# Week 4 – part 1 : Reduction of the Hodgkin-Huxley Model

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

# Neuronal Dynamics:
## Computational Neuroscience of Single Neurons

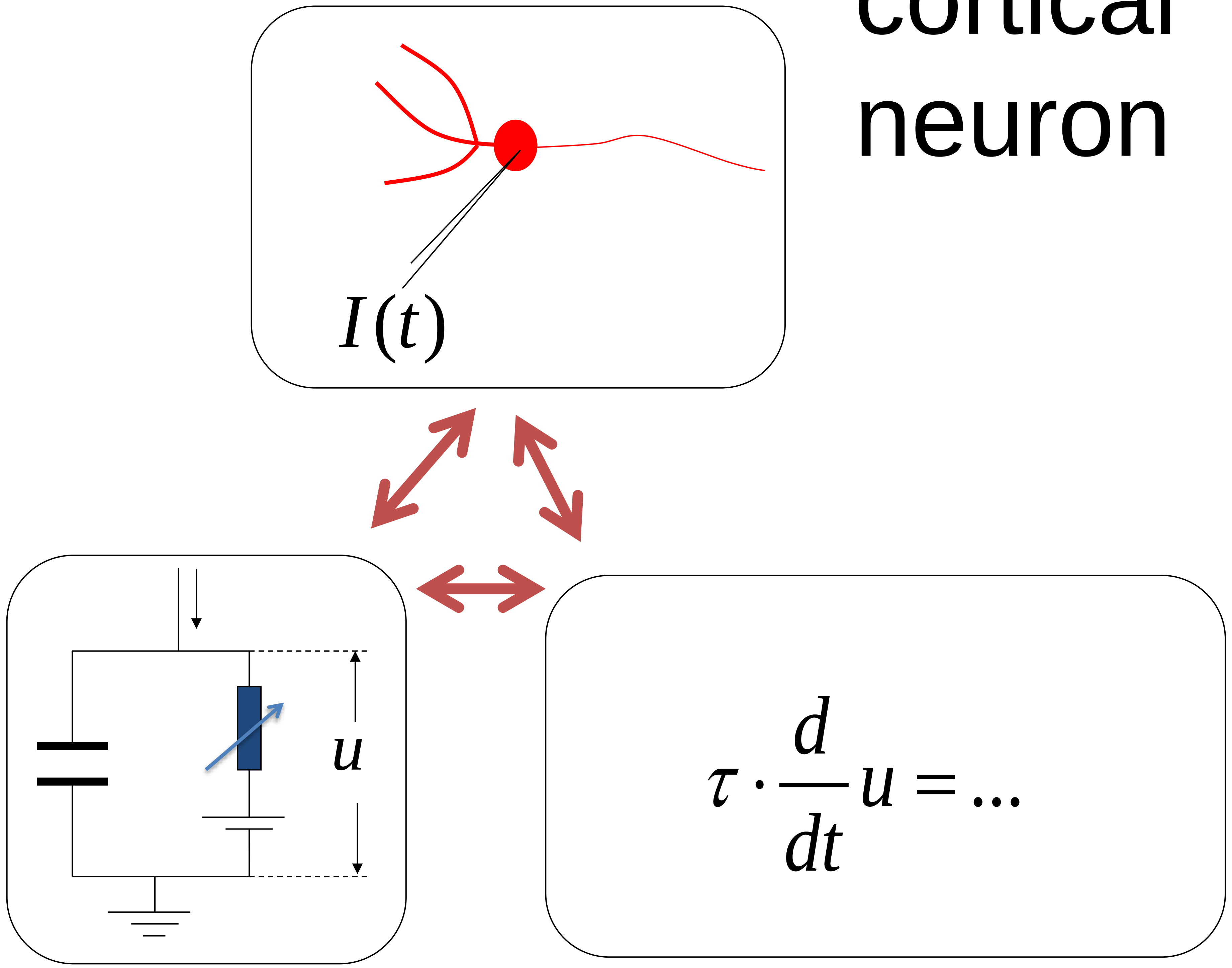
## Week 4 – Reducing detail:
## Two-dimensional neuron models

