Last Name .... First Name..................

Neural Networks and Biological Modelling Exam
3 July 2019

- Write your name in legible letters on top of this page.
- The exam lasts 160 min.
- Write **all** your answers in a legible way on the exam (no extra sheets).
- No documentation is allowed apart from 1 sheet A5 of **handwritten** notes.
- No calculator is allowed.
- Have your student card displayed before you on your desk.
- **Check that your exam has 13 pages**

Evaluation:

1. ..../15 pts
2. .... / 11 pts
3. .... /13 pts

Total: .... / 39 pts

1 Phase Plane Analysis (15 points)

An integrate-and-fire neuron model with adaptation is described by two differential equations

$$\frac{du}{dt} = F(u) - w + I \tag{1}$$

$$\tau \frac{dw}{dt} = -w + a \cdot (u - 1) \tag{2}$$

If $u > 5$ the variable u is reset to $u = 0$. The variable w is increased by an amount of 2 during the reset.

We take

$$F(u) = -(2u - 1) \quad \text{for } u \leq 2 \tag{3}$$

$$F(u) = 4u - 11 \quad \text{for } u > 2 \tag{4}$$

(a) Plot the nullclines in the phase-plane (u, w) for $I = 0$ and $a = 0.2$, using the space here

number of points: ....../ 2

(b) In the same graph, add representative arrows indicating qualitatively the flow in different regions of the phase-plane (you might assume $\tau = 1$).

number of points: ....../ 2

(c) In the same graph, indicate a trajectory in the phase-plane, after a stimulus $I = 2\delta(t)$ has been applied (δ denotes the Dirac delta function).

number of points: ....../ 1

(d) In the same graph, indicate a trajectory in the phase-plane, after a stimulus $I = 3\delta(t)$ has been applied.

number of points: ....../ 2

(e) Under the assumption of a **constant** current $I = I_0$, what is the minimal current I_C such that for $I_0 > I_C$ the neuron must show regular firing?

$I_C =$ ....

number of points: ....../ 1

(f) We assume $\tau << 1$ (e.g. $\tau = 0.01$) and approximate the system of two equations by one equation. Give this equation:

....

....

number of points: ....../ 1

(g) Consider your result in (f) in the regime $u < 1$ and $a = 1$. What is the value of the stable fixed point?

....

number of points: ....../ 1

(h) What is the time-scale (compute the eigenvalue) characterizing the approach to the stable fixed point?

....

number of points: ....../ 1

(i) What is the time-scale (compute the eigenvalues) characterizing the approach to the fixed point in the 2D original system?

....

number of points: ....../ 2

(j) In the space below, re-draw the 2D phase-plane (just the nullclines). Add representative arrows in the phase-plane with separation of time scales as in (f). Finally, add the two trajectories starting from $(u, w) = (2, 1)$ and $(u, w) = (4, 2)$.

number of points: ....../ 2

Free space for your calculations, do not use to write down solutions/answers.

2 Mean-field in a Network of Rate Models (12 points)

We study a network of N neurons using mean-field methods. Each neuron has a (rescaled unit-free) firing rate r

$$\tau_r \frac{dr}{dt} = -r + f(I) \tag{5}$$

with τ_r = 10ms and

$$f(x) = \frac{1}{2}(3x - x^3) \quad \text{for} \quad 0 \leq |x| \leq 1 \tag{6}$$

$$f(x) = -1 \qquad \text{for} \quad x < -1 \tag{7}$$

$$f(x) = +1 \qquad \text{for} \quad x > +1. \tag{8}$$

(a) Calculate the slope of f at $x = 0$ and $x = 1$

$f'(0)$= ....

$f'(1)$= ....

and plot the graph of $f(x)$ in the space here:

number of points: ....../ 1

(b) Assume that we have a large network of $N = 100000$ neurons, where each neurons is described by the equations above. Each neuron receives exactly K=10 000 input connections, chosen randomly from the N neurons. Each connection has a weight w/K. The set of neurons that send connections to neuron i is called n_i. The input to neuron i is

$$I_i(t) = \sum_{k \in n_i} \frac{w}{K} r_k \tag{9}$$

Here r_k is the rate of excitatory neuron k.

Using a mean-field argument, solve graphically for the expected activity r_i of neuron i in a stationary state of the network for two qualitatively different choices of w. Make two different graphs for the two different situations. Circle the steady state(s), if there are any.

Show both graphs in the space on the NEXT PAGE

SHOW TWO GRAPHS ON THIS PAGE

number of points: ....../ 4

(c) Give the analytical solution for the steady state(s) /fixed point(s) of the network using a mean-field argument.

Consider the following values for w and indicate stability of the fixed point(s)

case (i) $w = 1/2$

For $w = 1/2$ the mean activity r in the stationary state is (one or several solutions)
....

....

and the fixed point(s) are stable/unstable ....

number of points: ....../ 1

case (ii) $w = 3/4$

For $w = 3/4$ the mean activity r in the stationary state is (one or several solutions)
....

....

and the fixed point(s) are stable/unstable ....

number of points: ....../ 2

case (iii), $w = 2$

For $w = 2$ the mean activity r in the stationary state is (one or several solutions)
....

....

and the fixed point(s) are stable/unstable ....

number of points: ....../ 1

(d) Is the solution found with the above mean-field argument for the network of rate neurons an exact solution or an approximate solution requiring additional assumptions? Tick one of the boxes and give a reason for your answer.

[ ] The above mean-field solution is the exact solution for the network in the stationary state because

....

....

