Week 4 – part : Type I and Type II Neuron Models

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Neuronal Dynamics:
Computational Neuroscience of Single Neurons

Week 4 – Reducing detail:
Two-dimensional neuron models

Wulfram Gerstner

EPFL, Lausanne, Switzerland

√ 4.1 **From Hodgkin-Huxley to 2D**

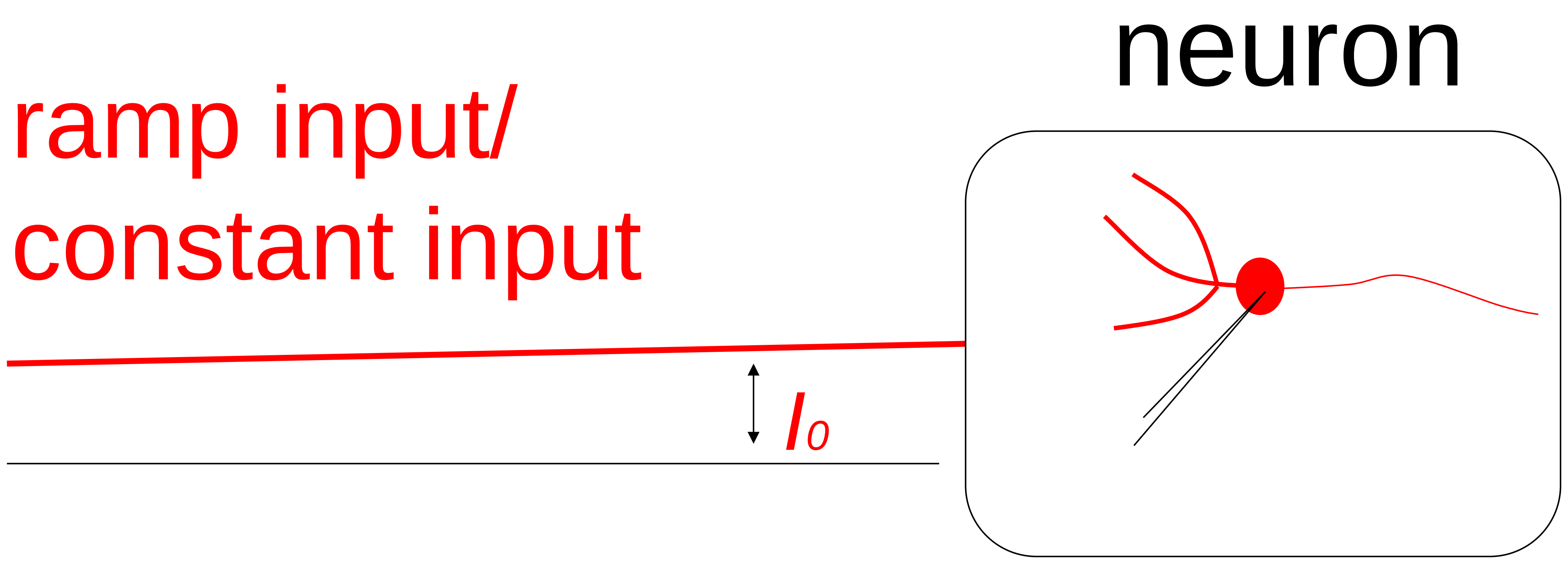
√ 4.2 **Phase Plane Analysis**

√ 4.3 **Analysis of a 2D Neuron Model**

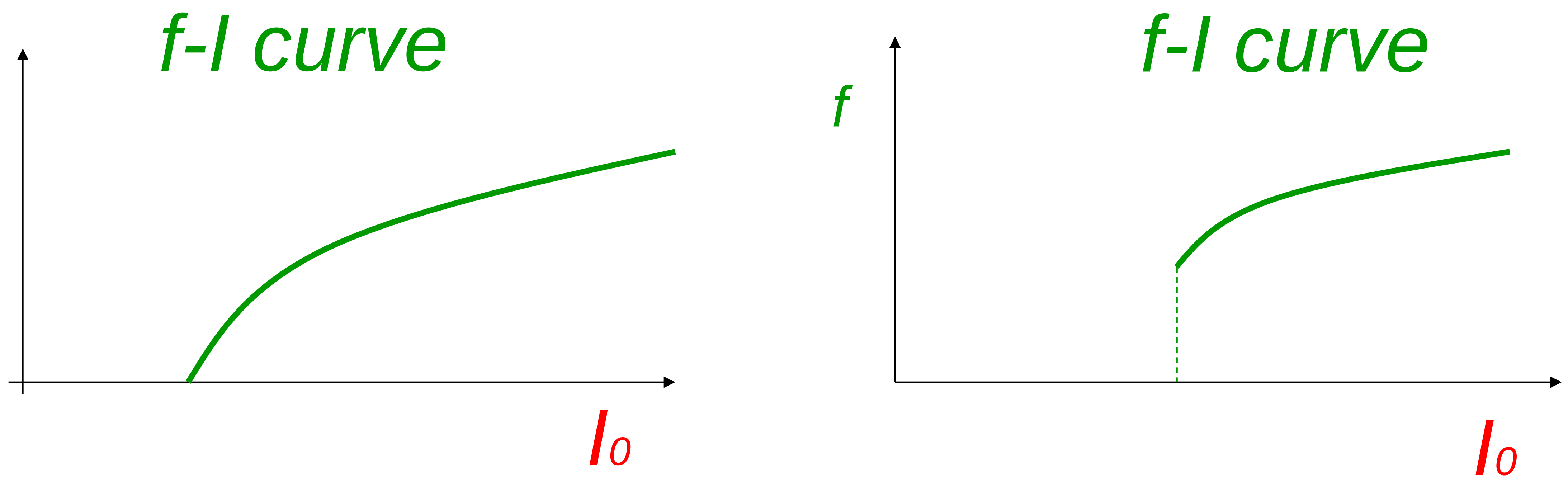
4.4 **Type I and II Neuron Models**
- where is the firing threshold?
- separation of time scales

4.5. **Nonlinear Integrate-and-fire**
- from two to one dimension

Neuronal Dynamics – 4.4. Type I and II Neuron Models



Type I and type II models



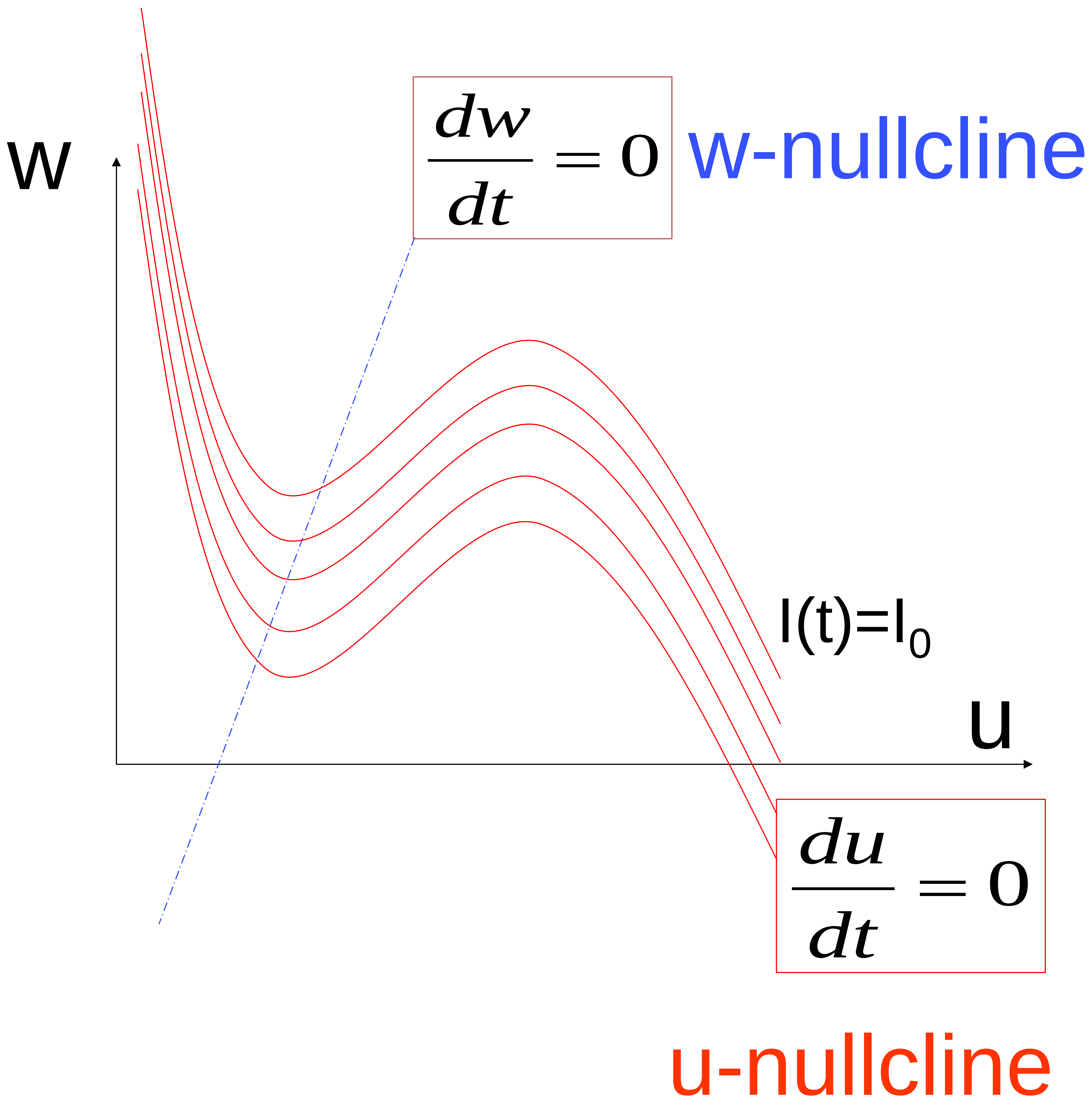
2 dimensional Neuron Models

stimulus

$$\tau \frac{du}{dt} = F(u,w) + I(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

apply constant stimulus I_0



FitzHugh Nagumo Model – limit cycle

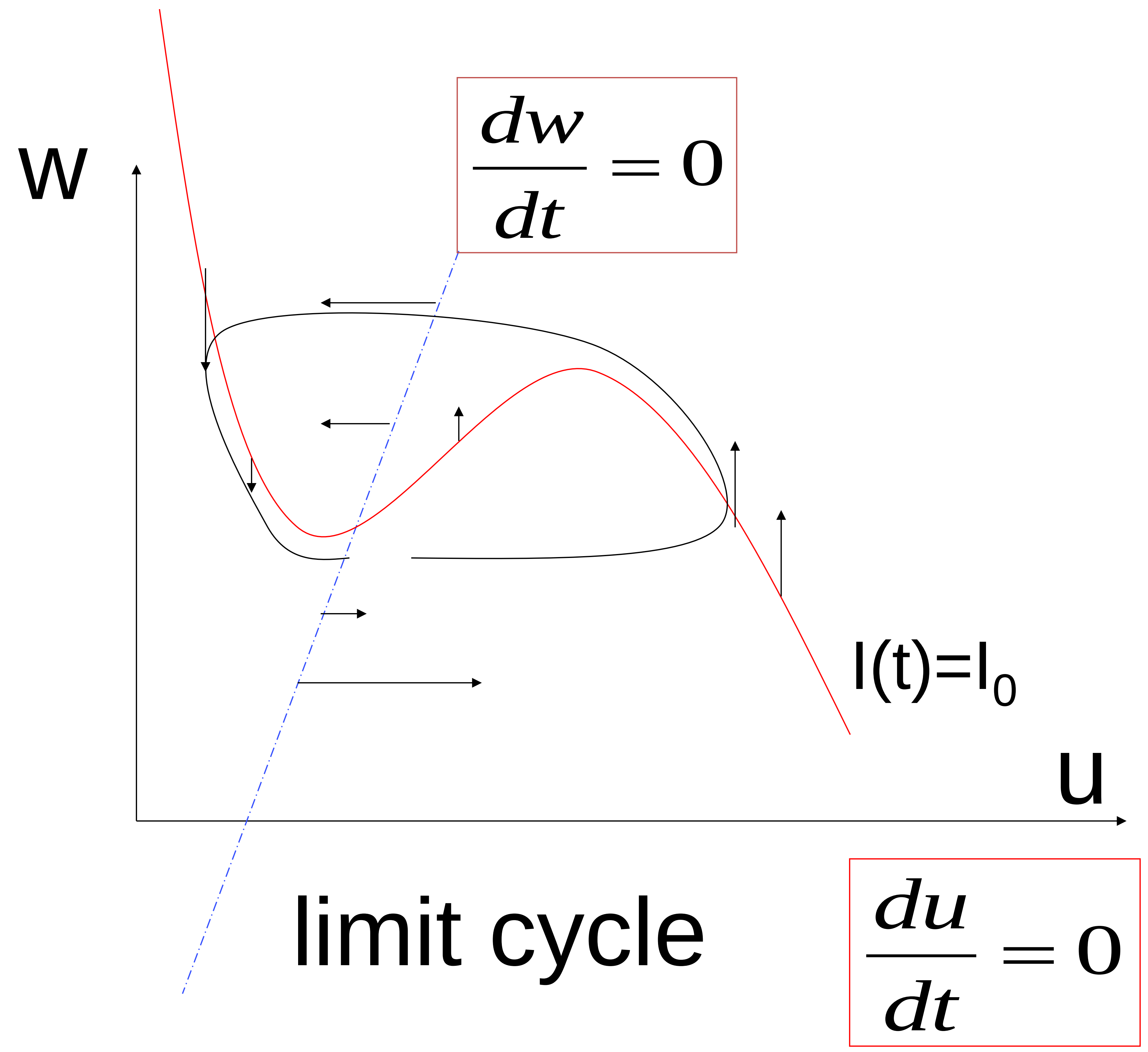
stimulus ↓

$$\tau \frac{du}{dt} = F(u,w) + I(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

-unstable fixed point
-closed boundary
with arrows pointing inside

→ limit cycle



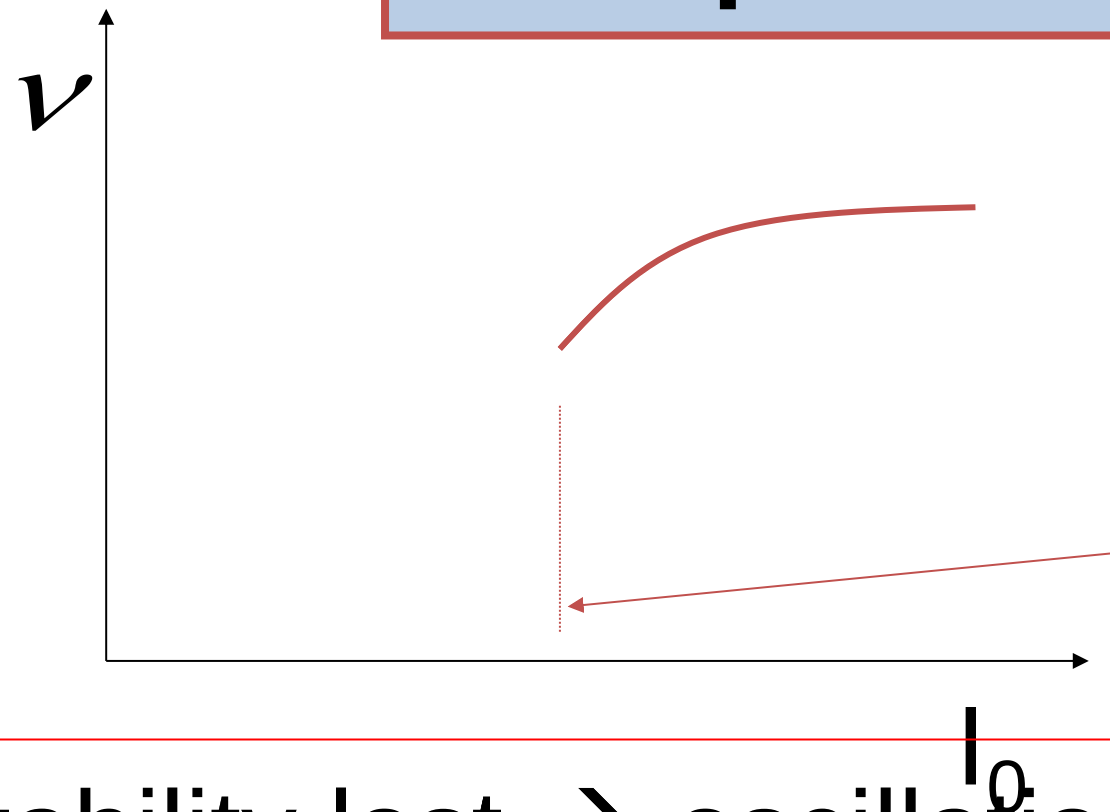
Type II Model
constant input

stimulus

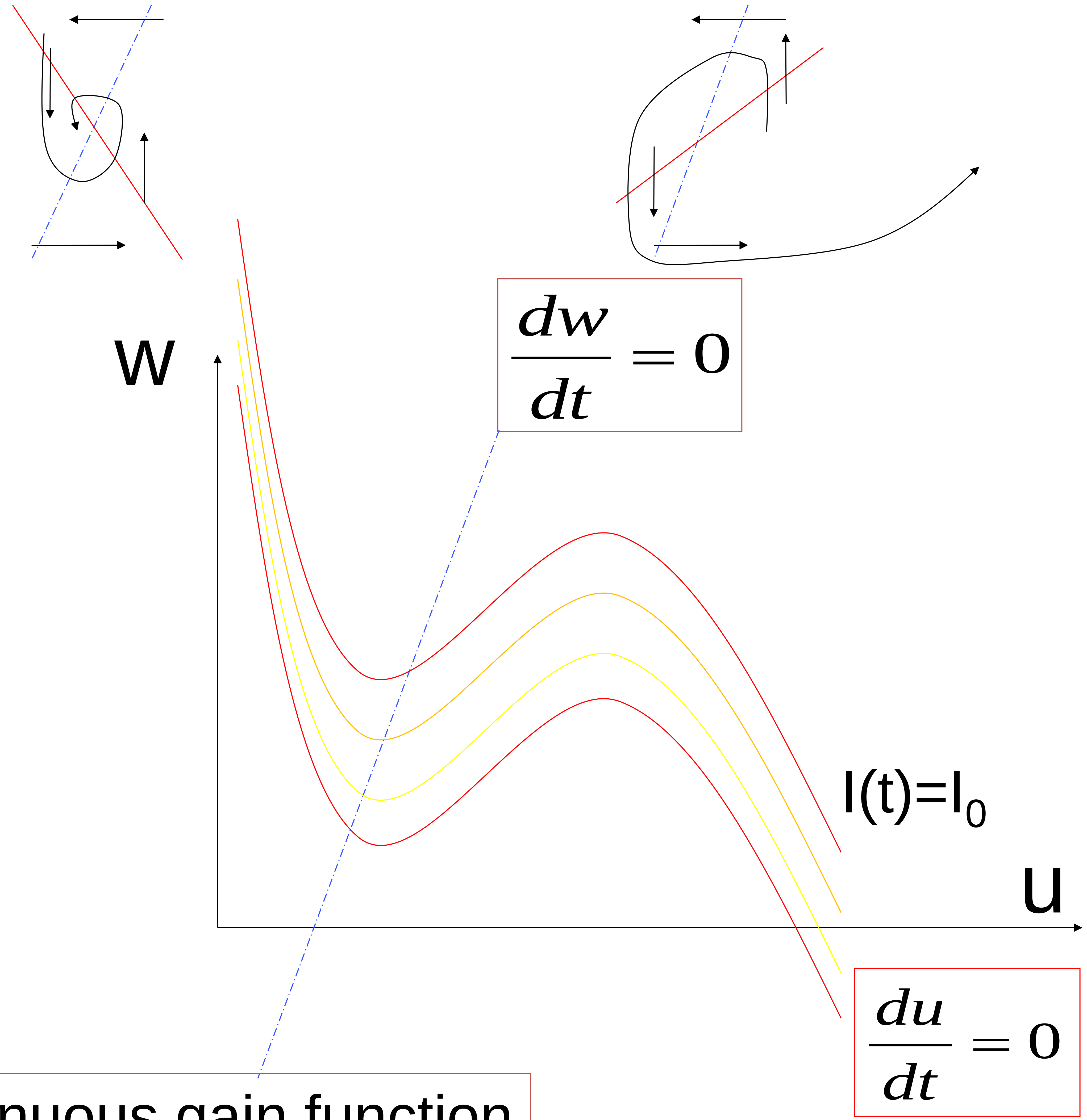
$$\tau \frac{du}{dt} = F(u,w) + I(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

Hopf bifurcation



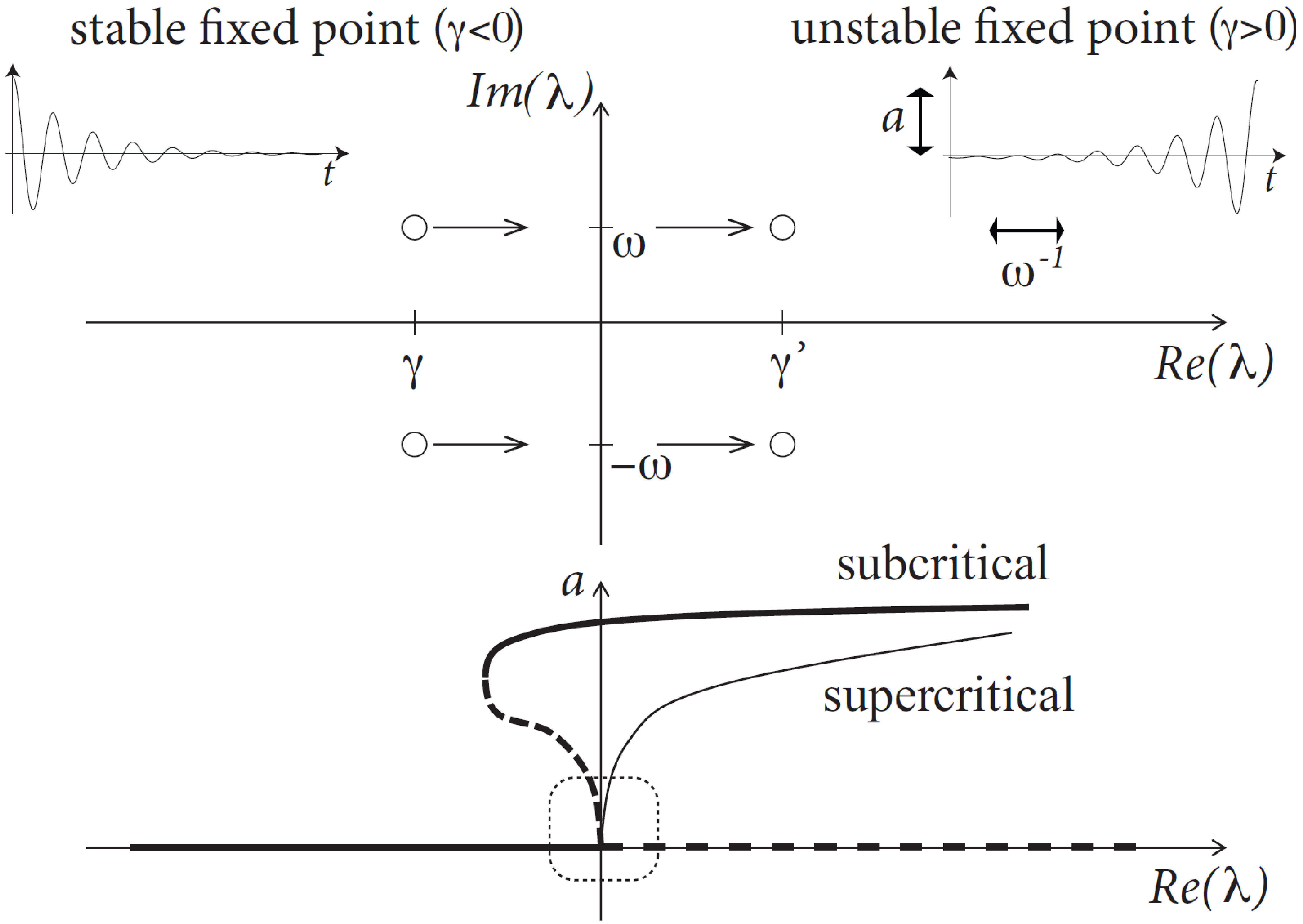
Discontinuous gain function



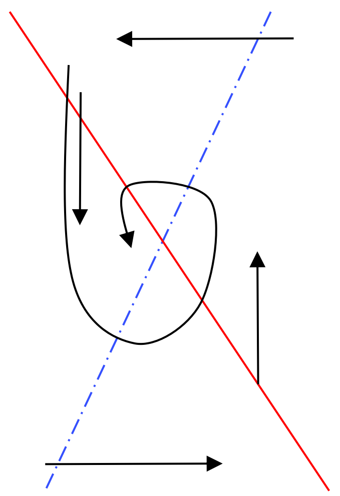
Stability lost → oscillation with finite frequency