Wulfram Gerstner

EPFL, Lausanne, Switzerland

# Neuronal Dynamics – 4.1. Review :Hodgkin-Huxley Model

cortical neuron




→Hodgkin-Huxley model

→Compartmental models

# Neuronal Dynamics – 4.1 Review :Hodgkin-Huxley Model

**Week 2:**

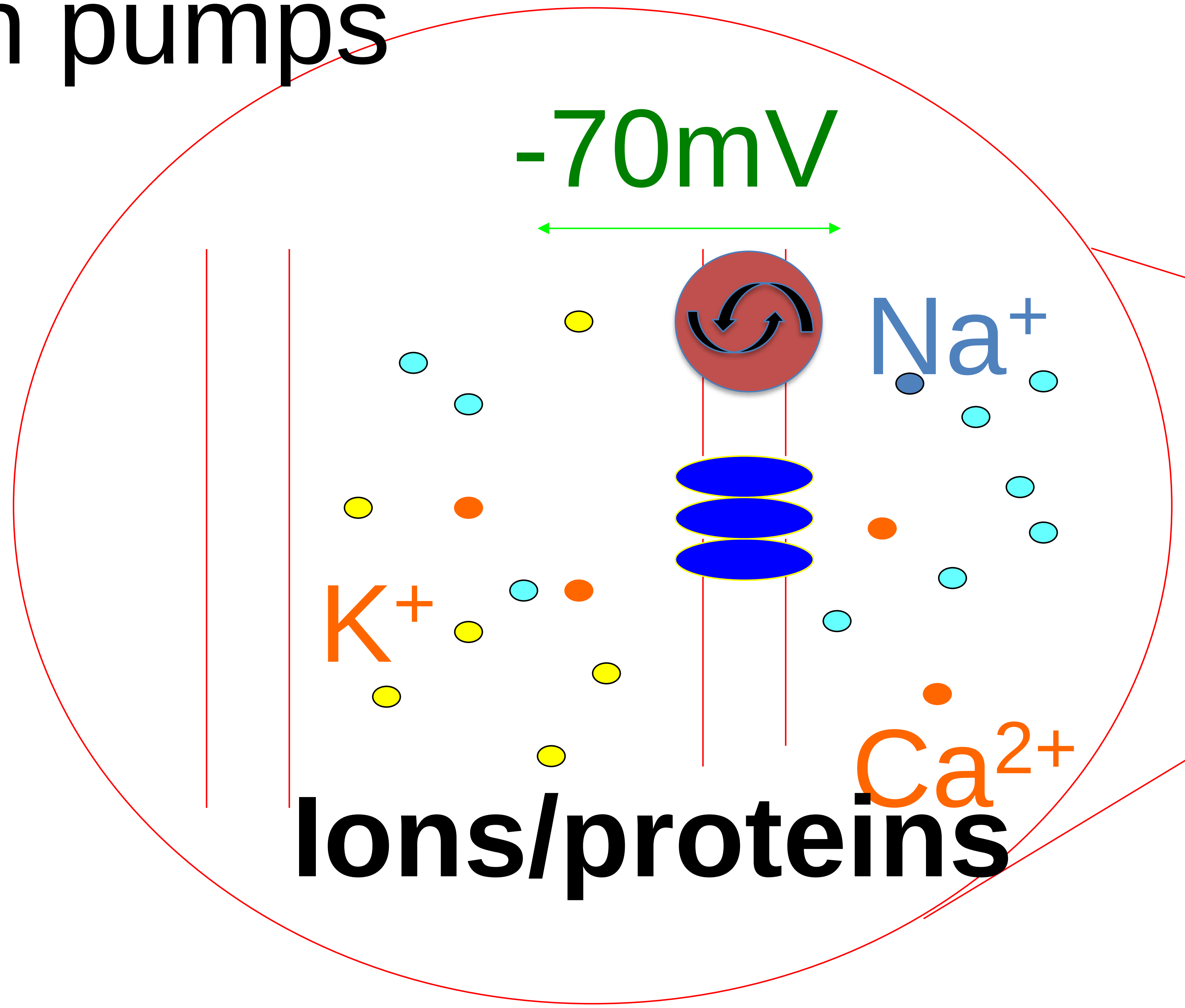
Cell membrane contains

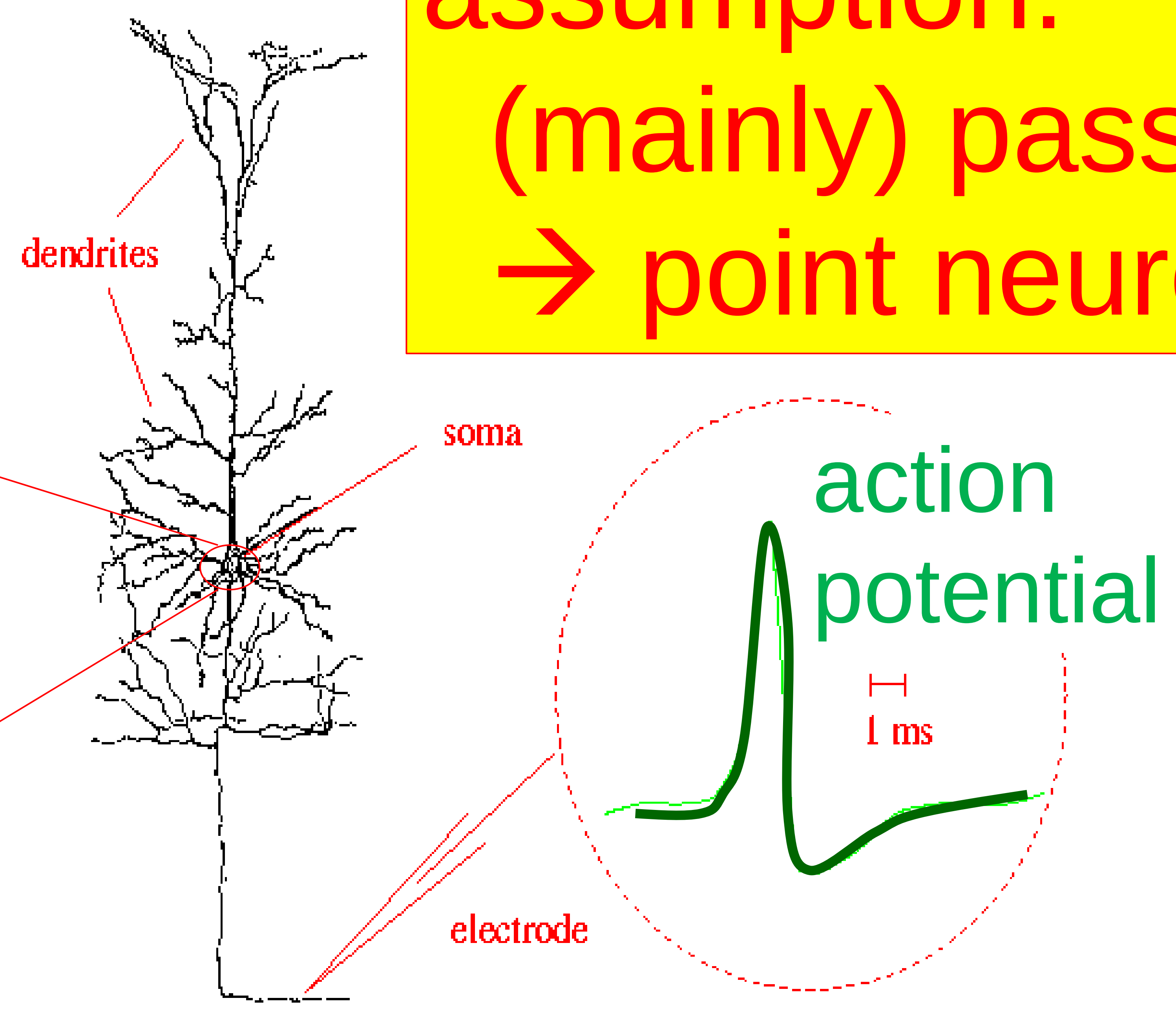
- ion channels
- ion pumps

Dendrites (week 3):

