Delta u)^2 \rangle =$ ....

where (1) is your starting point (which you are free to choose!) and (4) is the final result.

For the transitions from (1) to (2) to (3) to (4) I used the following reasoning or mathematical identities

from (1) to (2) ....

from (2) to (3) ....

from (3) to (4) ....

number of points: ....../[3]

The exam continues on the next page!

Space for calculations. Do not use to write down answers!

e) We now take the threshold ϑ of neurons in population C into account and use the continuity equation or Fokker-Planck equation which we write either in this form

$$\frac{\partial}{\partial t}p(u,t) = -\frac{\partial}{\partial u}J(u,t) + \delta(u)A(t) \tag{6}$$

or alternatively as

$$\frac{\partial}{\partial t}p(u,t) = -\frac{\partial}{\partial u}[\gamma(u,t)p(u,t)] + \sigma^2\frac{\partial^2}{\partial u^2}[p(u,t)] + \delta(u)A(t) \tag{7}$$

We assume as before that neurons in population A fire at $\nu_E = 10$Hz and those in B at $\nu_I = 20$Hz.

(i) What is the 'drift' $\gamma(u)$ for the population of leaky integrate-and-fire neurons in C, as a function of α ?

$\gamma(u,t) =$ ....

(ii) What is the 'flux' J in Eq. (6)?

$J(u,t) =$ ....

number of points: ....../[2]

f) How can you determine the value of the population activity $A(t)$ from the continuity equation or Fokker-Planck equation (give the procedure needed for the calculation, not the answer)

$A(t) =$ ....

number of points: ....../[1]

g) We now compare to simulations. In the first stimulation the parameter is $\alpha = 0.5mV$ and in the second simulation $\alpha = 1mV$. Your friend Bob says that both values look biologically plausible. Do you expect that one of the simulations has a larger value of the population activity than the other one or both the same? If there is a difference, which value is larger? Your statement must be supported by a mathematical or intuitive reason and relate to some of your results in a) - f)

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....

....

number of points: ....../[1]