Neuronal Dynamics – 4.4. Hopf bifurcation

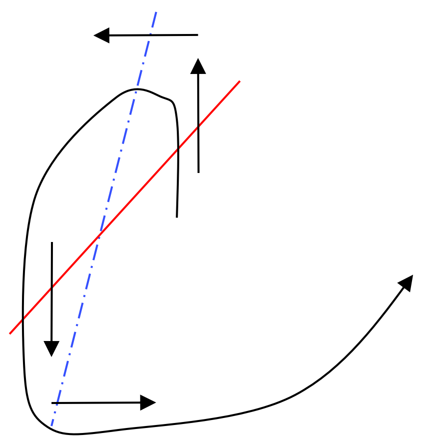
$$\lambda = \gamma + i\omega$$

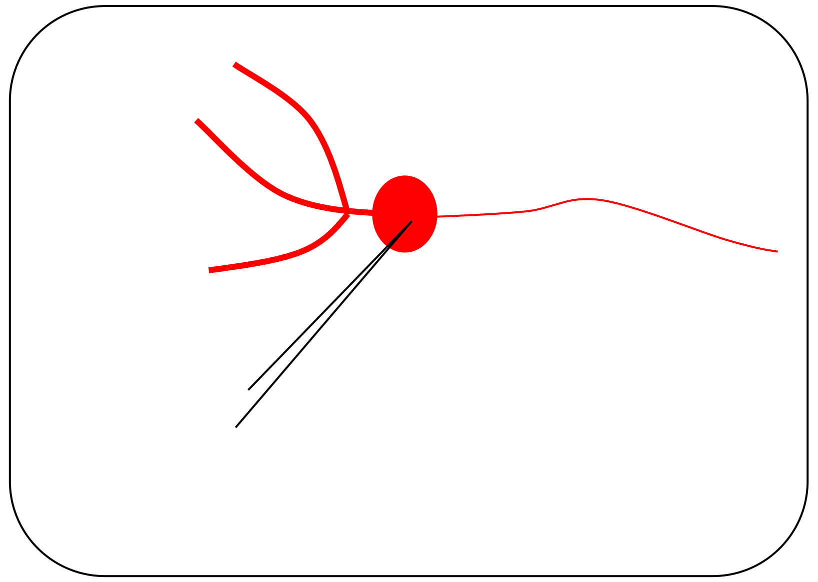


$\gamma < 0$

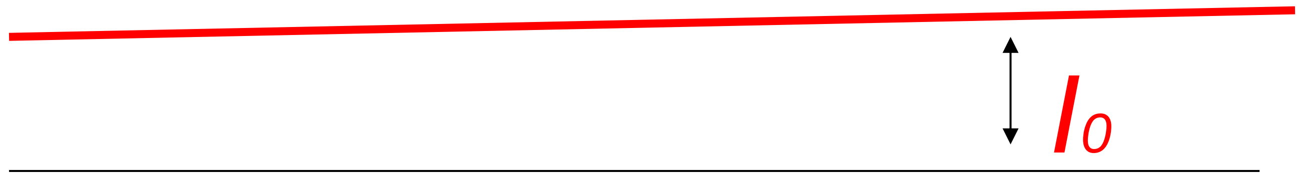


$\gamma > 0$

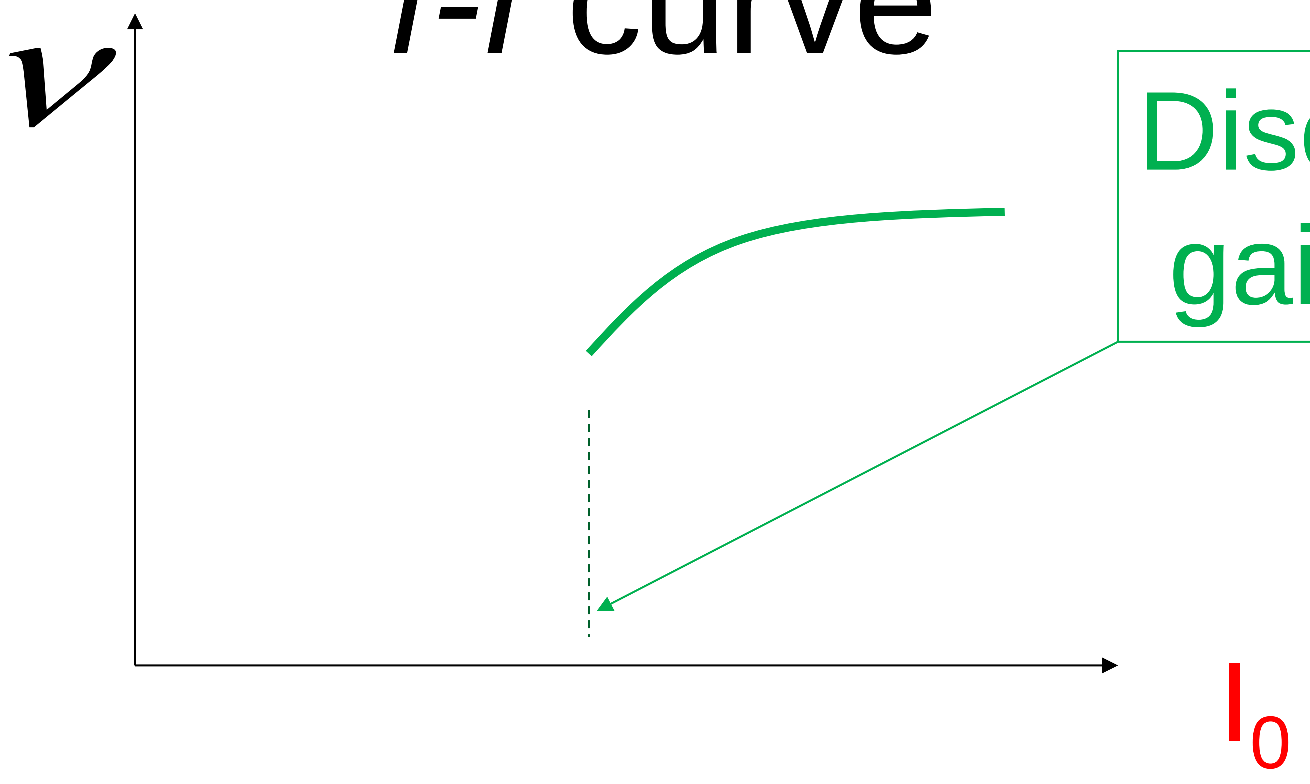




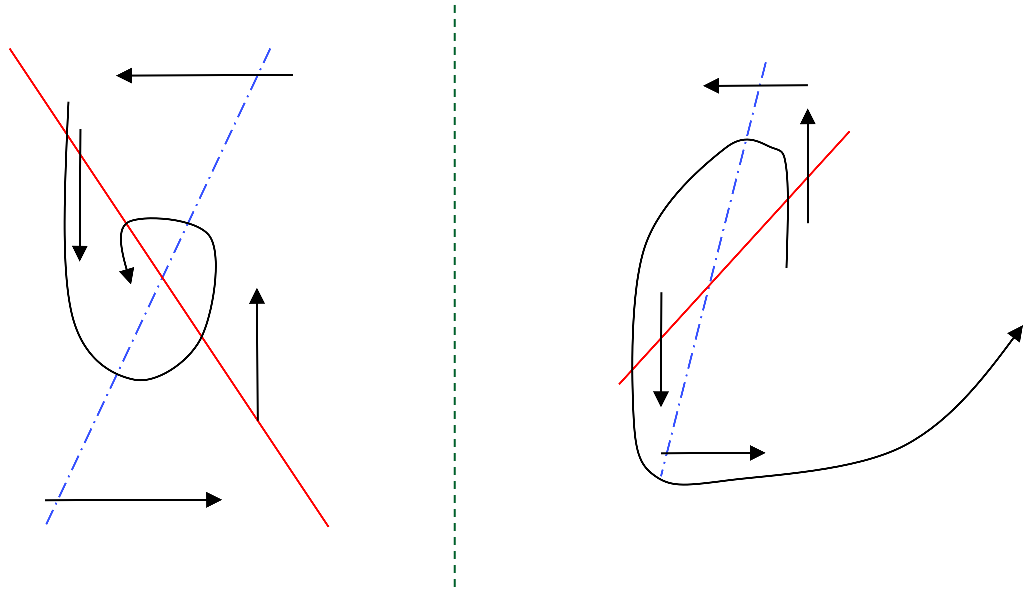
ramp input/
constant input



f-I curve



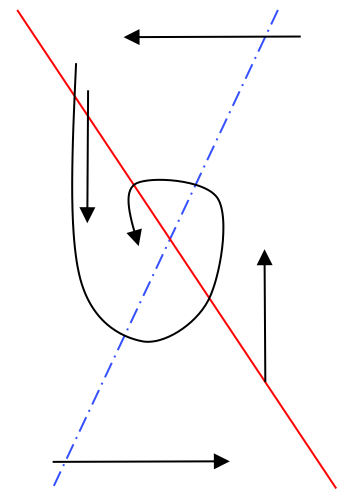
Discontinuous
gain function: Type II

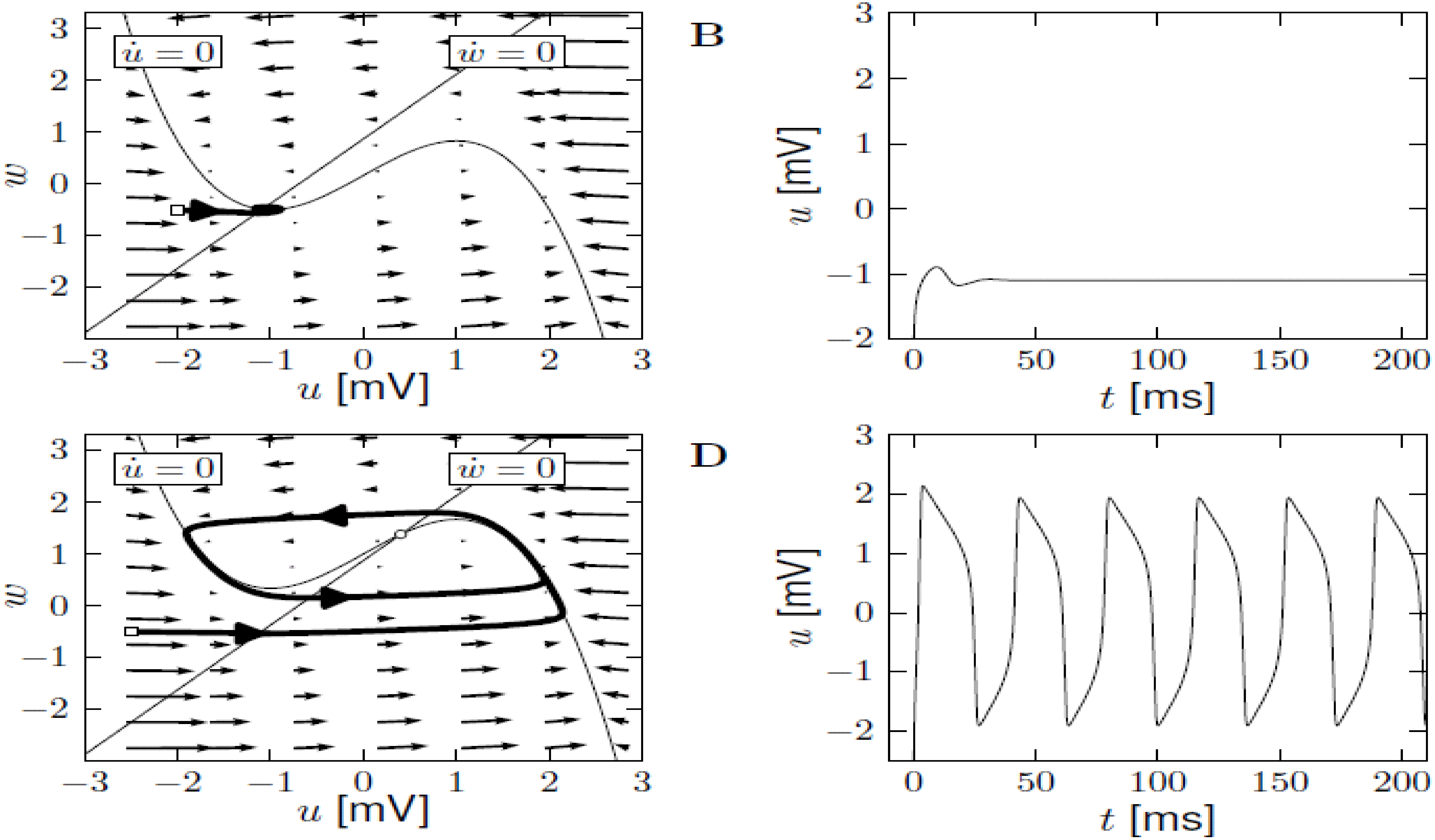


Stability lost → oscillation with finite frequency

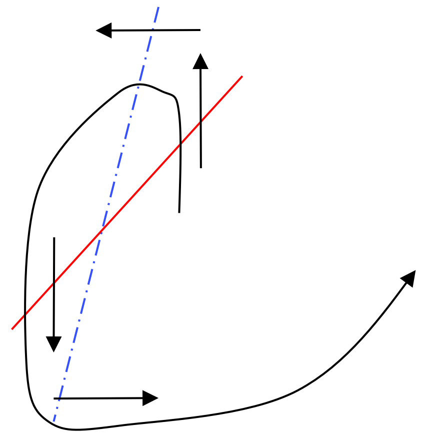
FitzHugh-Nagumo: type II Model – Hopf bifurcation

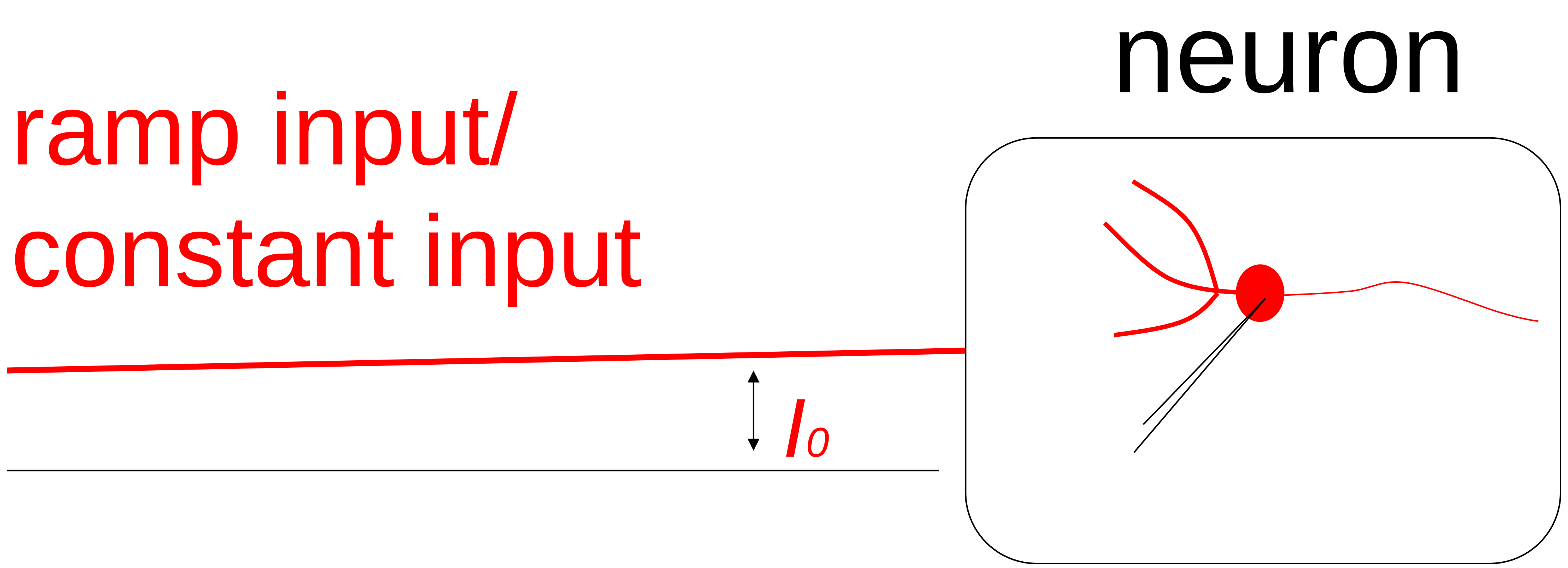
I=0



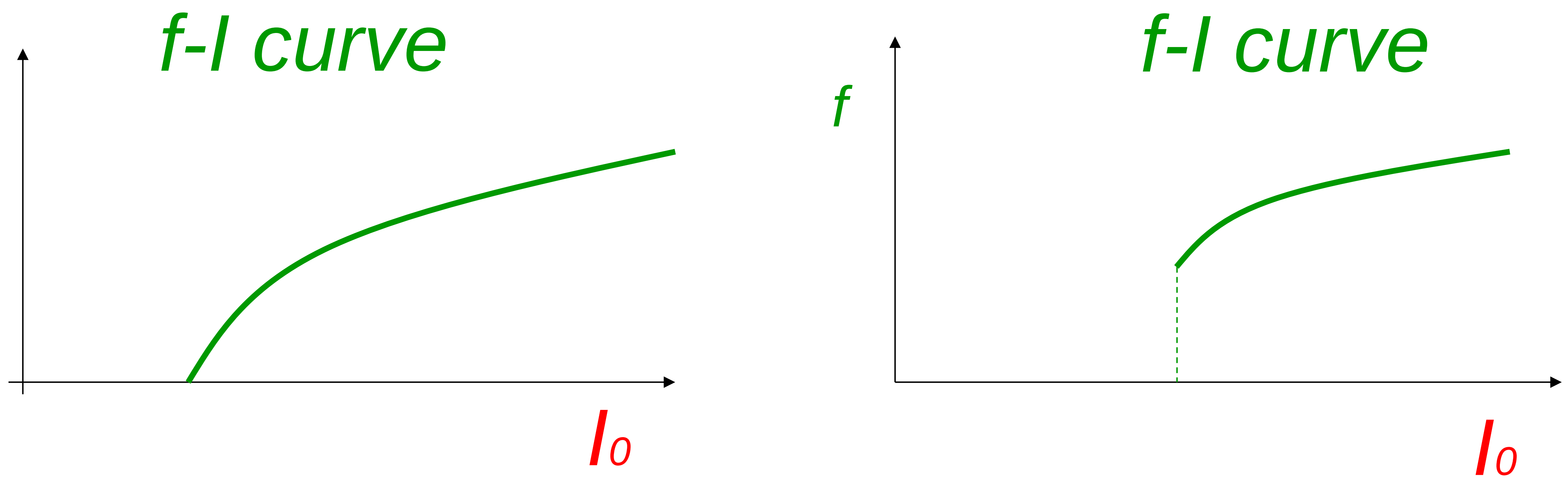


$I > I_c$





Type I and type II models



type I Model: 3 fixed points

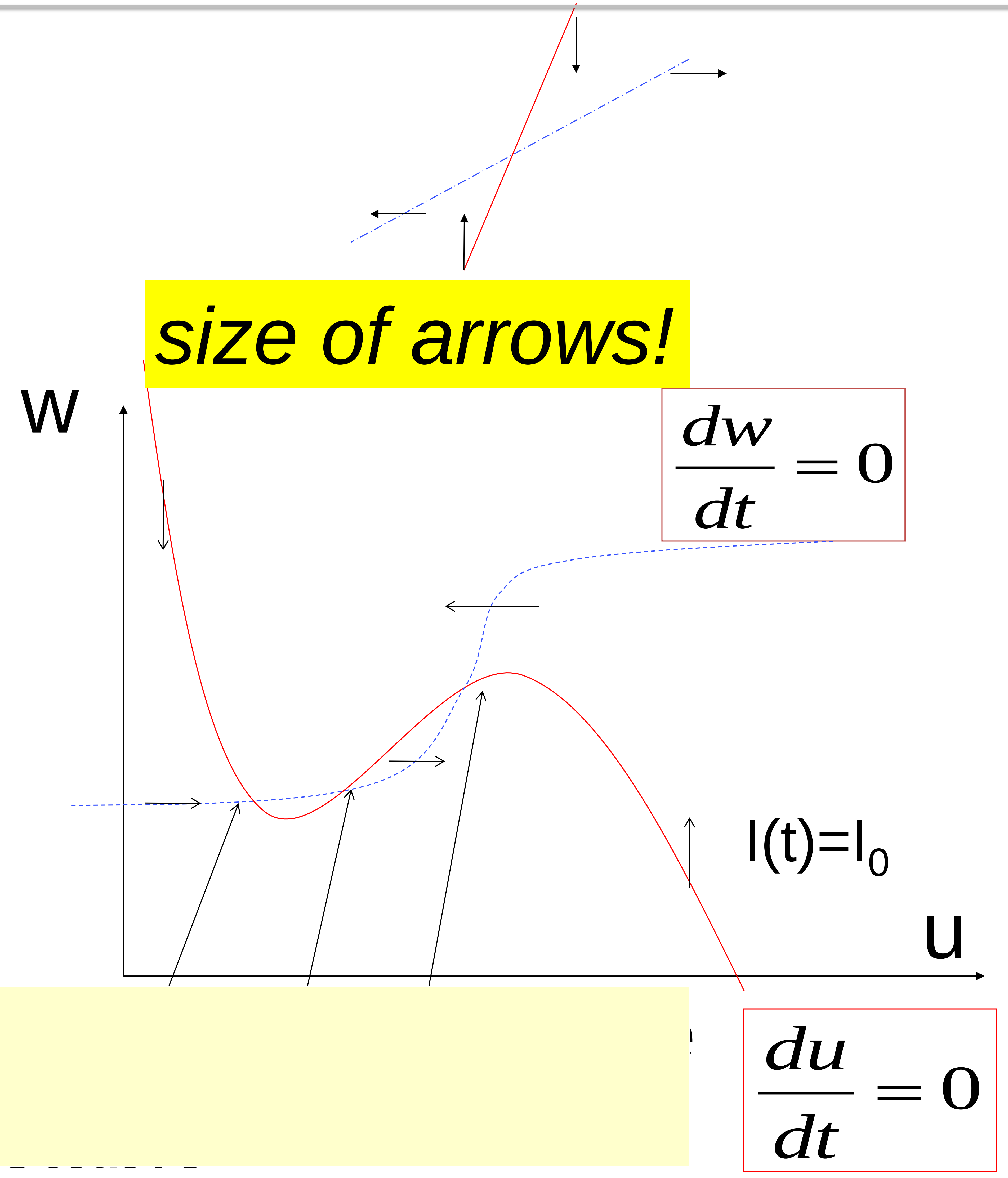
stimulus

$$\tau \frac{du}{dt} = F(u,w) + I(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