[ ] The above mean-field solution is only an approximate solution for the network in the stationary state because

....

....

Would your answer change if we take $K = N$ or $K = 1$? Why?

....

....

Would your answer change if the set n_i of neurons connecting to i is chosen with probability $p_i = 0.1$? Why?

....

....

number of points: ....../ 2

Free Space for calculations, do no use to write answers

3 Stochastic spike arrival and stochastic spike firing (13 points)

We consider a neural network composed of two populations. A first population with $N_{post} = 1000$ neurons. These neurons are called postsynaptic neurons, because each neuron i in the postsynaptic population receives input from exactly $N_{pre} = 100$ presynaptic neurons. There is a total number of 100 000 presynaptic neurons (second population) and each one of these makes a connection to exactly one of the postsynaptic neurons.

The firing of neurons in the postsynaptic population is described by an inhomogeneous Poisson process with instantaneous firing rate $\rho_i^{post}(t)$ related to the membrane potential:

$$\rho_i^{post}(t) = \rho_0 \left[u_i(t) - \epsilon\right]_+ \tag{10}$$

where $[x]_+ = x$ for $x > 0$ and zero otherwise. $\epsilon \ll 1$ is a small positive number and $\rho_0 = 10Hz$ is the scale of the firing rate (stochastic intensity) of the inhomogeneous Poisson process.

The membrane potential of a neuron i in the post-synaptic population is given by

$$\tau \frac{d}{dt} u_i = -u_i + w \cos(\kappa t) \sum_j S_j(t) \tag{11}$$

$S_j(t) = \sum_f \delta(t - t_j^f)$ denotes the spike train of neuron j. δ is the Dirac δ-function; τ, κ and w are fixed parameters. The periodic modulation with frequency κ summarizes the influence of oscillatory activity in other brain areas not described in the simplified model.

The spiking behaviour of a pre-synpatic neuron j is described by a Poisson process with constant instantaneous firing rate $\rho(t) = \rho^{pre}$ for $t > 0$ and zero for $t < 0$. We assume that for $t < t_0$ the potential of postsynaptic neurons vanishes, $u_i(t) = 0$ $\forall i$.

(a) STEP1. For known input spike trains $S_j(t)$, write the solution of Eq. (11) in the form of an integral $u_i(t) = \int_?^? \cdots dt'$, but do not evaluate the integral yet. (Of course, u_i should not appear on the right hand side!)

$u_i(t) =$ ....

STEP 2. Now, at a given time t take the expectation over input spike trains. Evaluate the expectation, but do not evaluate the integral yet:

$< u_i(t) >=$ ....

number of points: ....../ 2

(b) The main task here consists in calculating the expected value of the membrane potential of the postsynaptic neurons as a function of time. For $t > 0$, the expected membrane potential of the postsynaptic neurons will show a perodic modulation of the form

$$< u_i(t) >= \frac{a_1}{1+\kappa^2\tau^2}\left(a_2 \cos(\kappa t) + a_3 \sin(\kappa t) - e^{-t/\tau}\right) \tag{12}$$

where the expectation sign indicates averaging over all possible input spike trains up to time t. Equivalently, the solution can also be written with a phase shift β

$$< u_i(t) >= \frac{B_1}{1+\kappa^2\tau^2}\left(B_2 \cos(\kappa t + \beta) - e^{-t/\tau}\right) \tag{13}$$

Calculate EITHER the three parameters a_1, a_2, a_3 OR the three parameters B_1, B_2, β whatever looks more convenient for your calculation. The values of the three parameters are:

....

....

....

number of points: ....../ 3

Free space for calculation, do not use to write answers

(c) Even if you have not found expressions for the parameters a_1, a_2, a_3, you are able to do the following: Sketch (in arbitrary units) the expected membrane potential $< u(t) >$ and a typical observation of the POPULATION ACTIVITY $A(t)$ of the postsynaptic population as a function of time for THREE values of κ and describe the behaviour in a few words (such as, the population activity is fluctuating/regular/ periodic/ stochastic/ constant-mean/ rectified/ inexistent etc.

case (i) $\kappa << 1/\tau$: The population activity is

....

Sketch here: $< u(t) >$ and $A(t)$

case (ii) $\kappa = 1/\tau$: The population activity is

....

Sketch here $< u(t) >$ and $A(t)$

case (iii) $\kappa \gg 1/\tau$: The population activity is

....

Sketch here $< u(t) >$ and $A(t)$

number of points: ....../ 3

(d) Even if you have not found expressions for the parameters a_1, a_2, a_3, you are able to do the following: Sketch a typical SPIKE TRAIN of a SINGLE neuron in the postsynaptic population for two situations and describe the spiking behaviour in 2-3 words (such as: the spike train is irregular/ regular/ periodic/ stochastic/ constant-mean/ rectified/ inexistent etc.)

case (i) $\kappa = 1/\tau = 50Hz$ and $a_1 = 40$. The spike train is

....

Sketch here:

case (ii) $\kappa = 1/\tau = 50Hz$ and $a_1 = 1$: The spike train is

....

Sketch here

number of points: ....../ 2

Free space for calculation, do not use to write answers

(e) calculate the variance $\Delta^2 =< u_i^2(t) > - < u_i(t) >^2=$

....

....

number of points: ....../ 3

Two hints:
(i) $\cos^2(x) = \frac{1+\cos(2x)}{2}$

(ii) the auto correlation of Poisson process input spike trains is given by:
$< S(t)S(t') >= (\rho^{pre})^2 + \rho^{pre}\delta(t - t')$

Free space for calculation, do not use to write answers