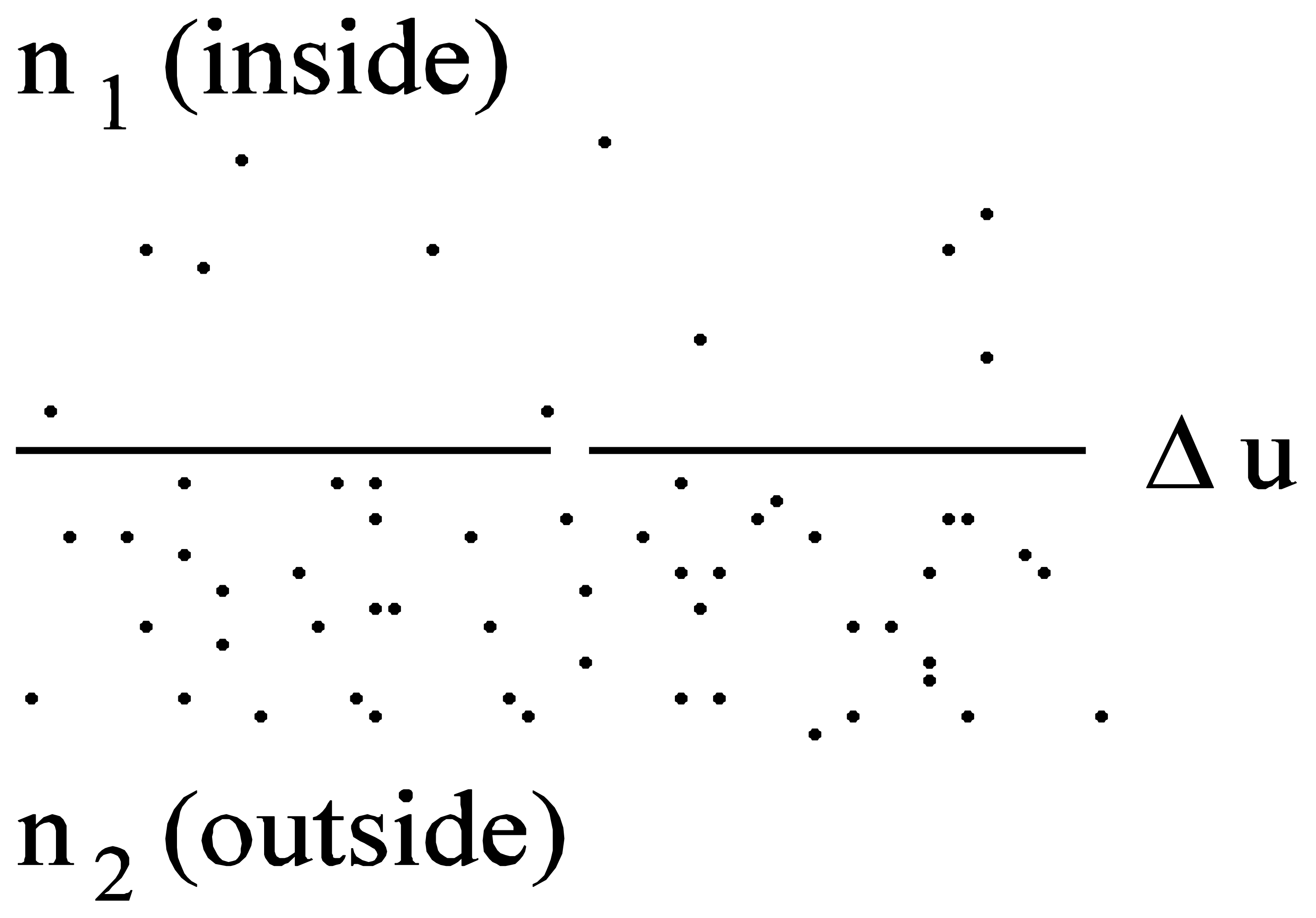
Active processes?