apply constant stimulus I_0

Saddle-node bifurcation



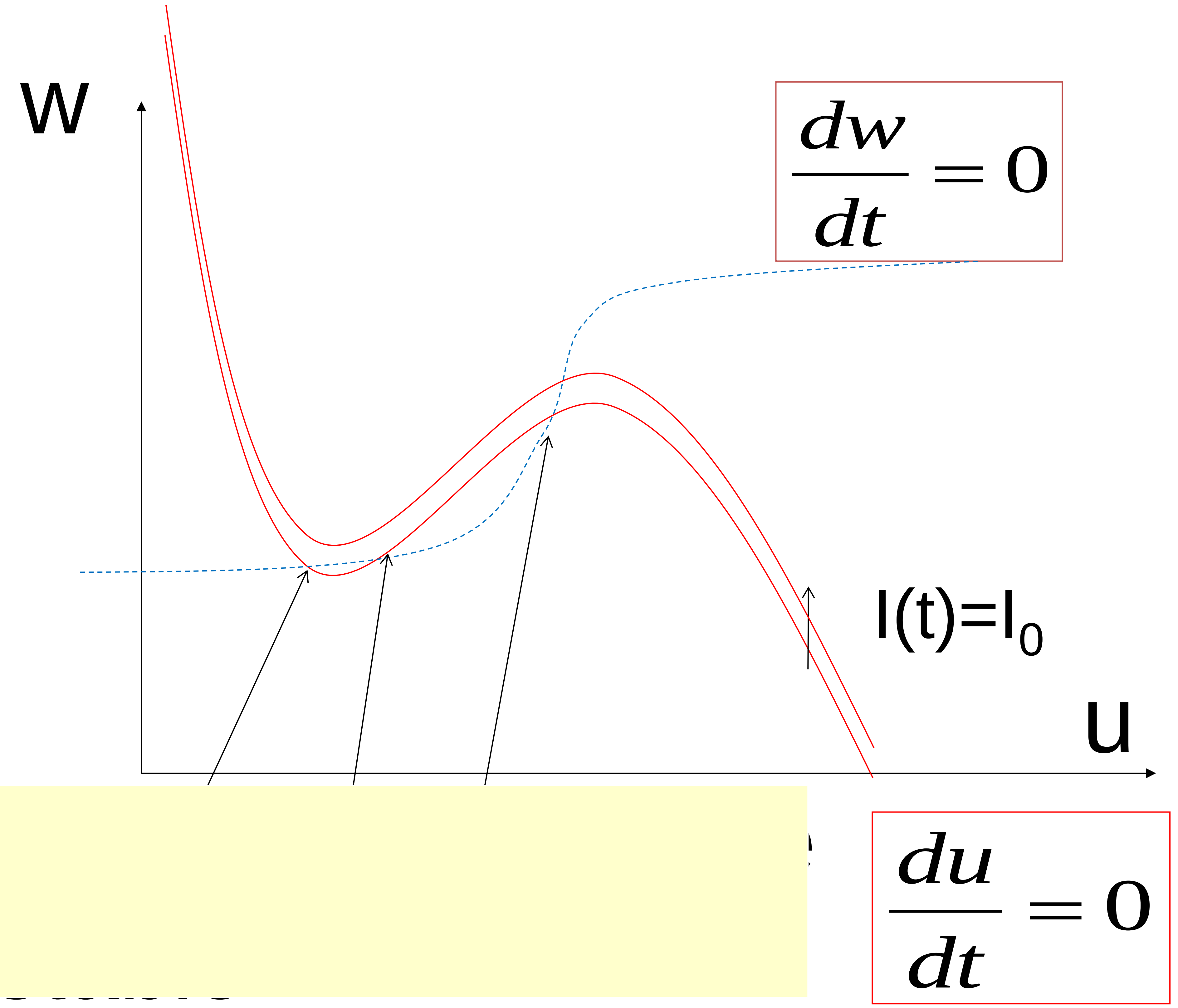
Saddle-node bifurcation

stimulus

$$\tau \frac{du}{dt} = F(u,w) + I(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

flow arrows

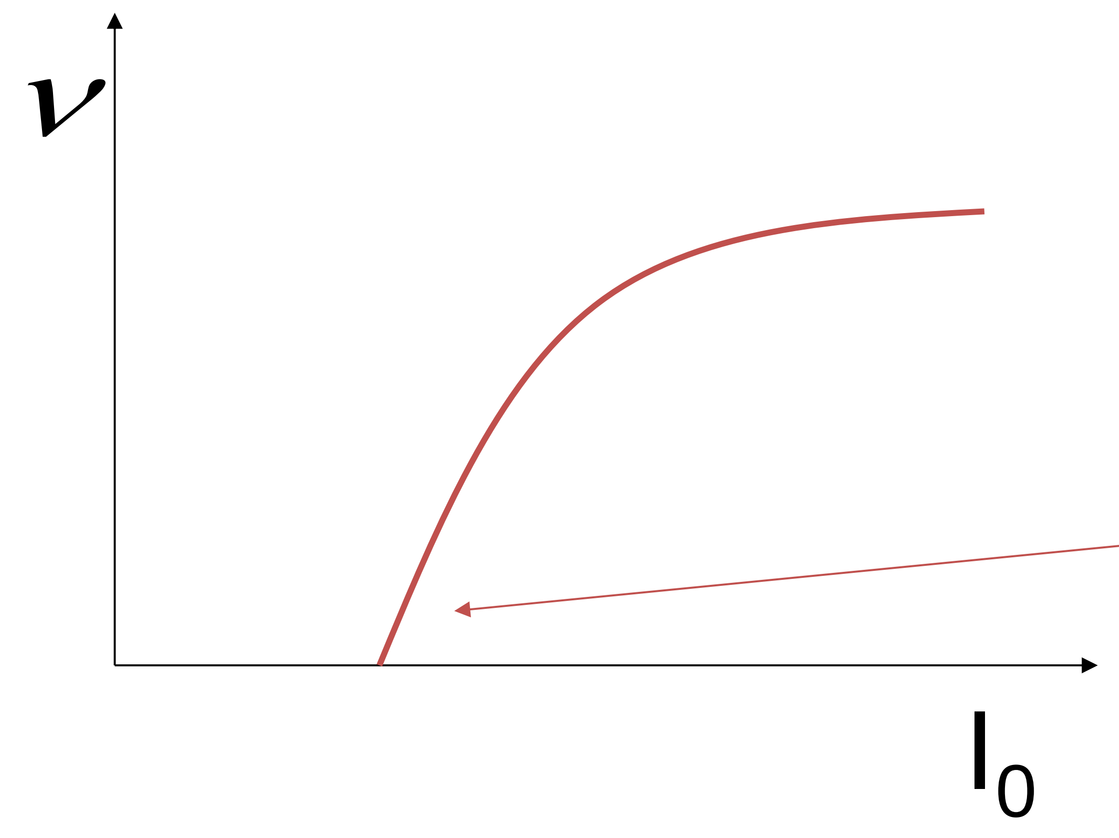


type I Model – constant input

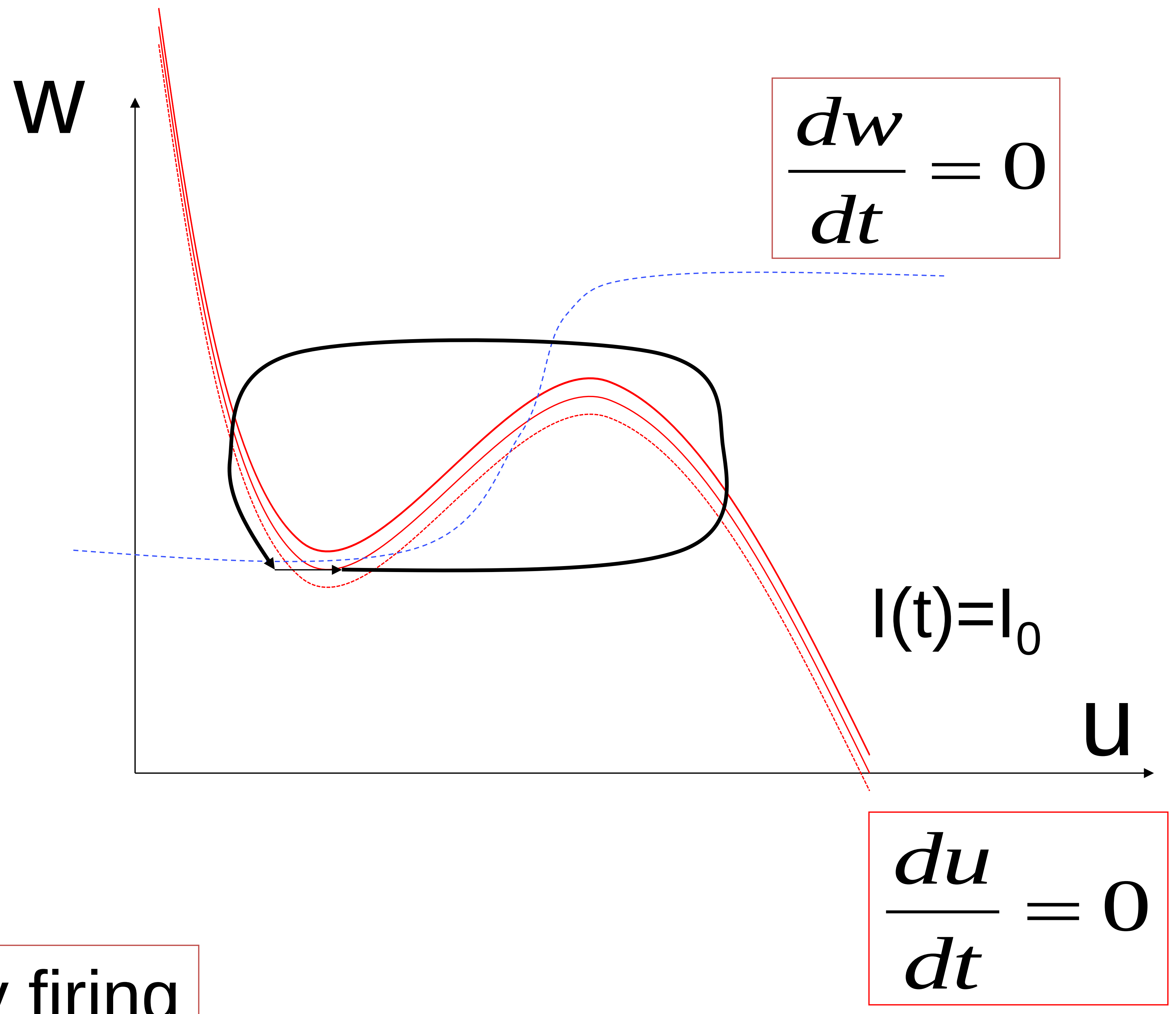
stimulus

$$\tau \frac{du}{dt} = F(u,w) + I(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$



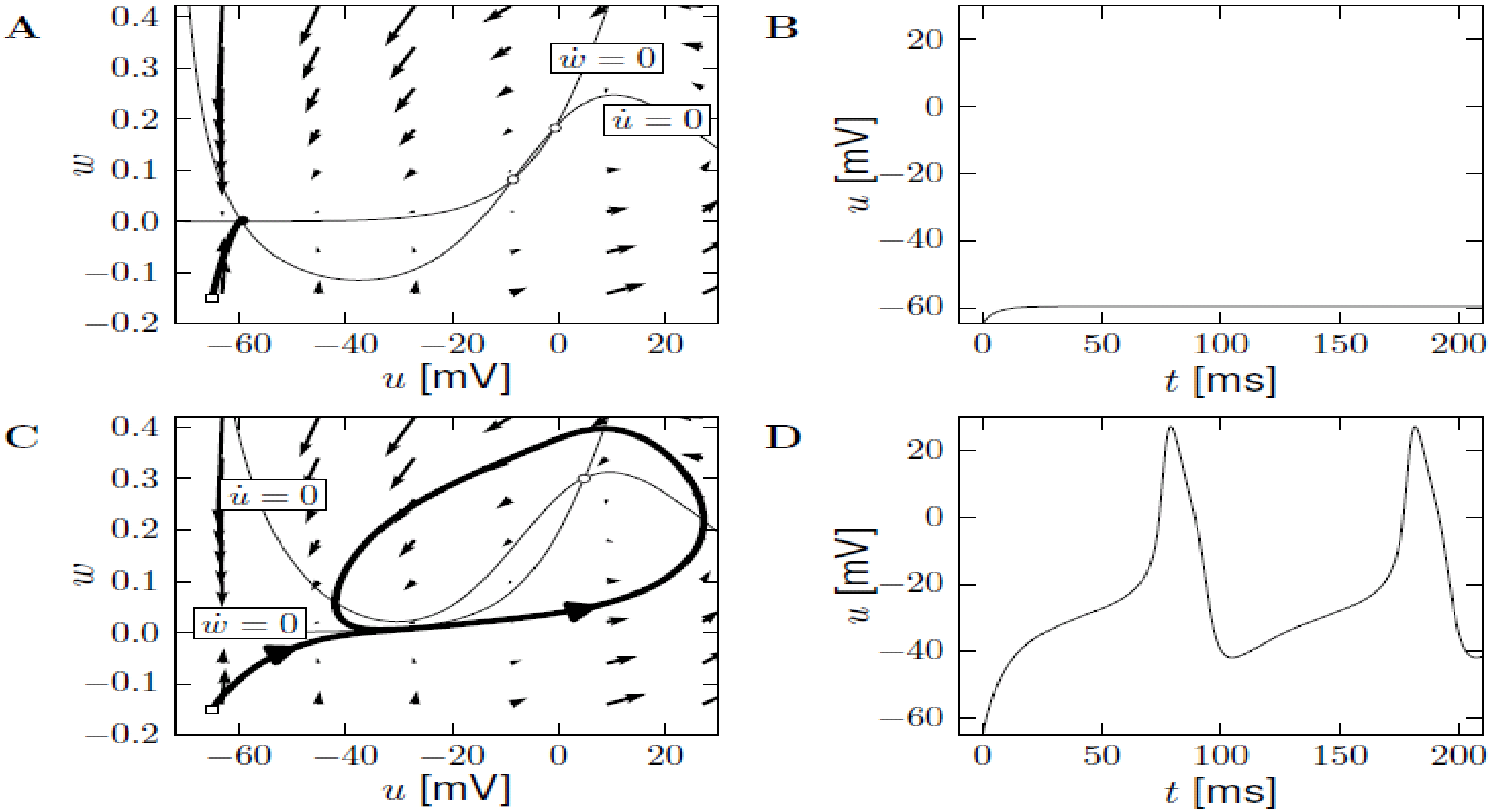
Low-frequency firing



Morris-Lecar, type I Model – constant input

I=0

$I>I_c$



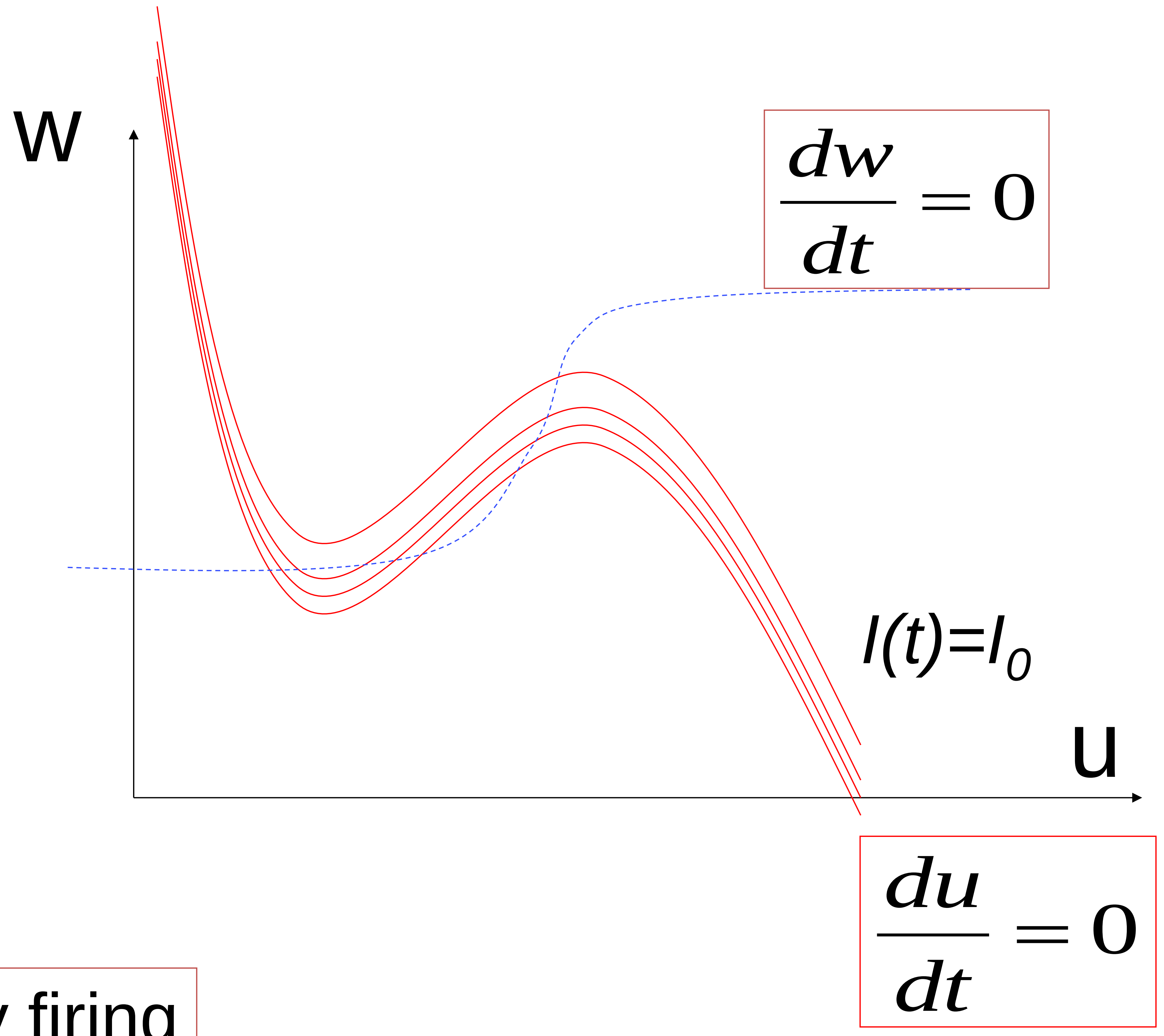
type I Model – constant input

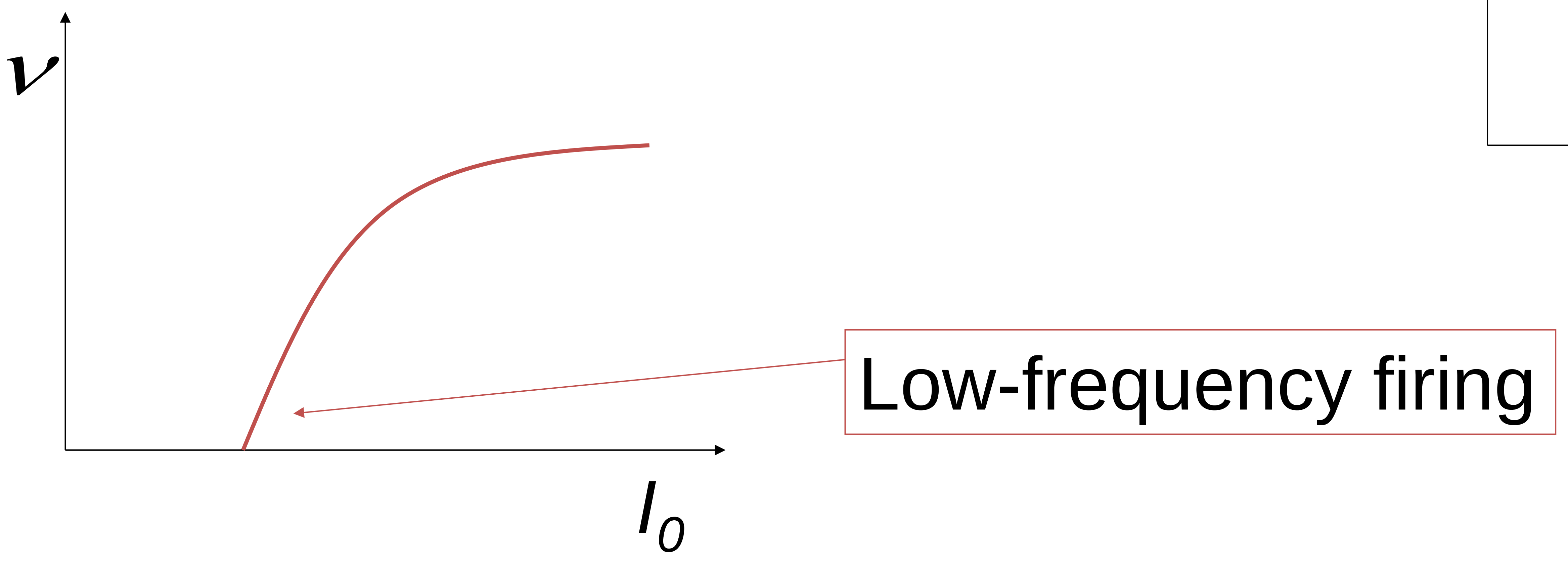
stimulus

$$\tau \frac{du}{dt} = F(u,w) + I(t)$$

$$\frac{dw}{dt} = -\frac{w - w_0(u)}{\tau_{eff}(u)}$$

$$w_0(u) = 0.5[1 + \tanh(\tfrac{u-\theta}{d})]$$





Type I and type II models

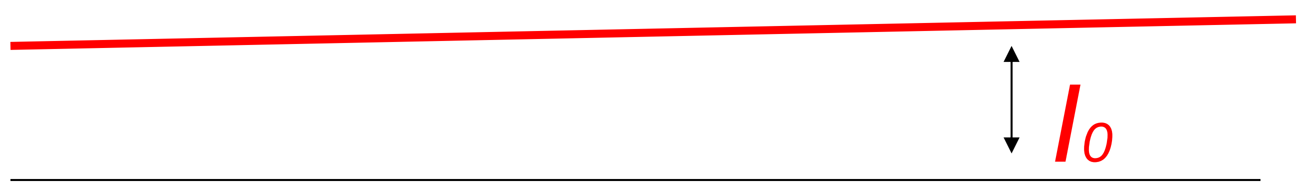
Response at firing threshold?