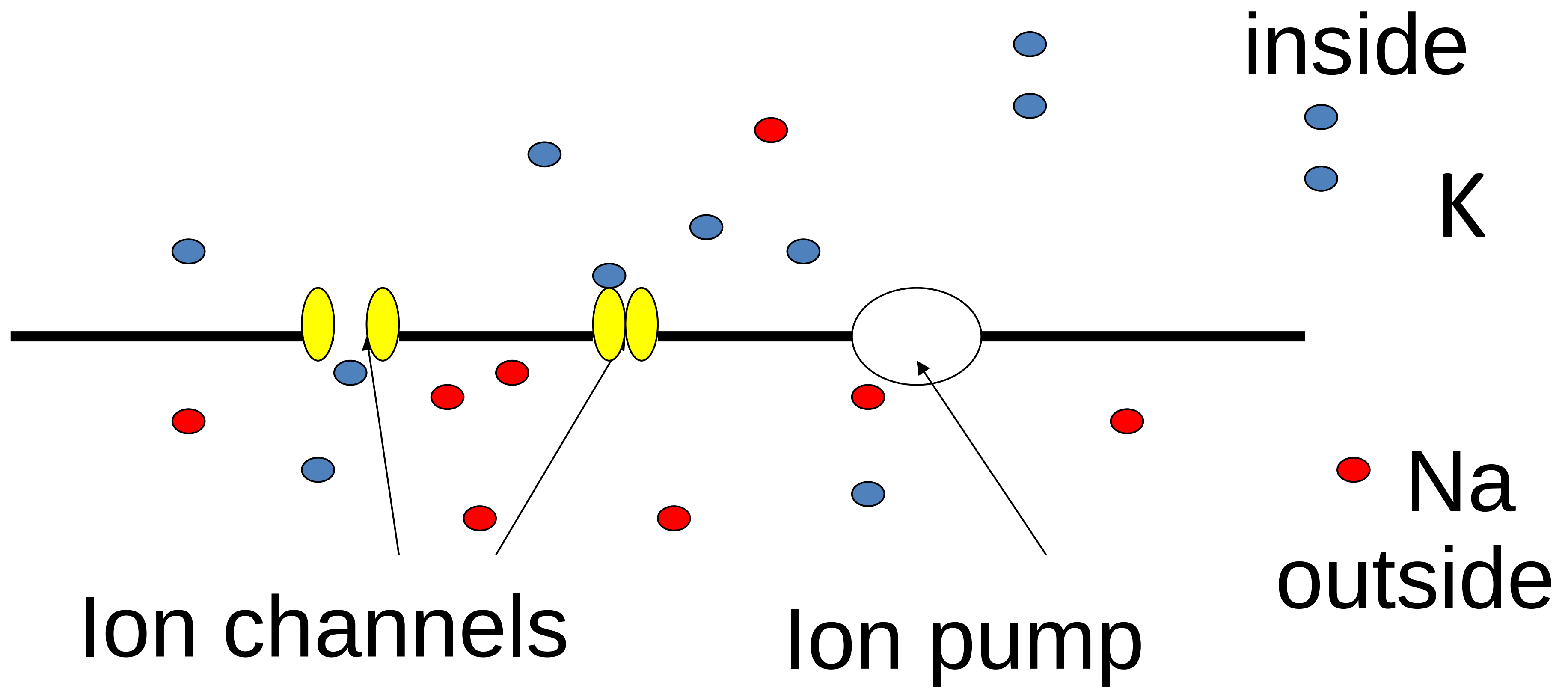
assumption:
(mainly) passive
→ point neuron