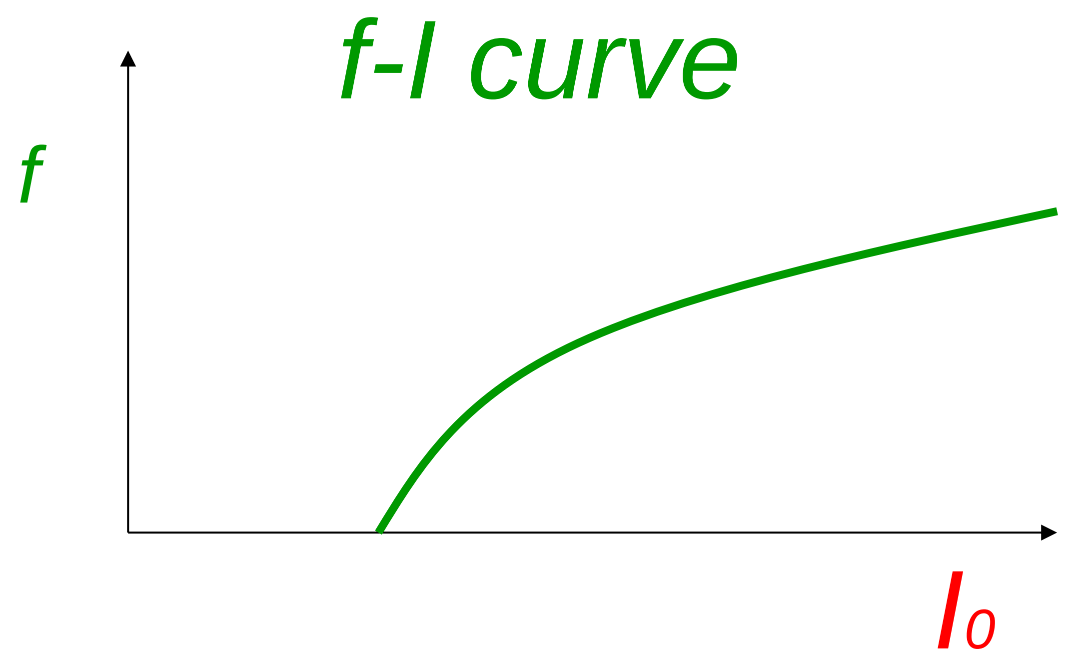
Type I type II

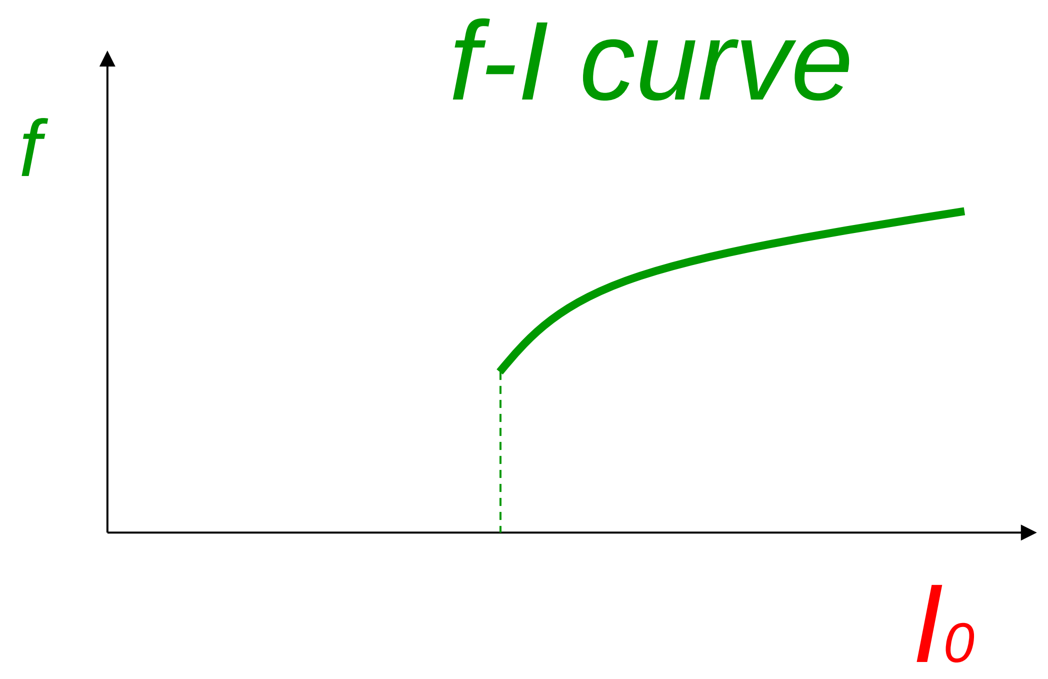
Saddle-Node
Onto limit cycle

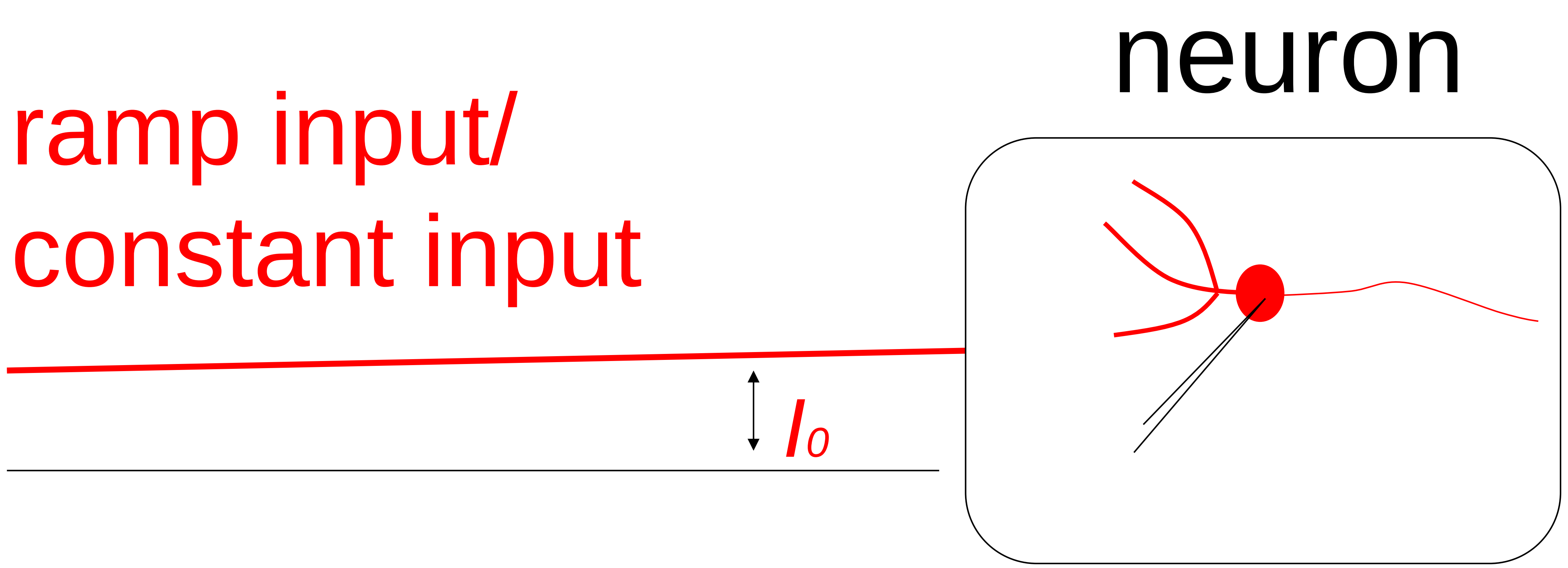
For example:
Subcritical Hopf

ramp input/
constant input

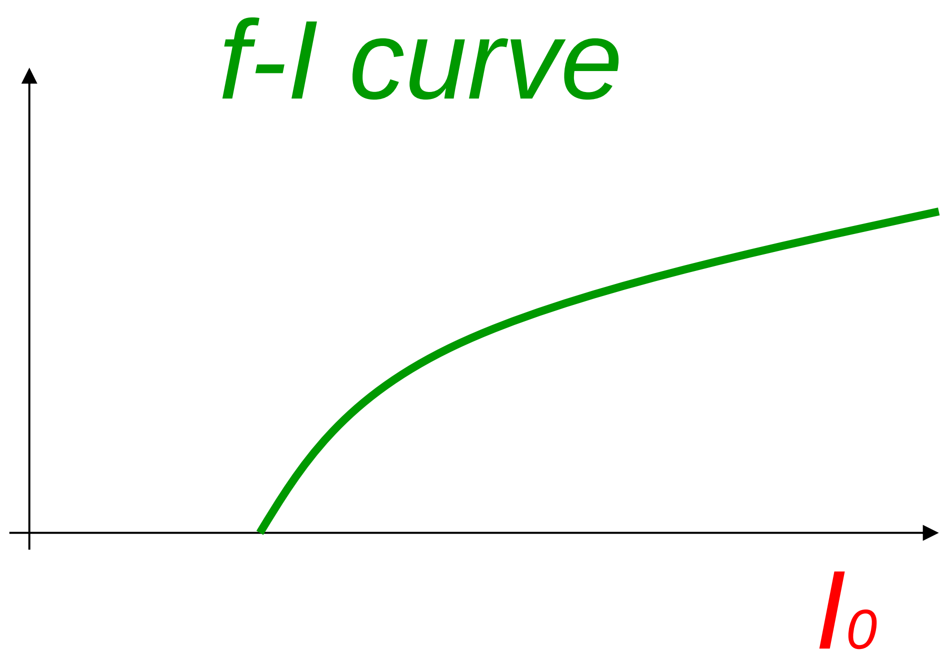


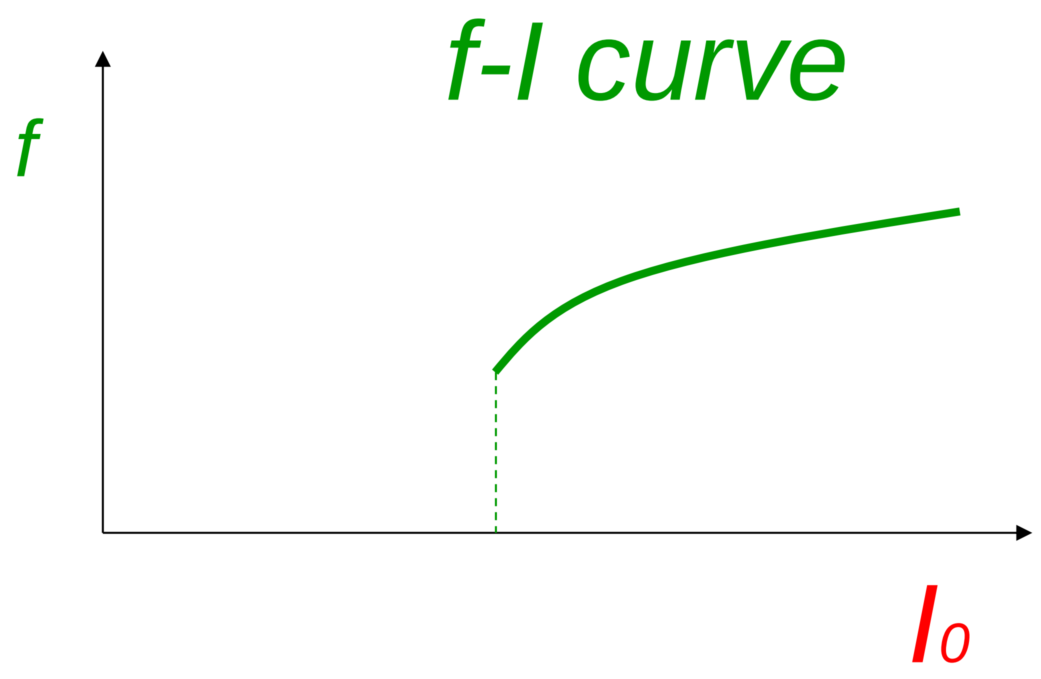






Type I and type II models





Week 4 – part 4b : Firing threshold in 2D models

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Neuronal Dynamics:
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√4.1 **From Hodgkin-Huxley to 2D**

√4.2 **Phase Plane Analysis**

√4.3 **Analysis of a 2D Neuron Model**

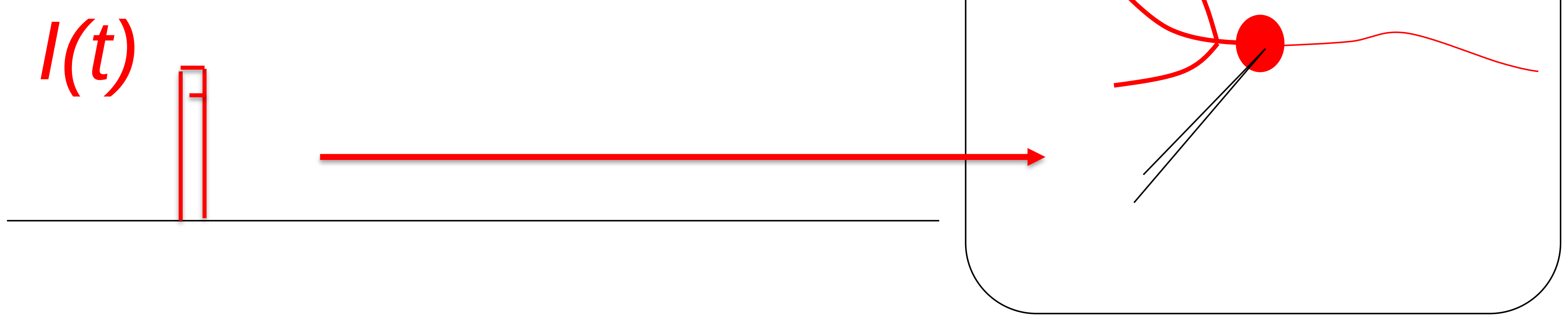
√4.4 **Type I and II Neuron Models**
- where is the firing threshold?

4.5. **Nonlinear Integrate-and-fire**
- from two to one dimension

Neuronal Dynamics – 4.4b Threshold in 2dim. Neuron Models

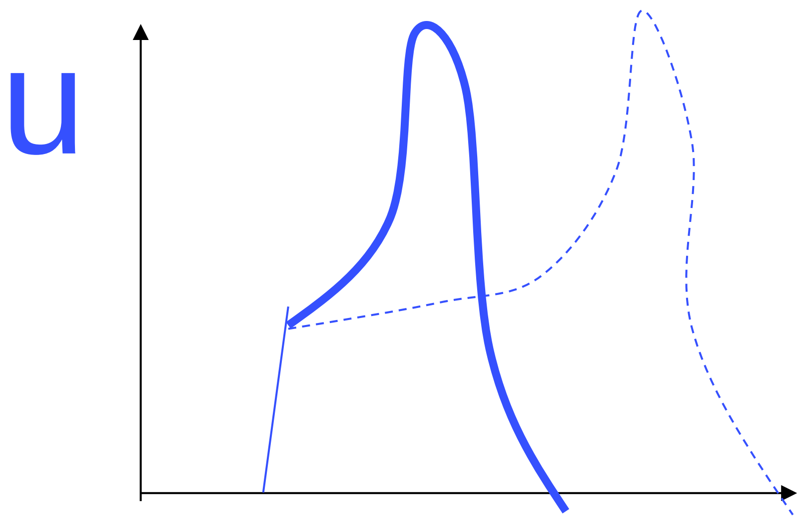
pulse input

neuron



Delayed spike

Reduced amplitude



Bifurcations in neural modeling,
Type I/II neuron models,
Canonical simplified models

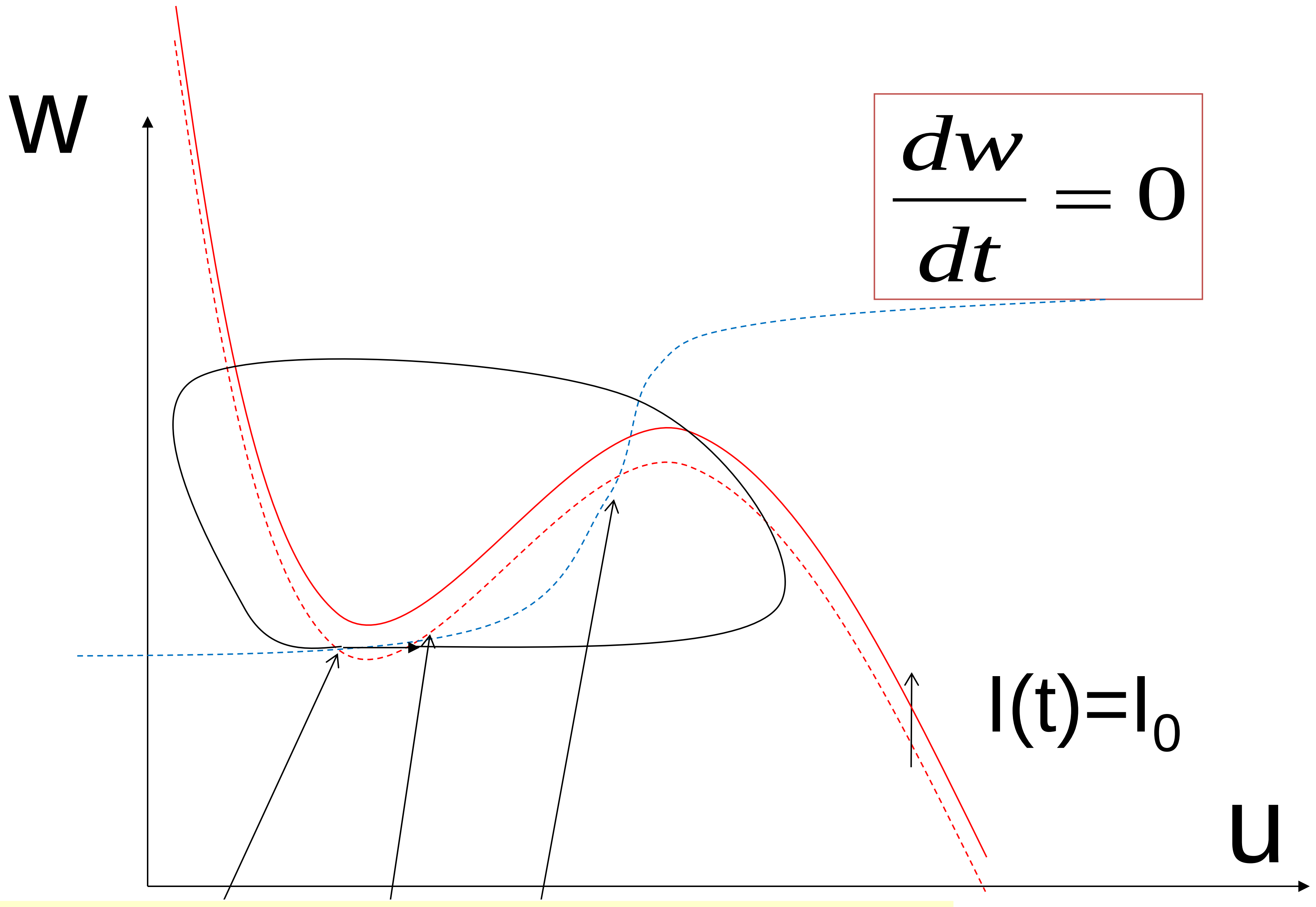
Nancy Koppell,
Bart Ermentrout,
John Rinzel,
Eugene Izhikevich
and many others

Saddle-node onto limit cycle bifurcation

stimulus

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$



$$\frac{dw}{dt} = 0$$

$$\frac{du}{dt} = 0$$

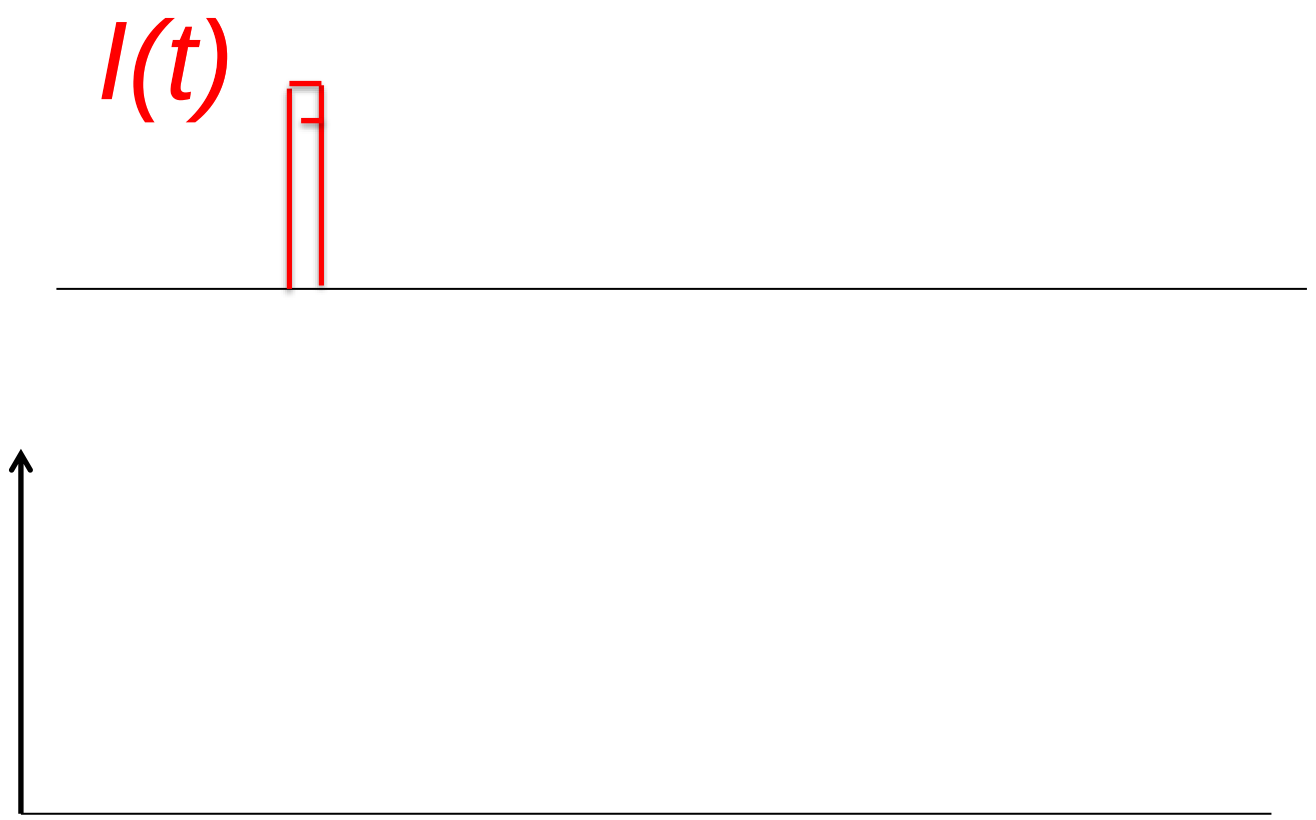
Neuronal Dynamics – 4.4b Pulse input

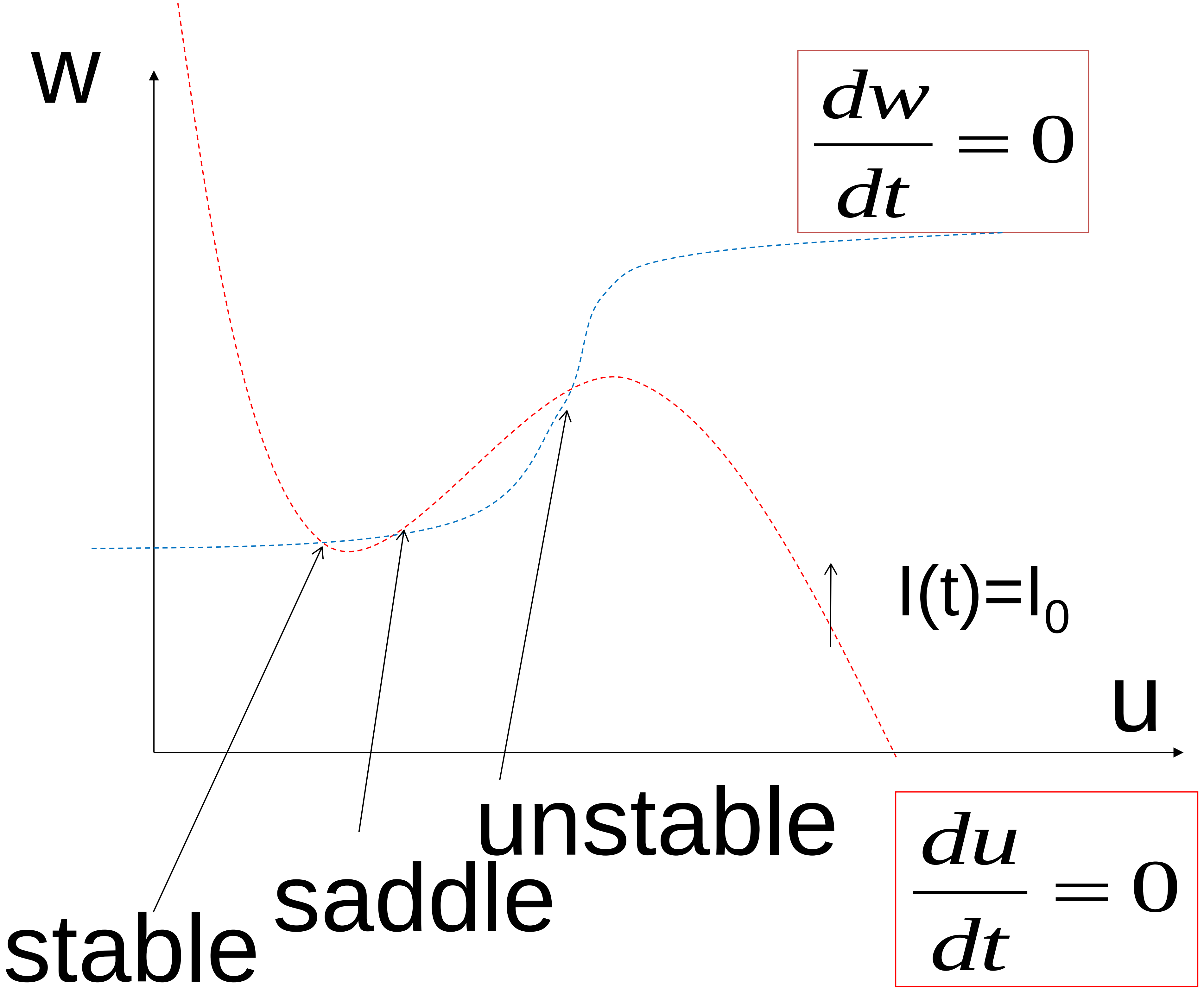
stimulus

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

pulse input





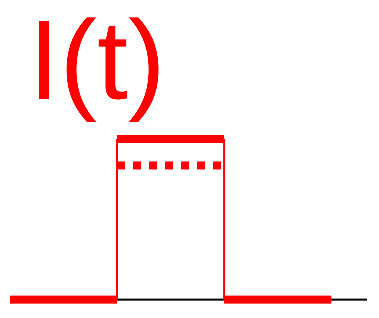
4.4b Type I model: Pulse input

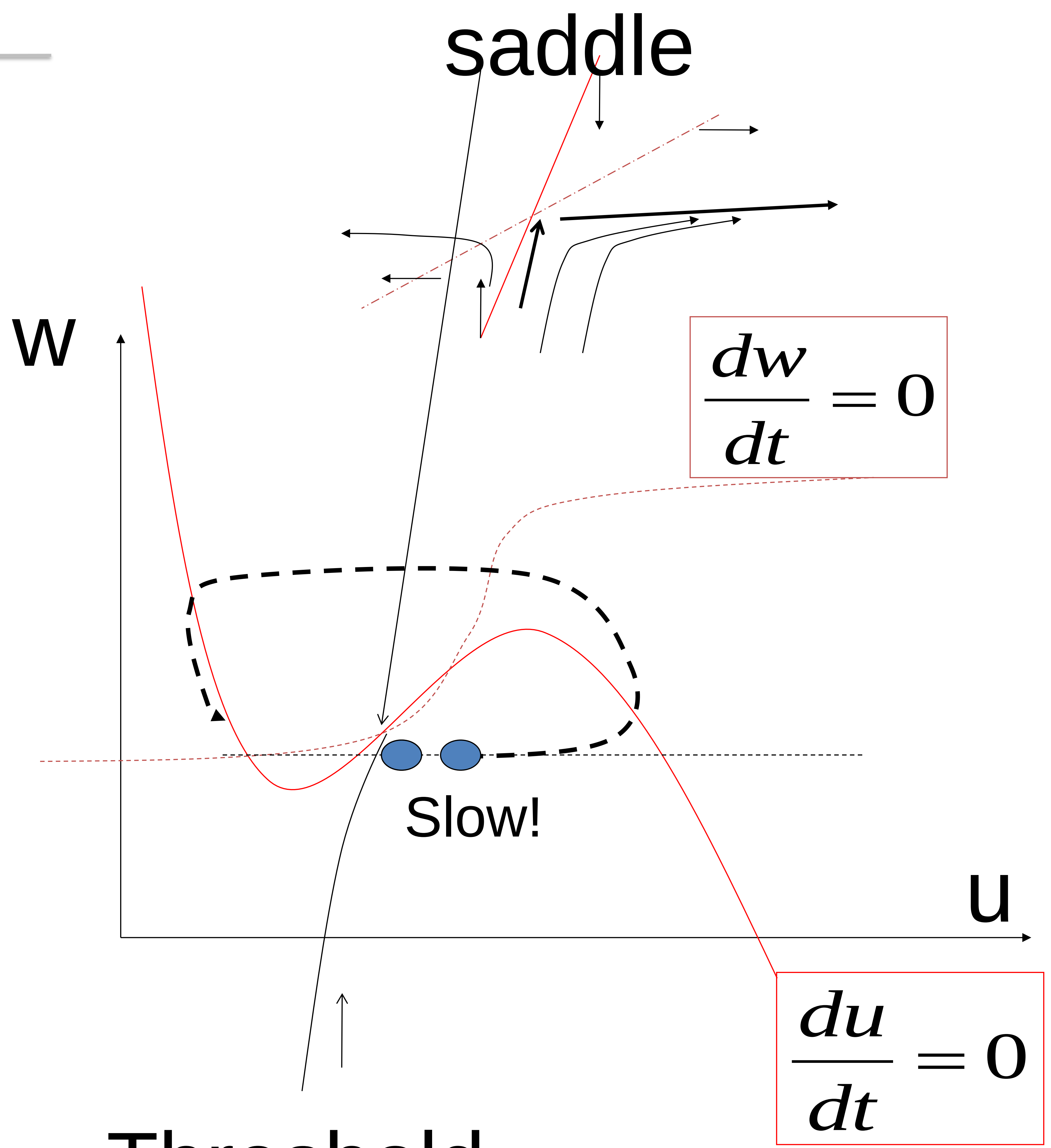
stimulus

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

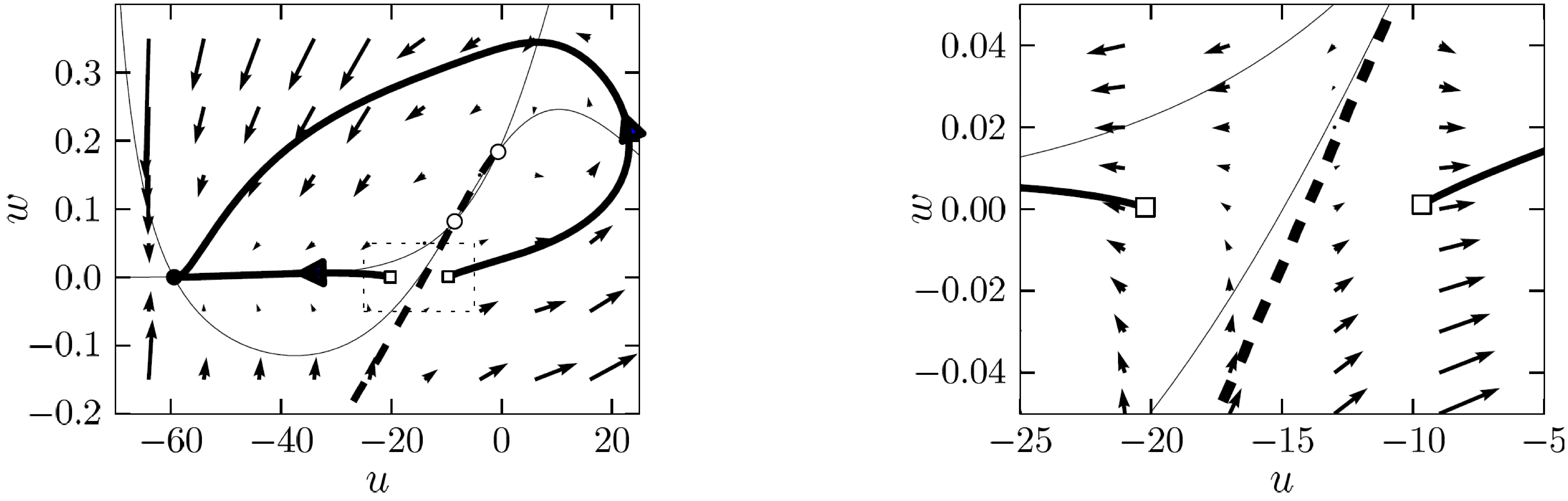
pulse input





Threshold
for pulse input

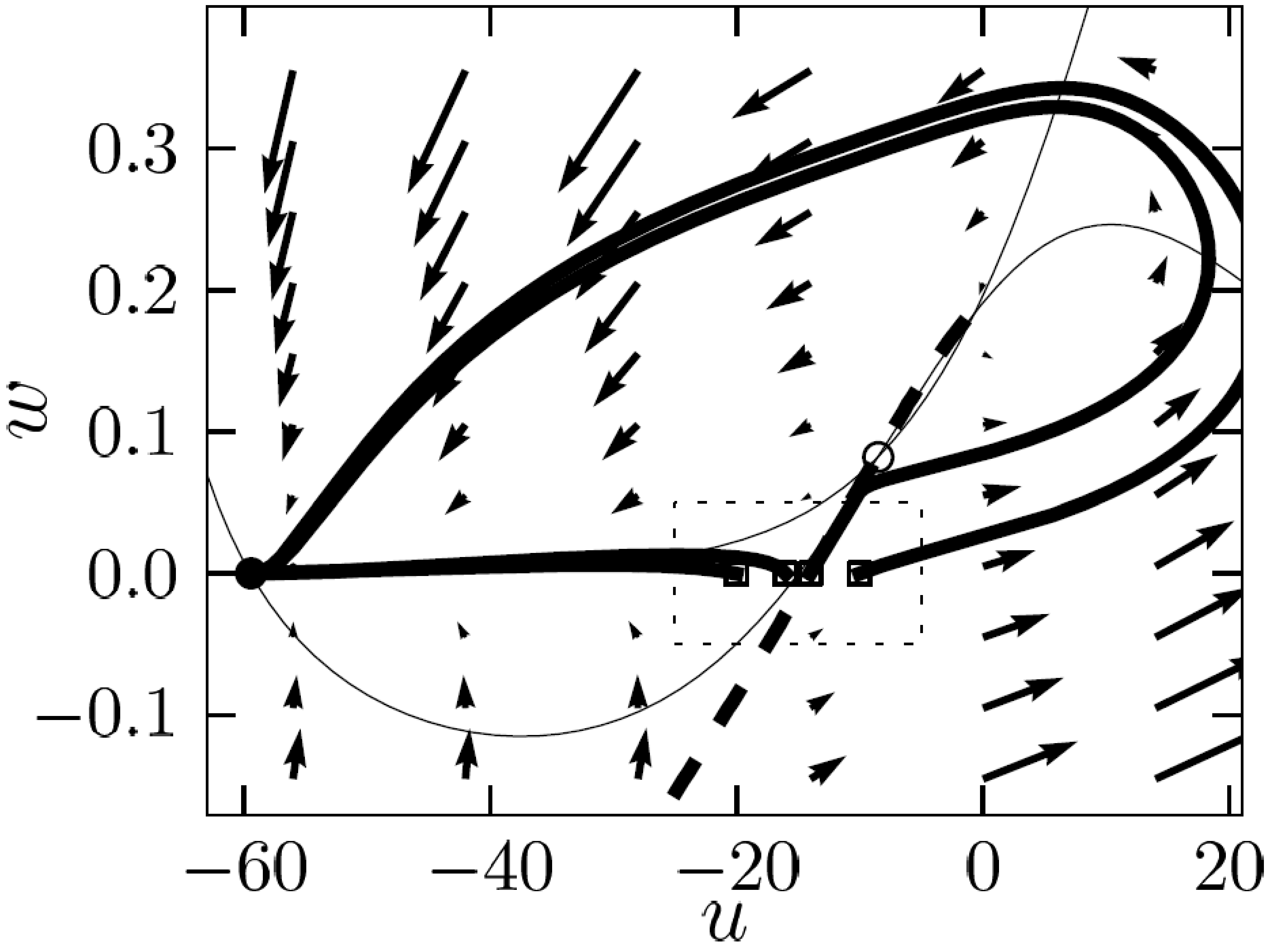
4.4b Type I model: Threshold for Pulse input

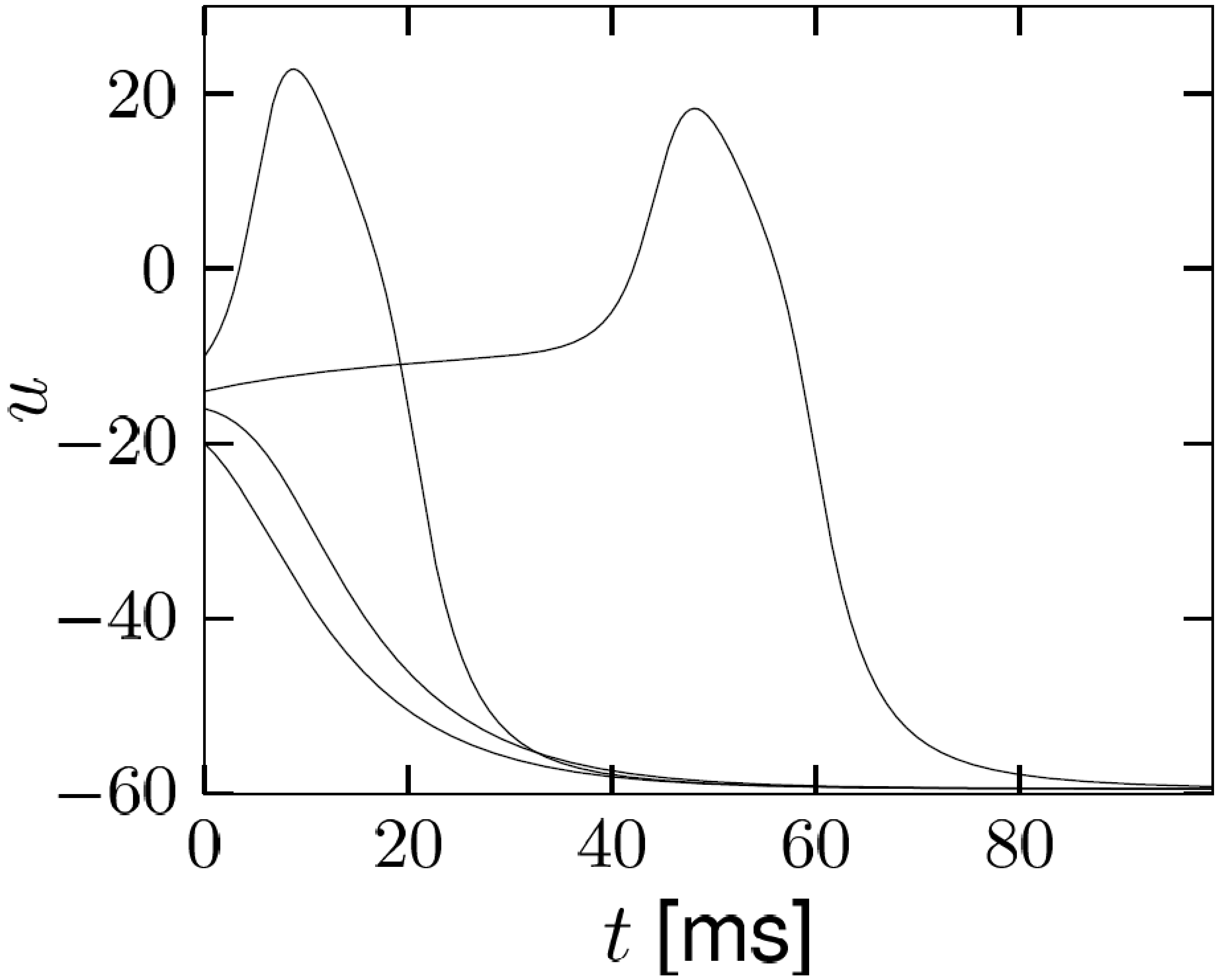


Stable manifold plays role of ‘Threshold’ (for pulse input)

Image: Neuronal Dynamics, Gerstner et al., Cambridge Univ. Press (2014)

4.4b Type I model: Delayed spike initation for Pulse input

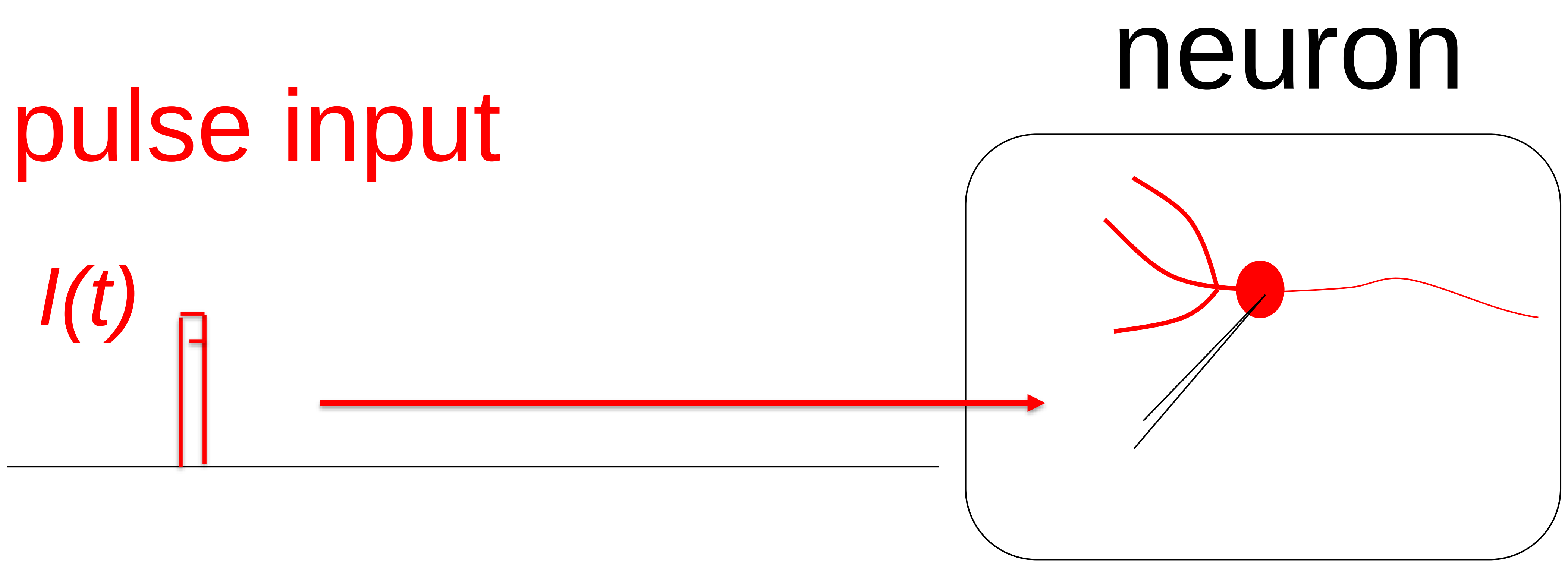




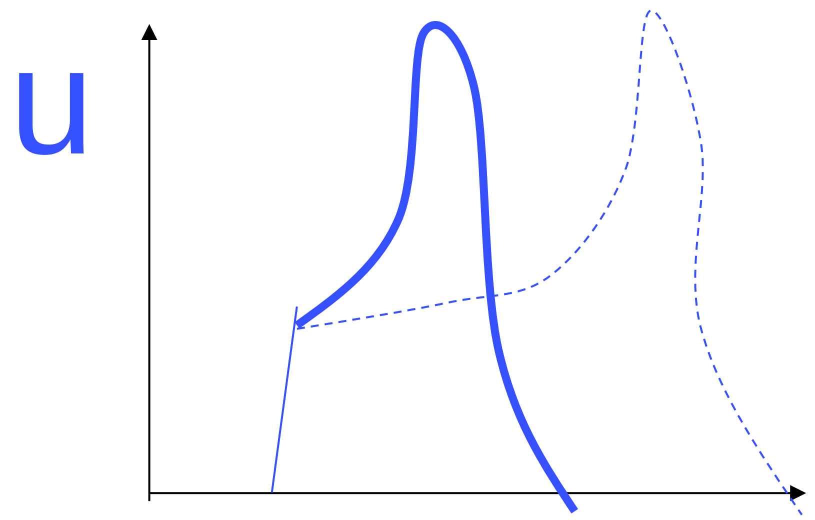
Delayed spike initiation close to 'Threshold' (for pulse input)

Image: Neuronal Dynamics, Gerstner et al., Cambridge Univ. Press (2014)

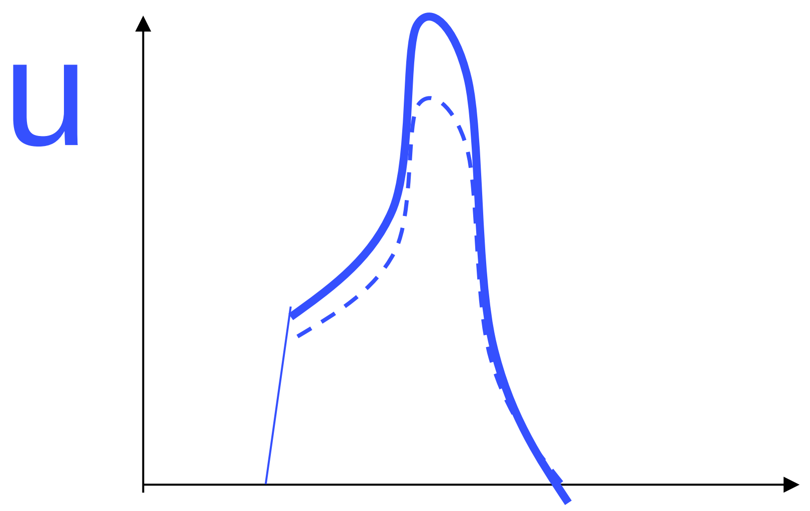
Neuronal Dynamics – 4.4b Threshold in 2dim. Neuron Models



Delayed spike



Reduced amplitude



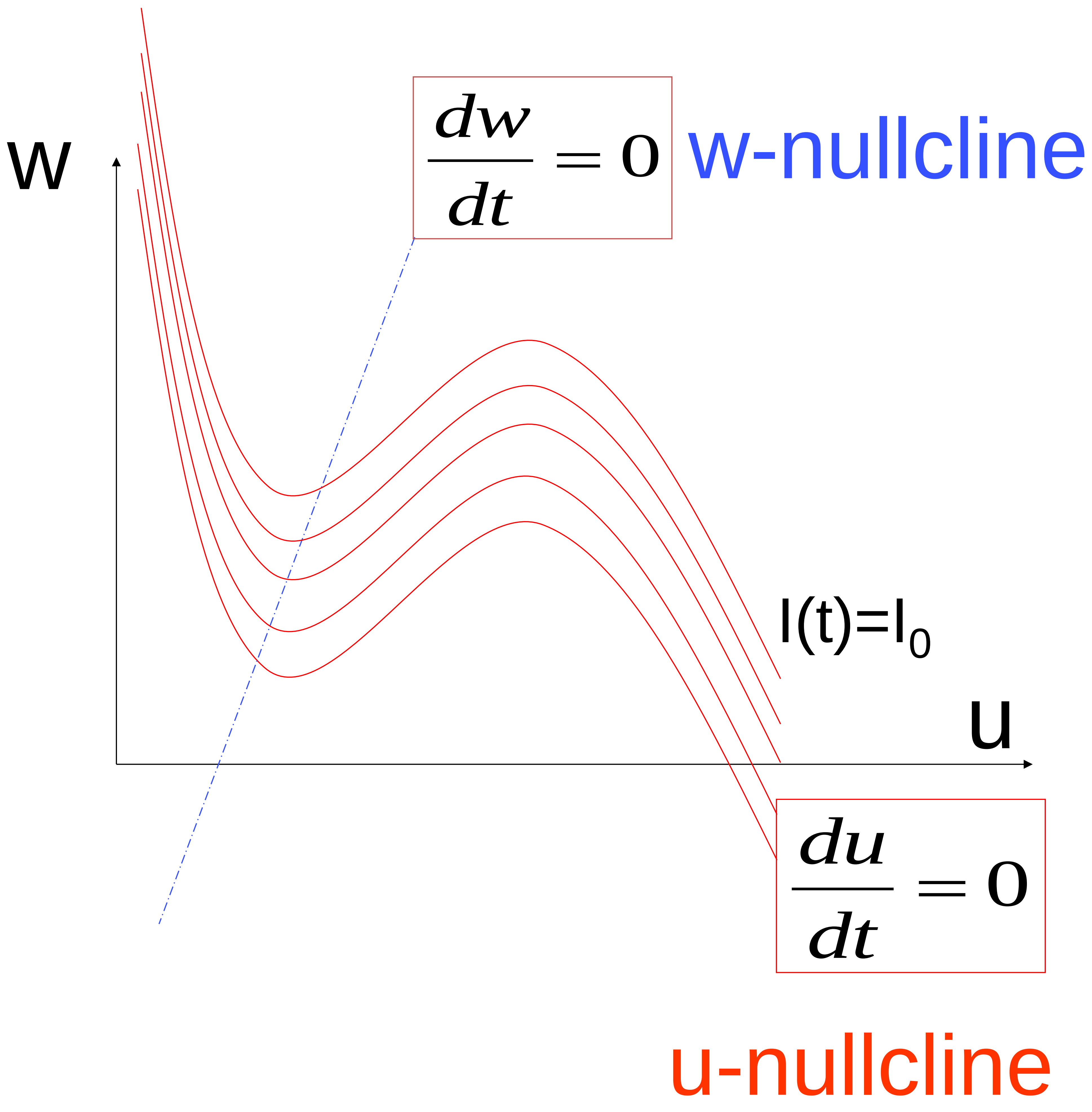
FitzHugh-Nagumo Model: Hopf bifurcation

stimulus

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

apply constant stimulus I_0



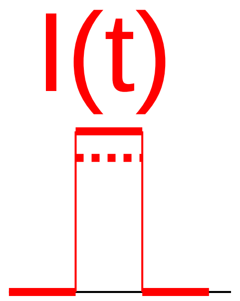
FitzHugh-Nagumo Model - pulse input

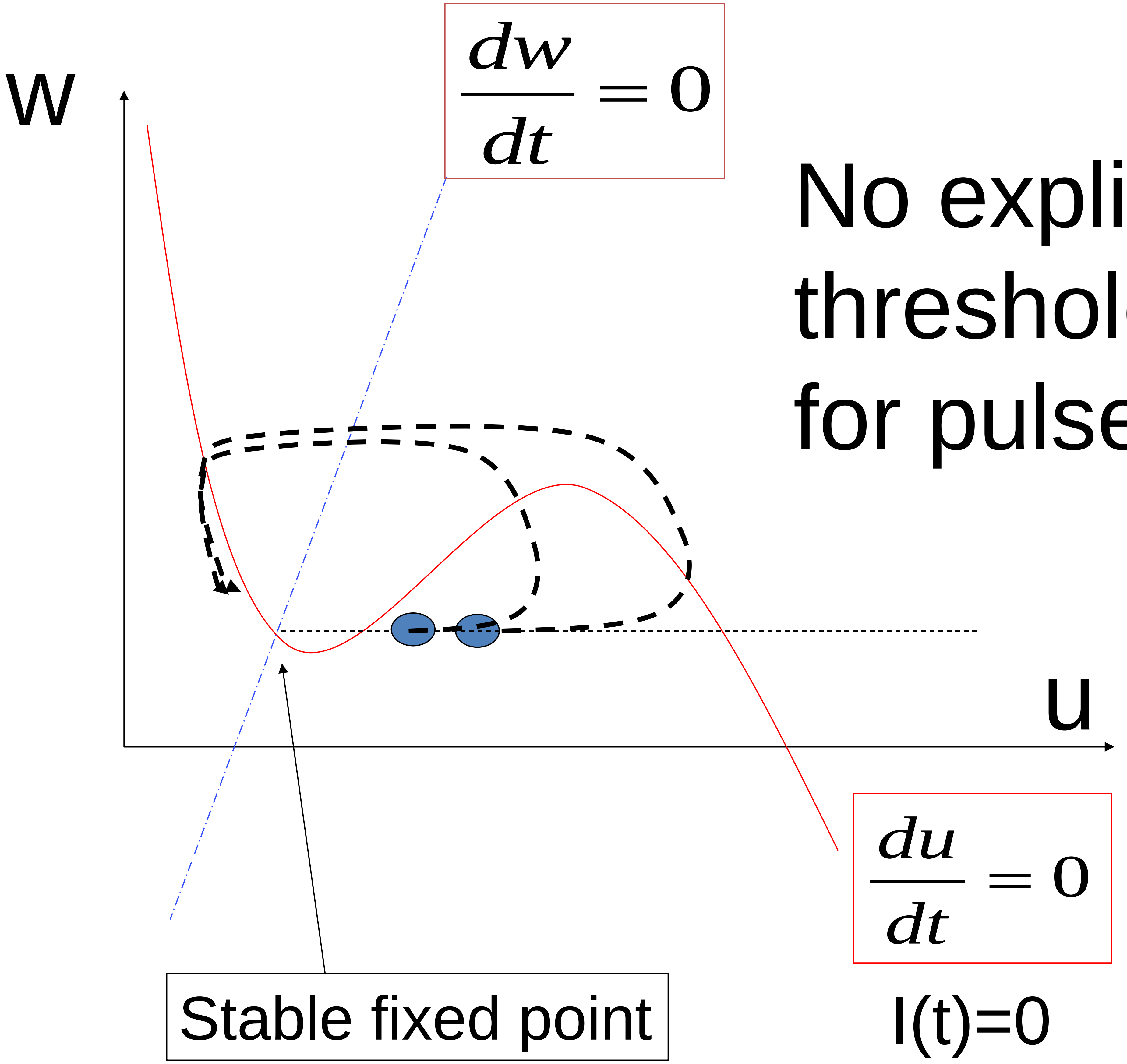
stimulus

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

pulse input





No explicit threshold for pulse input

FitzHugh-Nagumo Model - pulse input threshold?

stimulus

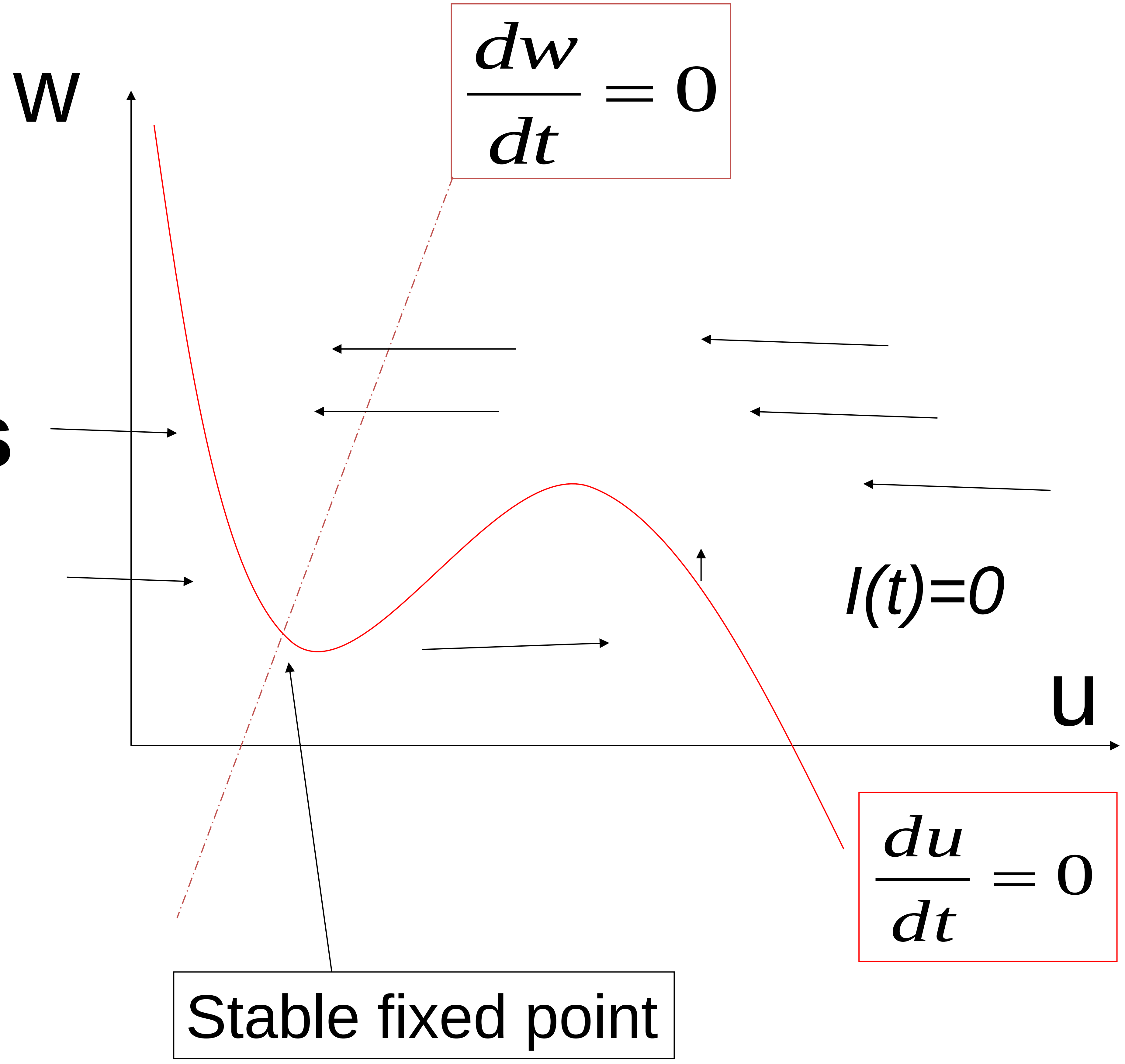
$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

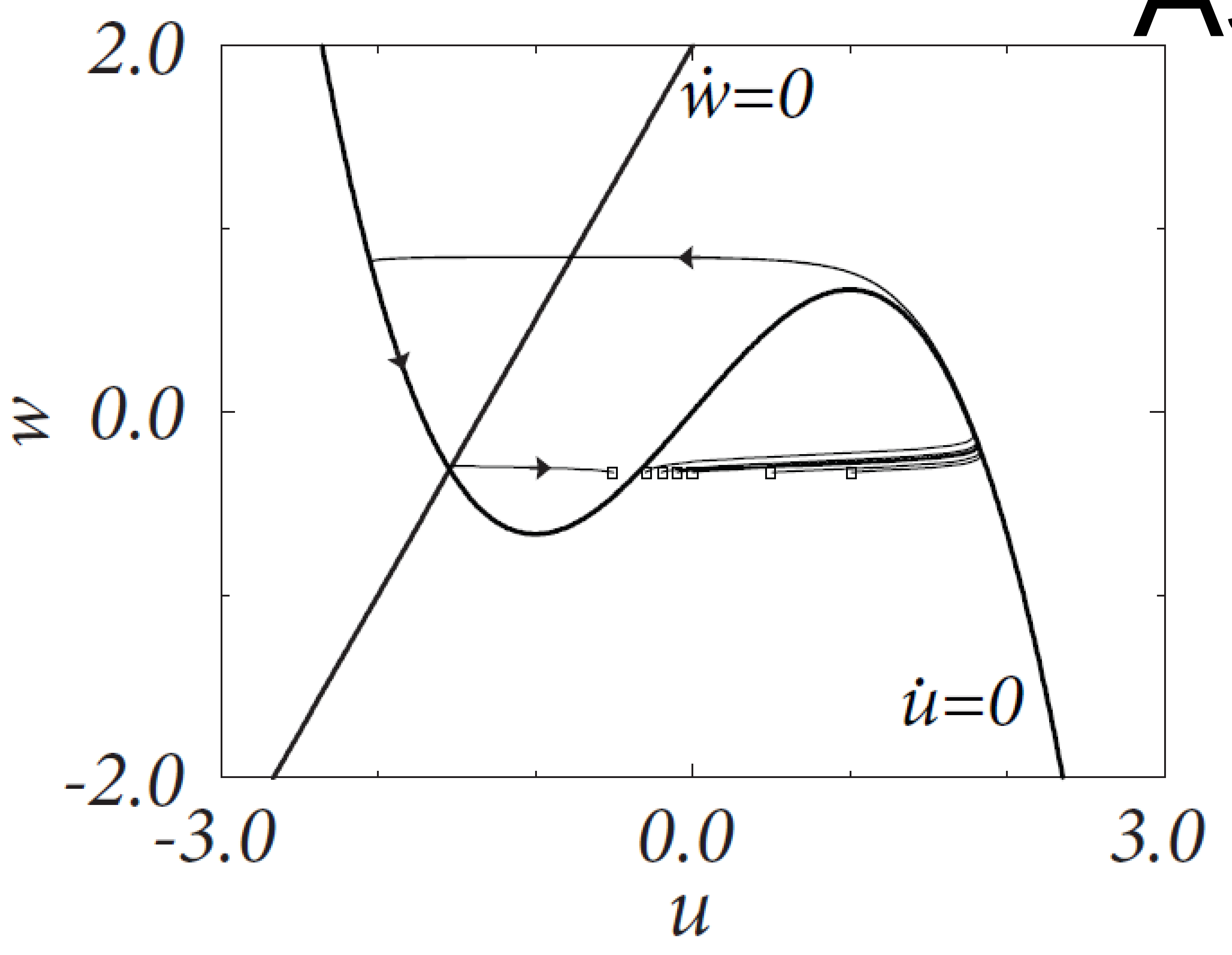
Separation of time scales

$$\tau_w >> \tau_u$$

pulse input

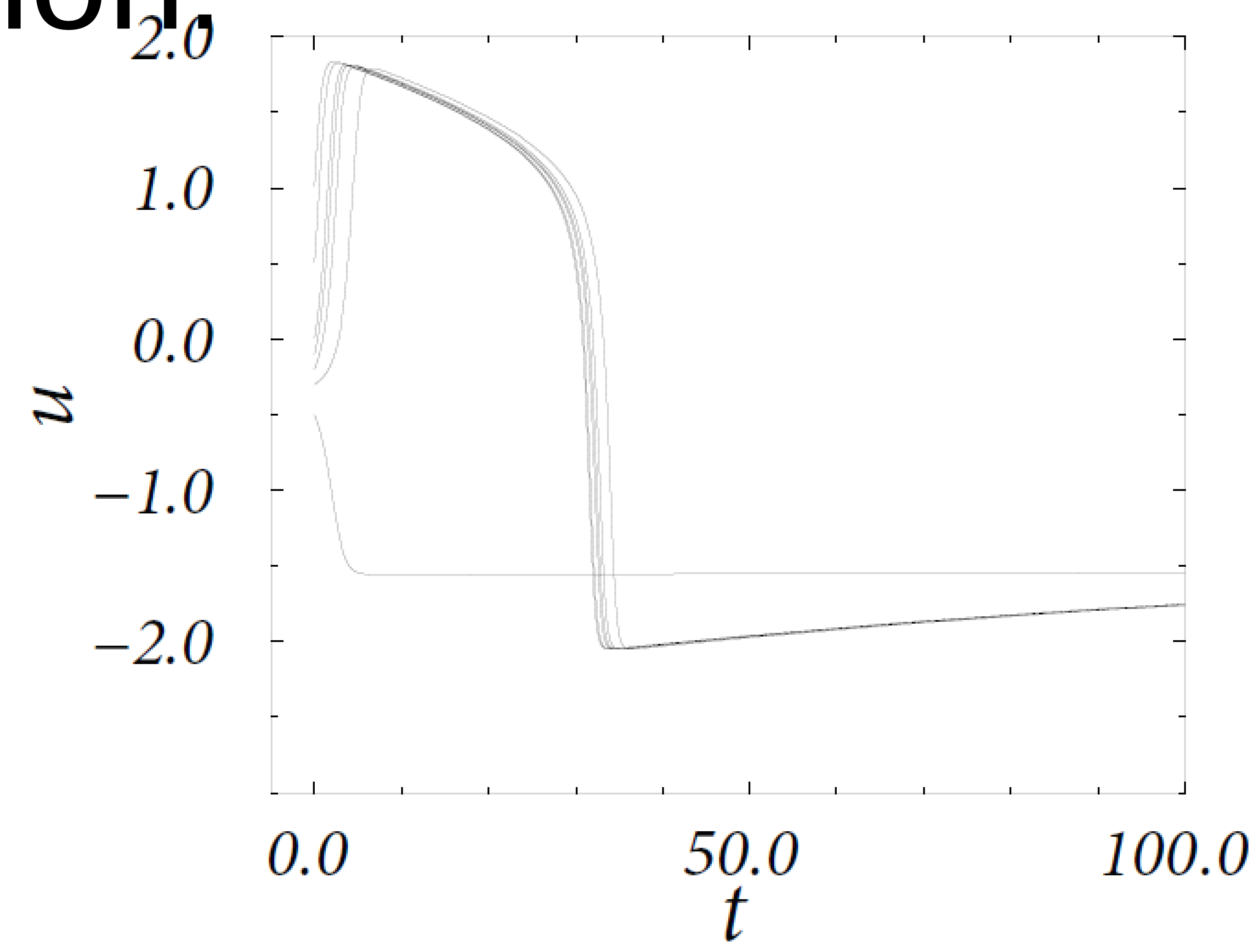


4.4b FitzHugh-Nagumo model: Threshold for Pulse input



Assumption:

$\tau_w >> \tau_u$



Middle branch of u-nullcline plays role of 'Threshold' (for pulse input)

Image: Neuronal Dynamics, Gerstner et al., Cambridge Univ. Press (2014)

4.4b Detour: Separation fo time scales in 2dim models

stimulus

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

Assumption:

$$\tau_w >> \tau_u$$

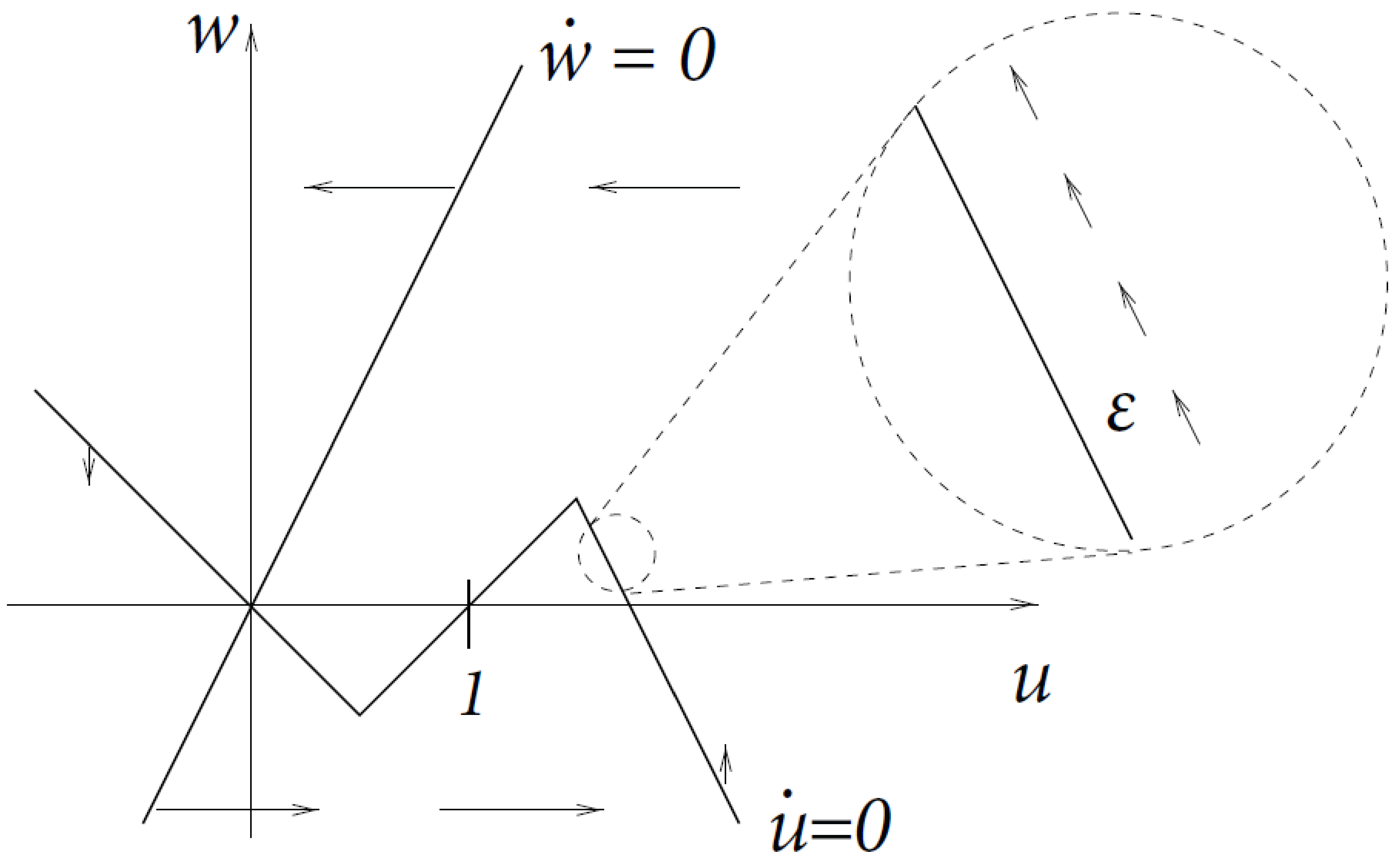
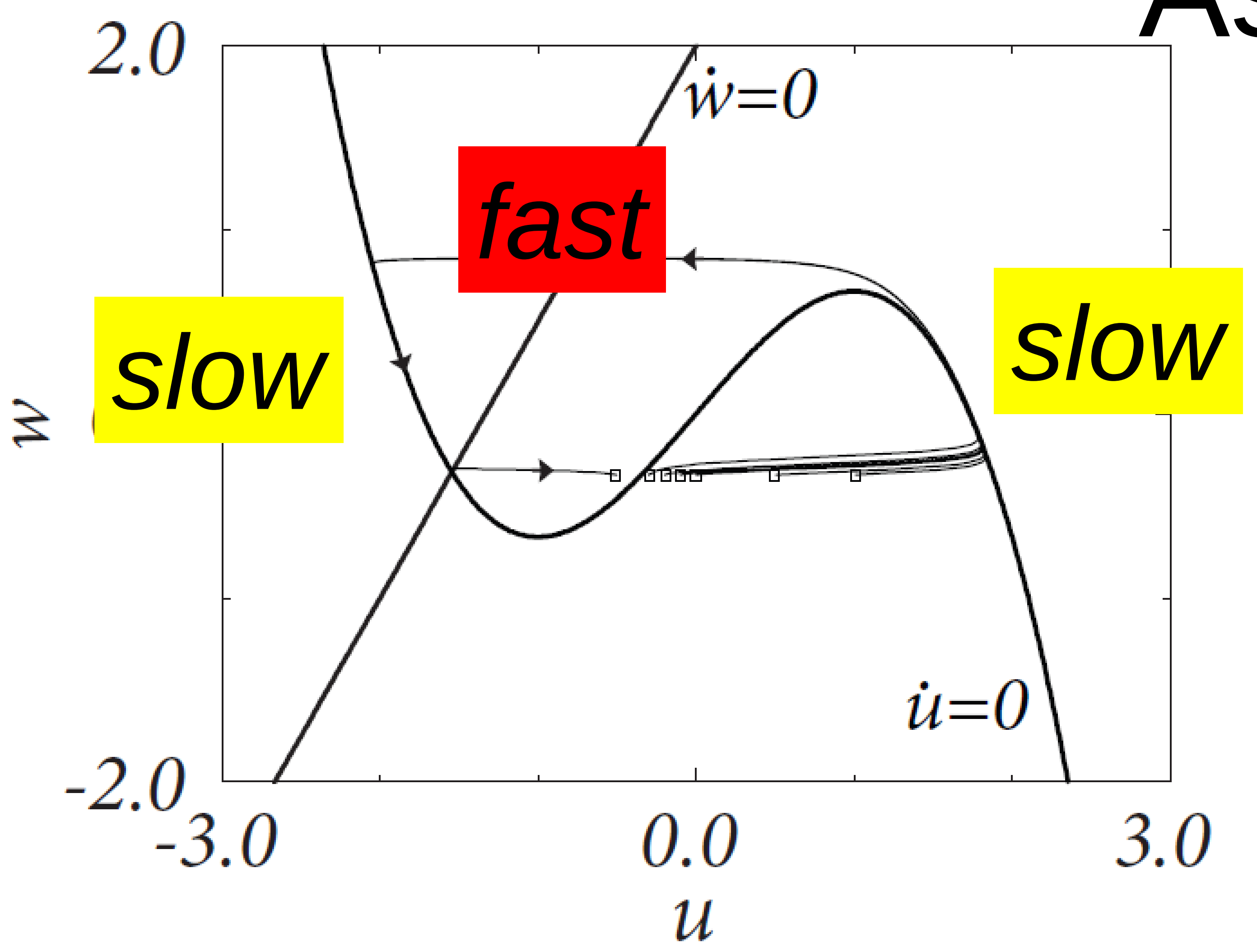


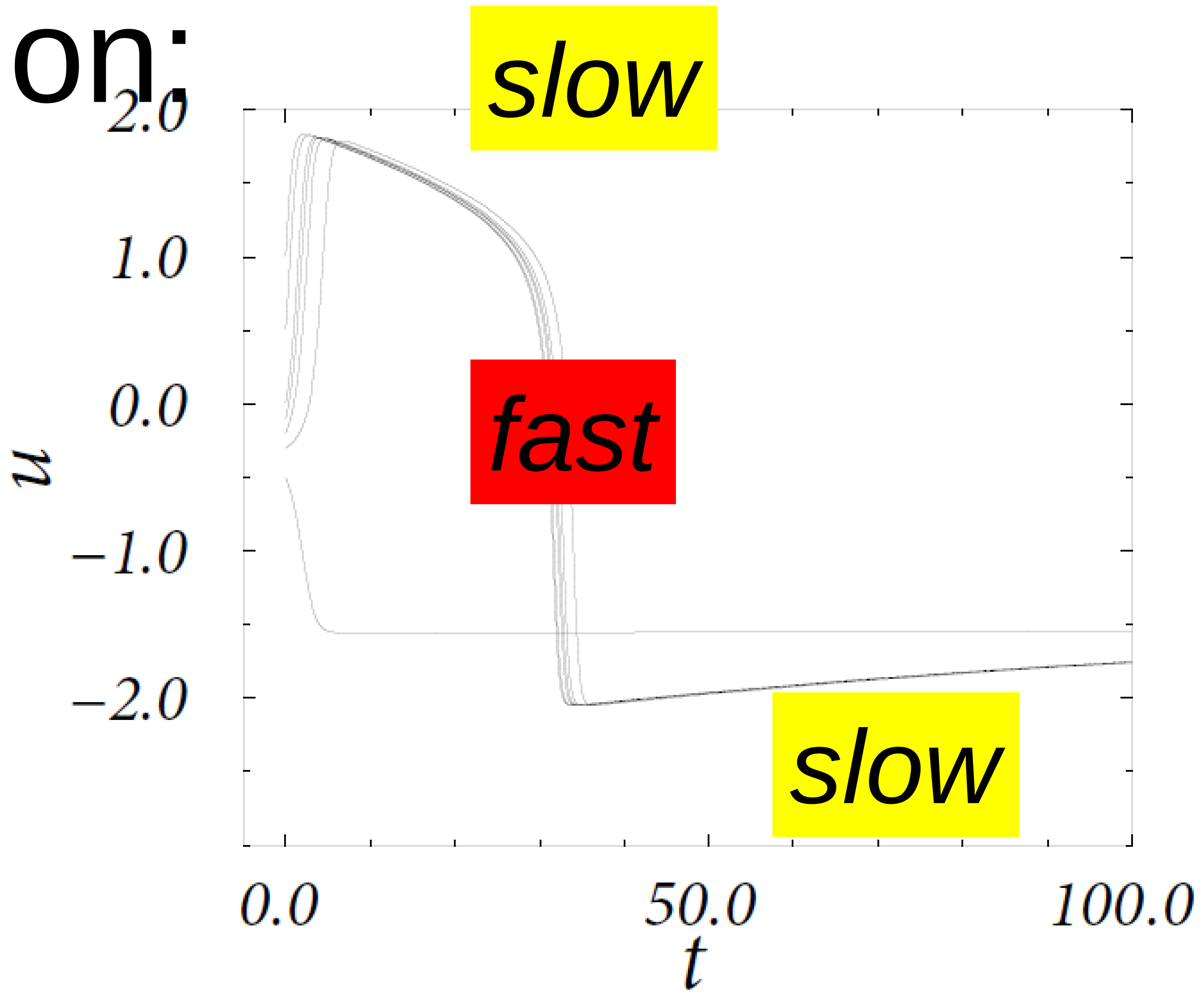
Image: Neuronal Dynamics, Gerstner et al., Cambridge Univ. Press (2014)

4.4b FitzHugh-Nagumo model: Threshold for Pulse input



Assumption:

$\tau_w >> \tau_u$

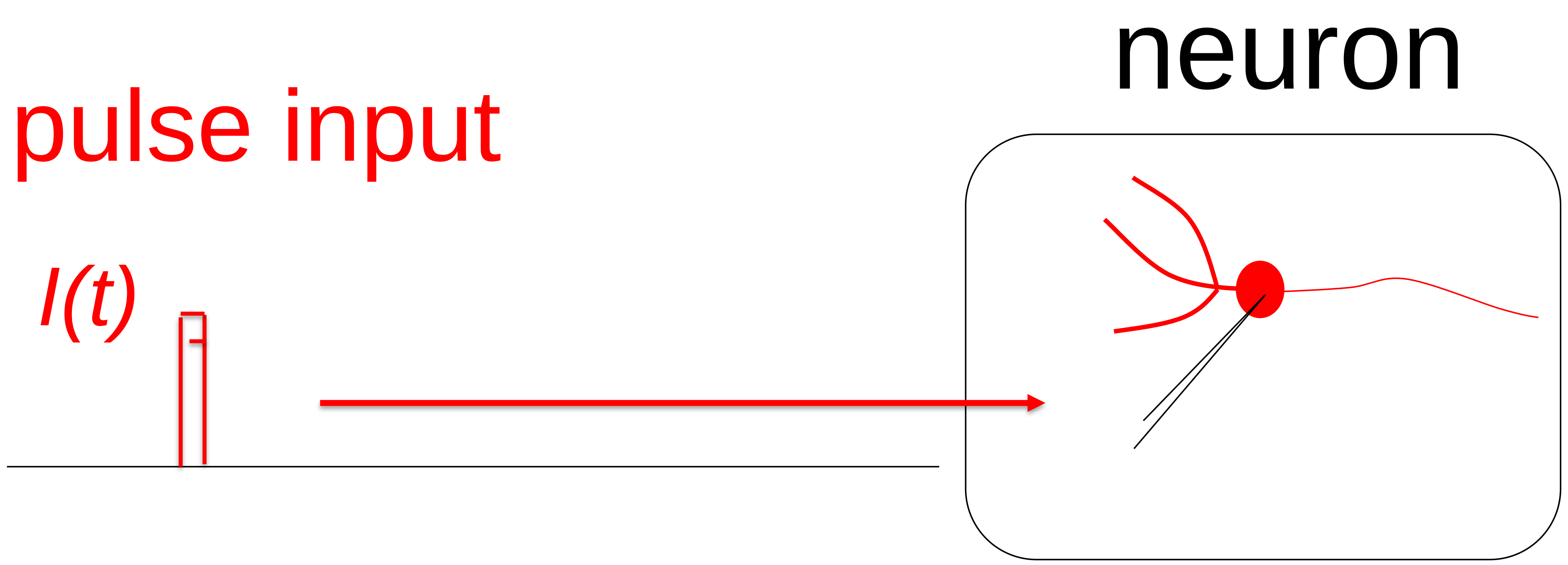


trajectory

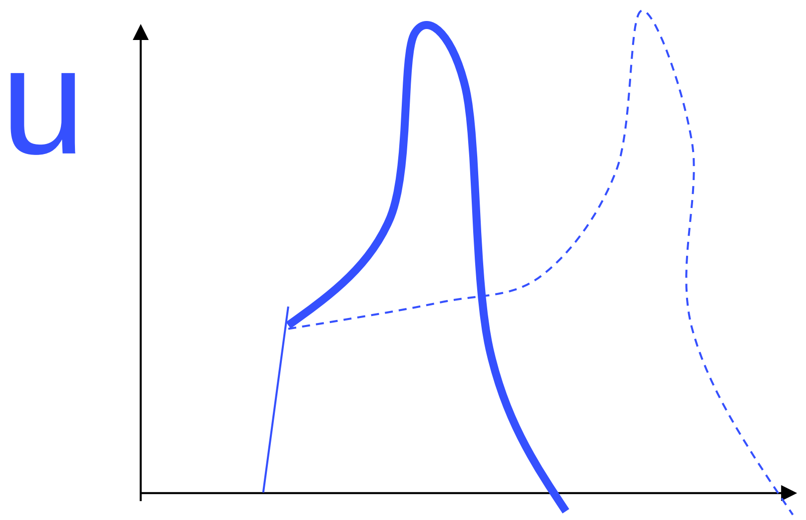
- -follows u-nullcline: slow
- -jumps between branches: fast

Image: Neuronal Dynamics, Gerstner et al., Cambridge Univ. Press (2014)

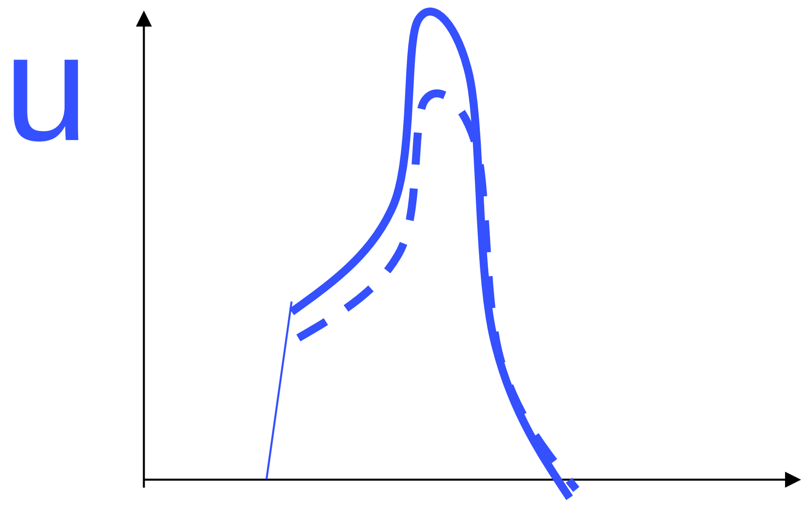
Neuronal Dynamics – 4.4b Threshold in 2dim. Neuron Models



Delayed spike



Reduced amplitude



Neuronal Dynamics – 4.4 Literature

Reading: W. Gerstner, W.M. Kistler, R. Naud and L. Paninski,
Neuronal Dynamics: from single neurons to networks and models of cognition. Chapter 4*: Introduction*. Cambridge Univ. Press, 2014
OR W. Gerstner and W.M. Kistler, *Spiking Neuron Models*, Ch.3. Cambridge 2002
OR J. Rinzel and G.B. Ermentrout, (1989). Analysis of neuronal excitability and oscillations. In Koch, C. Segev, I., editors, *Methods in neuronal modeling*. MIT Press, Cambridge, MA.

Selected references.

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-Fourcaud-Trocme, N., Hansel, D., van Vreeswijk, C., and Brunel, N. (2003). *How spike generation mechanisms determine the neuronal response to fluctuating input.* J. Neuroscience, 23:11628-11640.

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- E.M. Izhikevich, Dynamical Systems in Neuroscience, MIT Press (2007)

Neuronal Dynamics – Quiz 4.6.

A. Threshold in a 2-dimensional neuron model with saddle-node bifurcation

[ ] The voltage threshold for repetitive firing is always the same as the voltage threshold for pulse input.

[ ] in the regime below the saddle-node bifurcation, the voltage threshold for repetitive firing is given by the stable manifold of the saddle.

[ ] in the regime below the saddle-node bifurcation, the voltage threshold for repetitive firing is given by the middle branch of the u-nullcline.

[ ] in the regime below the saddle-node bifurcation, the voltage threshold for action potential firing in response to a short pulse input is given by the middle branch of the u-nullcline.

[ ] in the regime below the saddle-node bifurcation, the voltage threshold for action potential firing in response to a short pulse input is given by the stable manifold of the saddle point.

B. Threshold in a 2-dimensional neuron model with subcritical Hopf bifurcation

[ ]in the regime below the bifurcation, the voltage threshold for action potential firing in response to a short pulse input is given by the stable manifold of the saddle point.

[ ] in the regime below the bifurcation, a voltage threshold for action potential firing in response to a short pulse input exists only if $\tau_w >> \tau_u$

Week 4 – part 4b : Firing threshold in 2D models

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

Neuronal Dynamics:
Computational Neuroscience of Single Neurons

Week 4 – Reducing detail: Two-dimensional neuron models

Wulfram Gerstner

EPFL, Lausanne, Switzerland

√4.1 From Hodgkin-Huxley to 2D

√4.2 Phase Plane Analysis

√4.3 Analysis of a 2D Neuron Model

√4.4 Type I and II Neuron Models
- where is the firing threshold?

4.5. Nonlinear Integrate-and-fire
- from two to one dimension

2-dimensional equation

stimulus

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

slow!

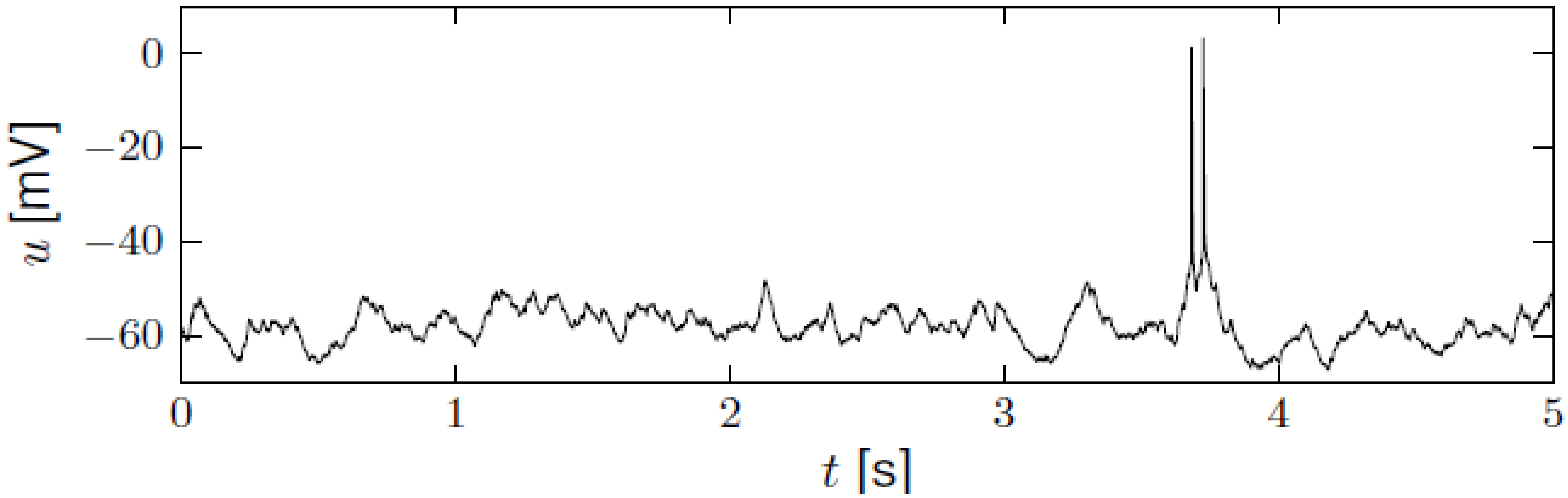
Separation of time scales

-*w* is nearly constant
(most of the time)

Neuronal Dynamics – 4.2 sparse activity in vivo

Spontaneous activity *in vivo*

awake mouse, cortex, freely whisking,



-spikes are rare events

Crochet et al., 2011

-membrane potential fluctuates around ‘rest’

Aims of Modeling: - predict spike initation times
- predict subthreshold voltage

Neuronal Dynamics – 4.5. Further reduction to 1 dimension

stimulus

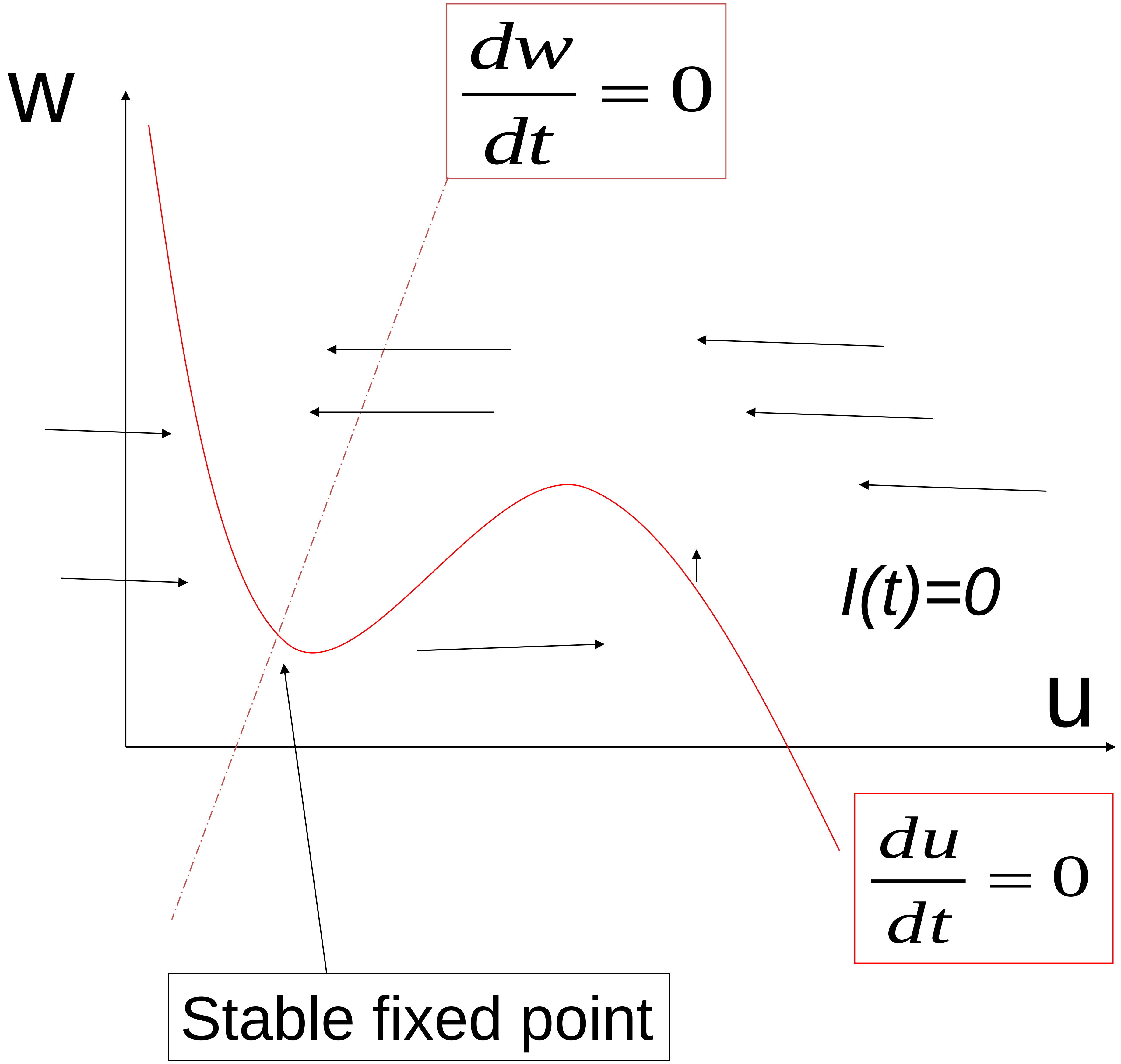
$$\tau \frac{du}{dt} = F(u,w) + I(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

Separation of time scales

$$\tau_w >> \tau_u$$

→ Flux nearly horizontal



Hodgkin-Huxley reduced to 2dim

$$\tau \frac{du}{dt} = F(u,w) + I(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

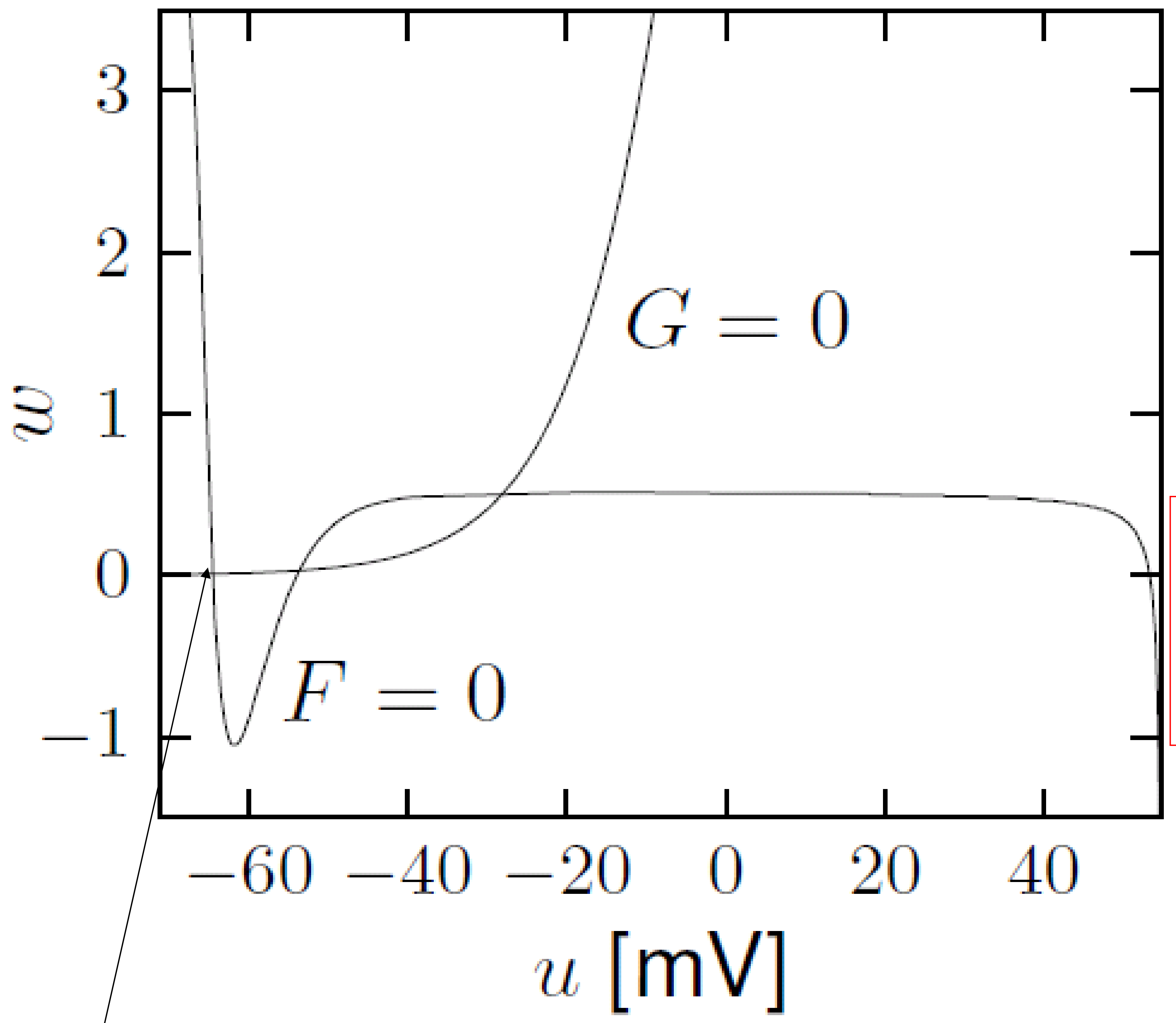
Separation of time scales

$$\tau_w >> \tau_u$$

$$\tau_w \frac{dw}{dt} \approx 0 \rightarrow \ w \approx w_{rest}$$

$$\tau \frac{du}{dt} = F(u, w_{rest}) + RI(t)$$

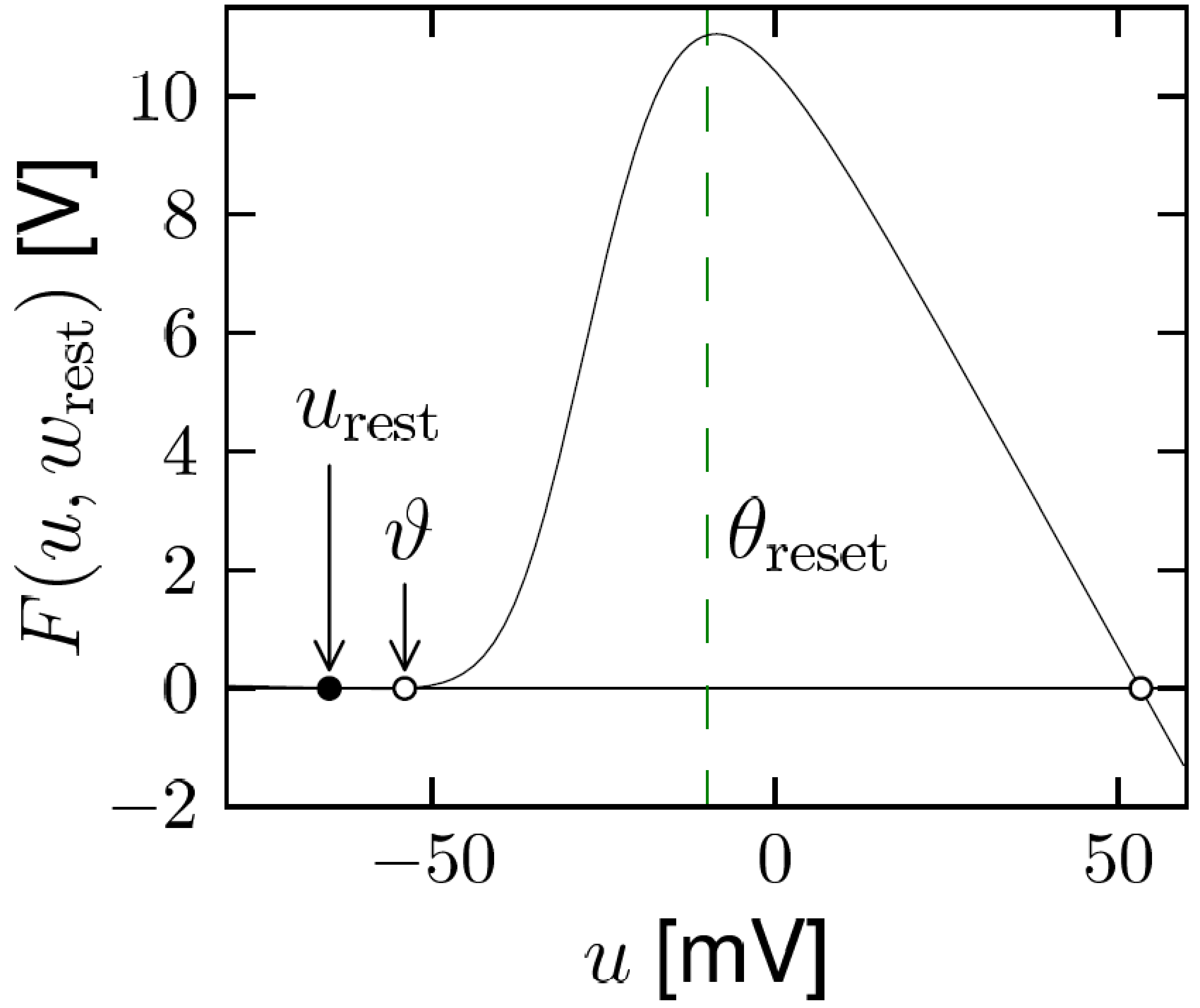
$$\frac{dw}{dt} = 0$$

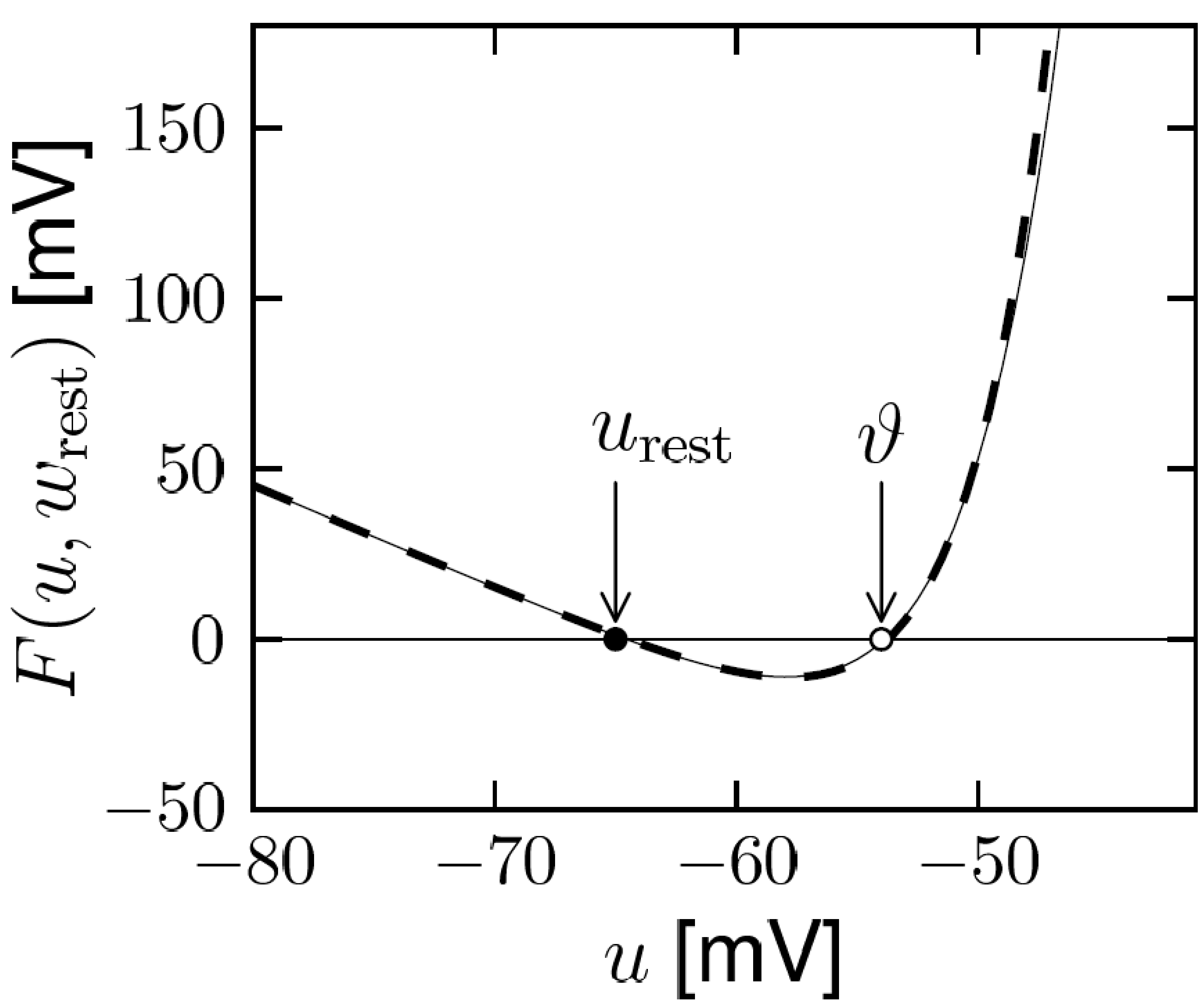


$$\frac{du}{dt} = 0$$

Stable fixed point

Neuronal Dynamics – 4.5. Nonlinear Integrate-and-Fire Model





$$\tau \frac{du}{dt} = F(u, w_{rest}) + RI(t) = f(u) + RI(t)$$

→ Nonlinear I&F (see week 1!)

Image: Neuronal Dynamics, Gerstner et al., Cambridge Univ. Press (2014)

Exponential integrate-and-fire model (EIF)

$$f(u) = -(u - u_{rest}) + \Delta \exp\left(\frac{u-\vartheta}{\Delta}\right)$$

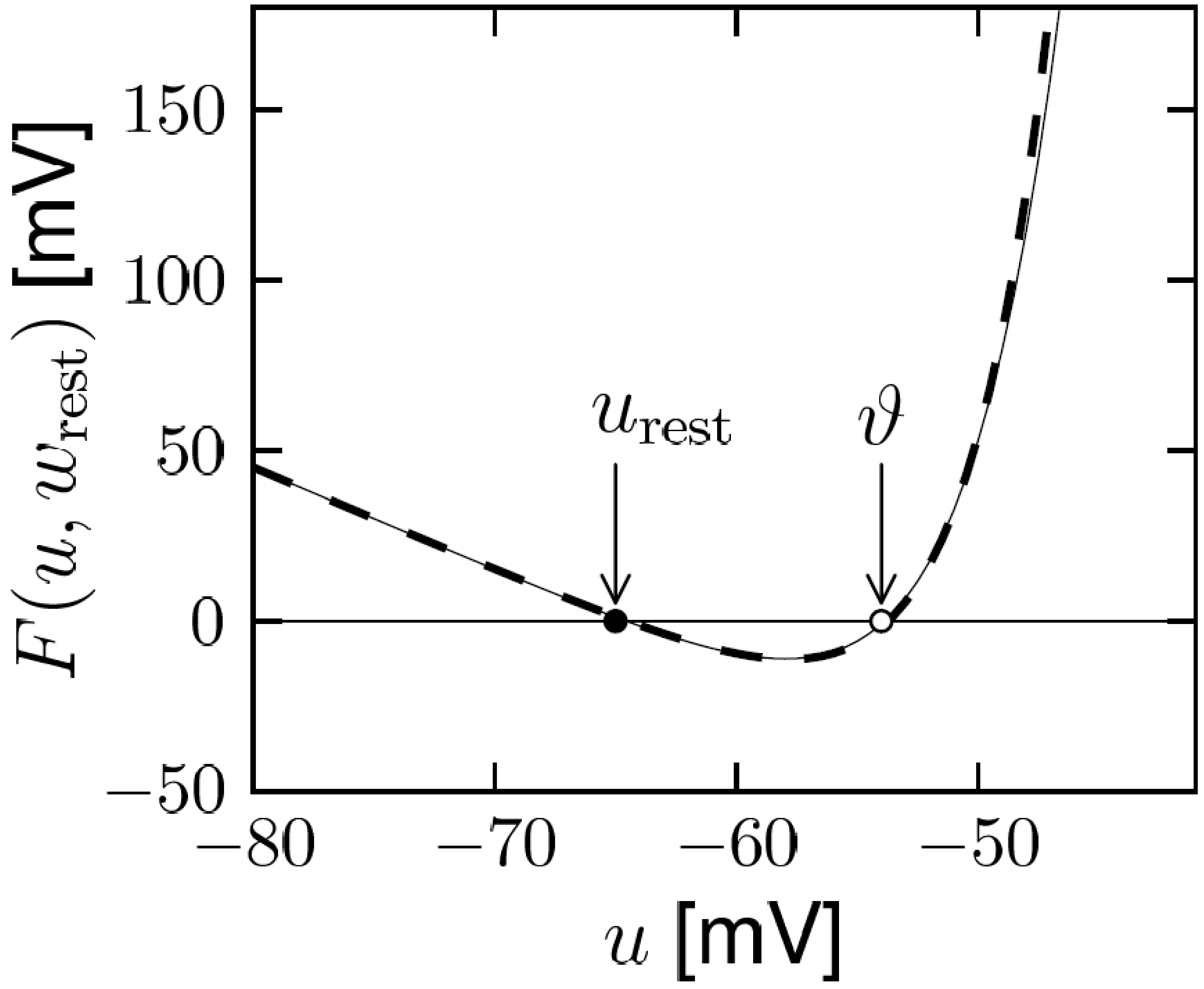


Image: Neuronal Dynamics, Gerstner et al., Cambridge Univ. Press (2014)

$$\tau \frac{du}{dt} = F(u, w_{rest}) + RI(t) = f(u) + RI(t)$$

→ Nonlinear I&F (see week 1!)

Exponential integrate-and-fire model (EIF)

$$f(u) = -(u - u_{rest}) + \Delta \exp\left(\frac{u-\vartheta}{\Delta}\right)$$

linear

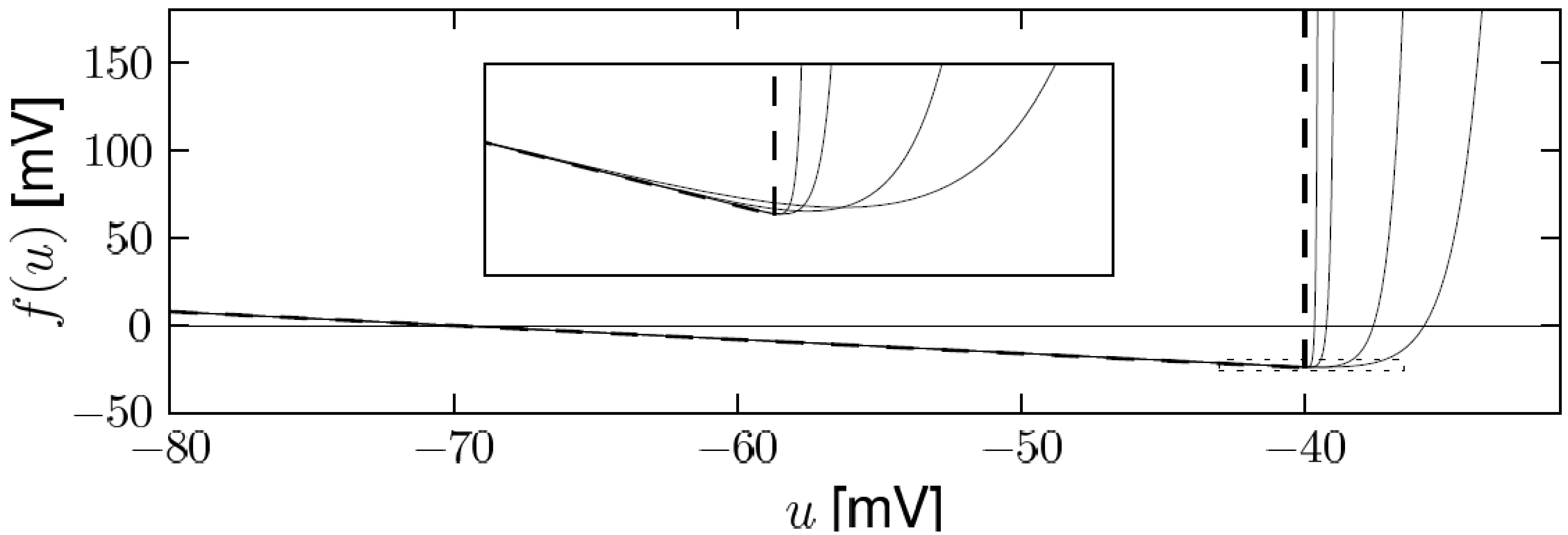
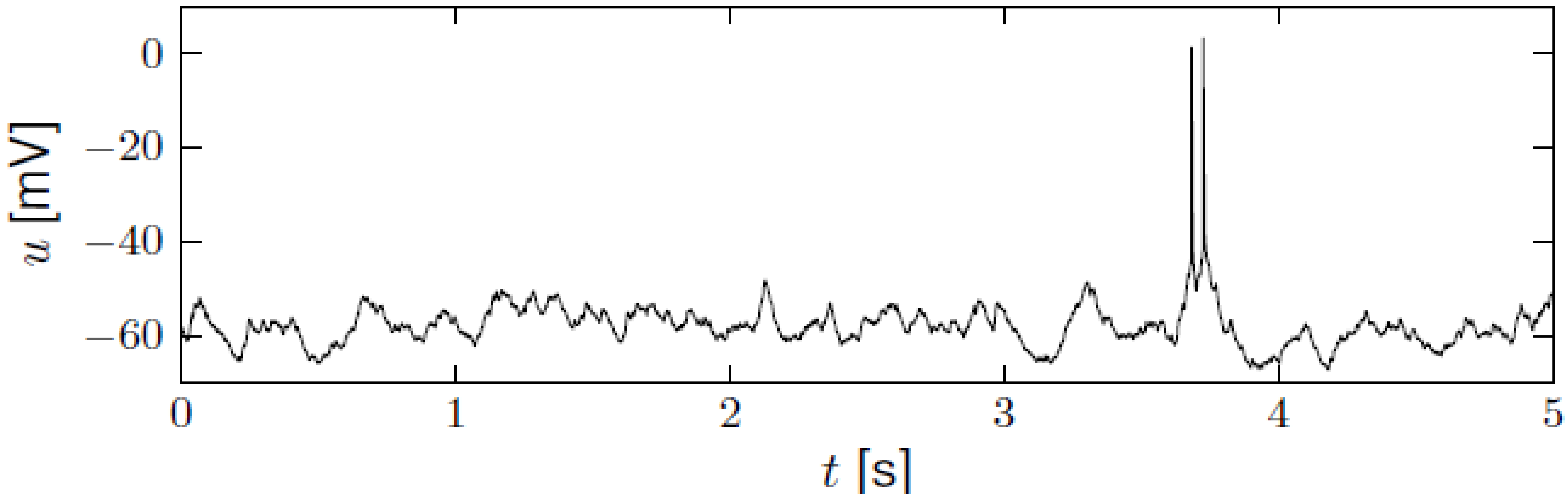


Image: Neuronal Dynamics, Gerstner et al., Cambridge Univ. Press (2014)

Neuronal Dynamics – 4.5 sparse activity in vivo

Spontaneous activity *in vivo*

awake mouse, cortex, freely whisking,



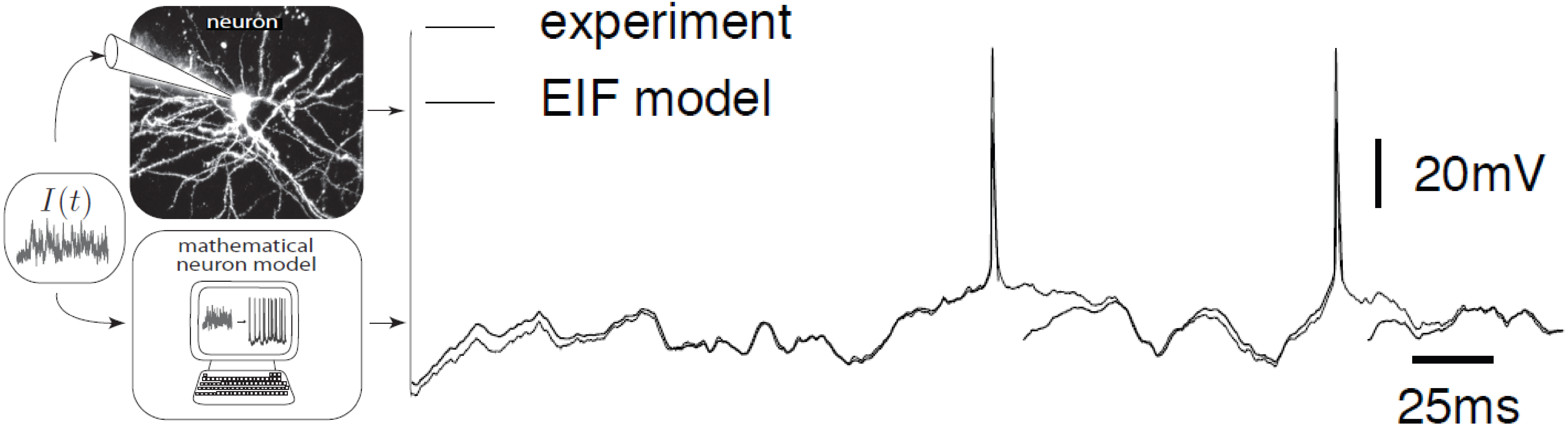
-spikes are rare events

Crochet et al., 2011

-membrane potential fluctuates around 'rest'

Aims of Modeling: - predict spike initation times
- predict subthreshold voltage

Neuronal Dynamics – 4.5.How good are integrate-and-fire models?



Badel et al., 2008

Aims: - predict spike initation times
- predict subthreshold voltage

Add adaptation and refractoriness (week 7)

Direct derivation from Hodgkin-Huxley

$$C\frac{du}{dt} = -g_{Na}\, m^3\, h\,(u - E_{Na}) - g_K n^4 (u - E_K) - g_l (u - E_l) + I(t)$$

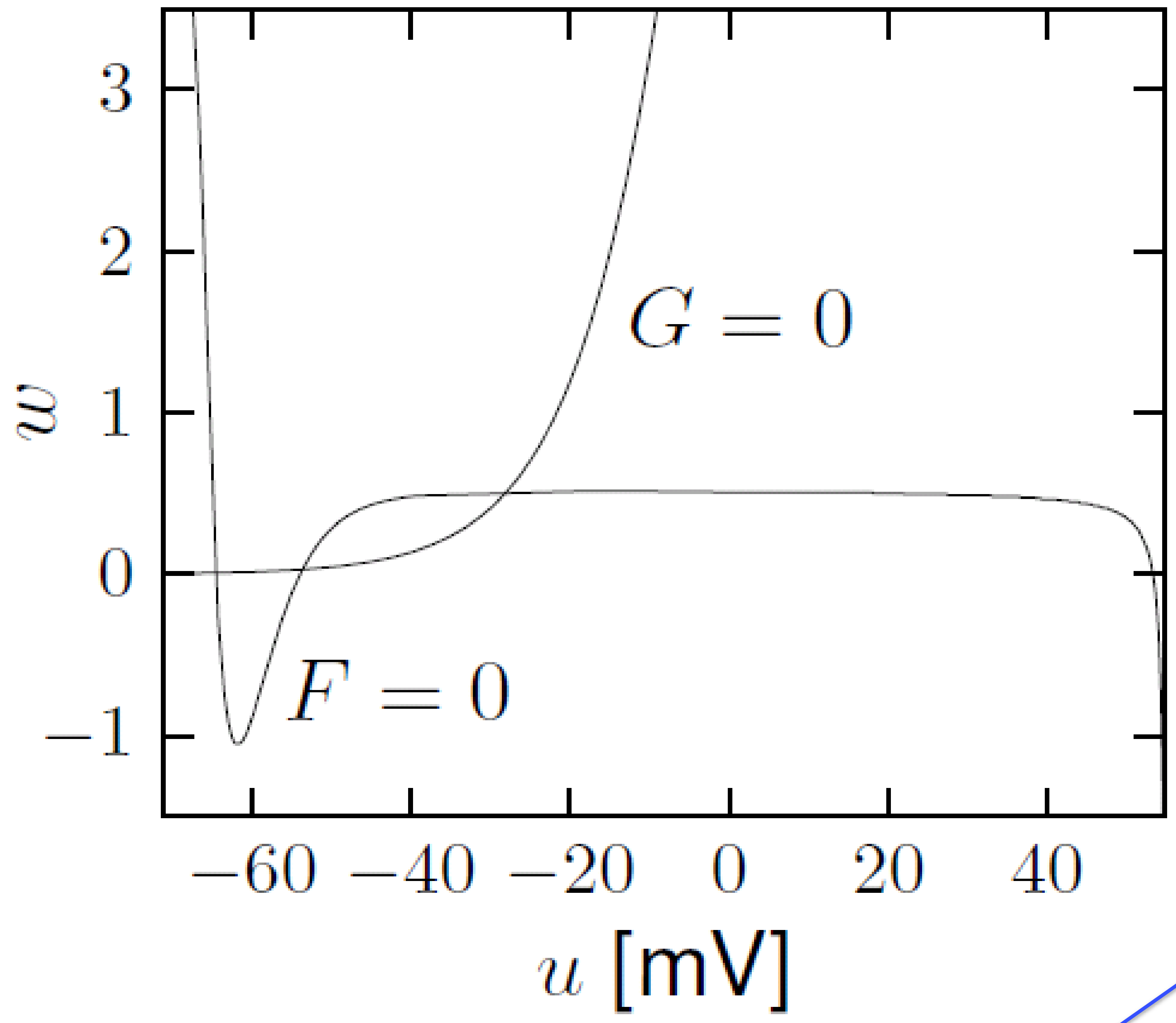
$$C\frac{du}{dt} = -g_{Na}[m_0(u)]^3 h_{rest}(u - E_{Na}) - g_K [n_{rest}]^4 (u - E_K) - g_l (u - E_l) + I(t)$$

Fourcaud-Trocme et al, J. Neurosci. 2003

$$f(u) = -(u - u_{rest}) + \Delta \exp\left(\frac{u-\vartheta}{\Delta}\right)$$

$$\tau\frac{du}{dt} = F(u, h_{rest}, n_{rest}) + RI(t) = f(u) + RI(t)$$

gives expon. I&F



2-dimensional equation

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

Relevant during spike and downswing of AP

Separation of time scales

-*w* is constant (if not firing)

$$\tau \frac{du}{dt} = f(u) + RI(t)$$

threshold+reset for firing

2-dimensional equation

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

Separation of time scales

-*w* is constant (if not firing)

$$\tau \frac{du}{dt} = f(u) + RI(t)$$

Linear plus exponential

Neuronal Dynamics – Quiz 4.5.

A. Exponential integrate-and-fire model.

The model can be derived

[ ] from a 2-dimensional model, assuming that the auxiliary variable w is constant.

[ ] from the HH model, assuming that the gating variables h and n are constant.

[ ] from the HH model, assuming that the gating variables m is constant.

[ ] from the HH model, assuming that the gating variables m is instantaneous.

B. Reset.

[ ] In a 2-dimensional model, the auxiliary variable w is necessary to implement a reset of the voltage after a spike

[ ] In a nonlinear integrate-and-fire model, the auxiliary variable w is necessary to implement a reset of the voltage after a spike

[ ] In a nonlinear integrate-and-fire model, a reset of the voltage after a spike is implemented algorithmically/explicitly

Neuronal Dynamics – 4.5 Literature

Reading: W. Gerstner, W.M. Kistler, R. Naud and L. Paninski,
Neuronal Dynamics: from single neurons to networks and models of cognition. Chapter 4 *(Dimension Reduction and Phase Plane analysis)*
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4.5. Summary: from HH to generalized integrate-and-fire

- The **reduction of the Hodgkin-Huxley (HH) model** from 4 to 2 dimensions generates nonlinear nullclines with several intersections.
- If we zoom in on the two left-most intersections the u-nullcline looks similar to a superposition of a linear and an exponential term
- Between (rare) spike events, the w-variable has always time to go back to resting potential. Hence during spike-initiation we can consider the w-variable as constant.
- This gives rise to the **exponential integrate-and-fire model**

- **Adaptation** means that for constant input the interspike intervals increase over time – we will add adaptation variables later
- The standard HH-model shows no (or very little) adaptation
- More complicated Hodgklin-Huxley type models would have additional variables (describing other ion channels) that cause adaptation
- In integrate-and-fire models, these additional adaptation variables can often be approximated by a linear dynamics for new variables w_k