# Neuronal Dynamics – 4.1. Review :Hodgkin-Huxley Model



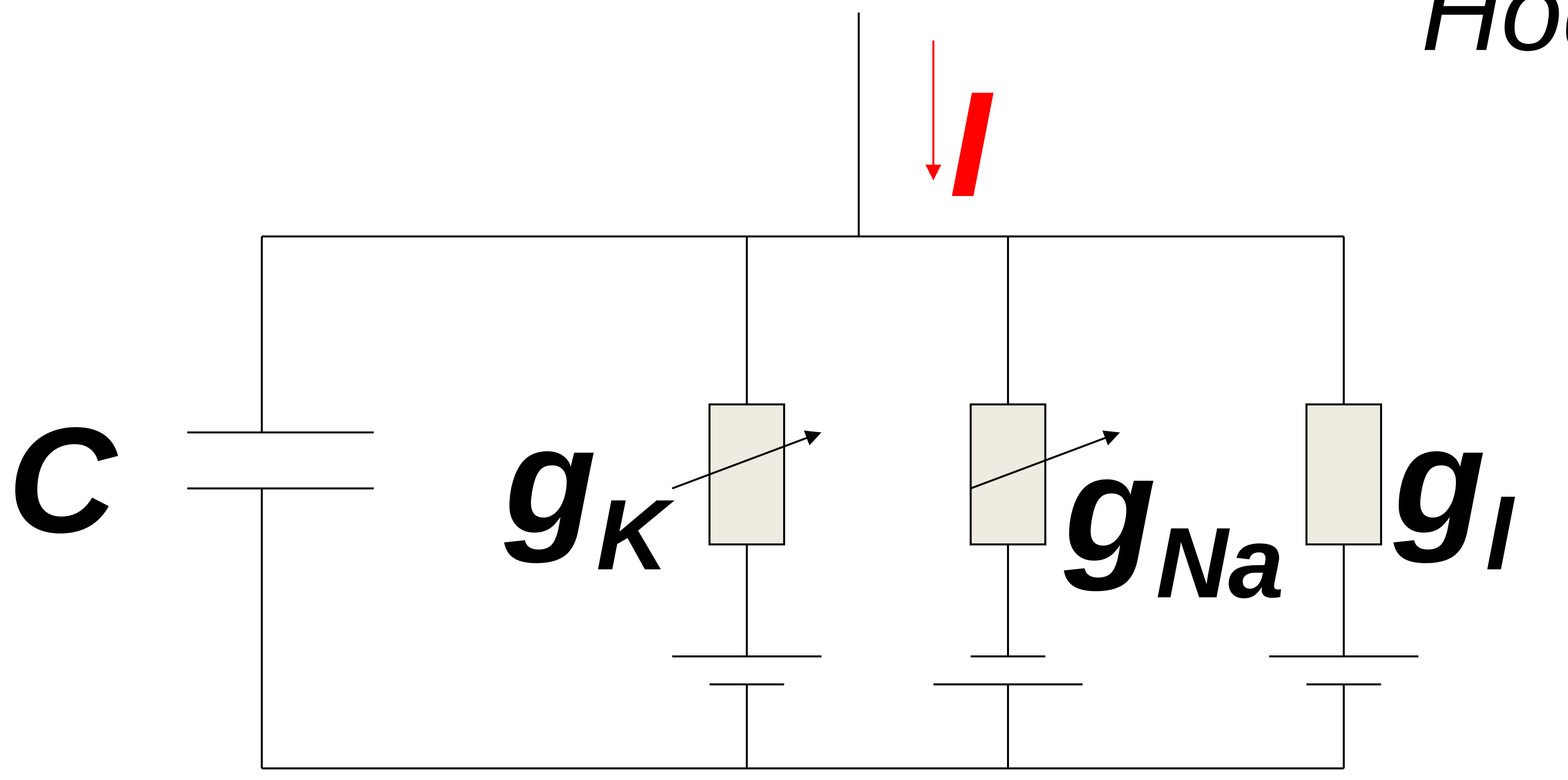

$$\Delta u = u_1 - u_2 = \frac{-kT}{q} \ln \frac{n(u_1)}{n(u_2)}$$

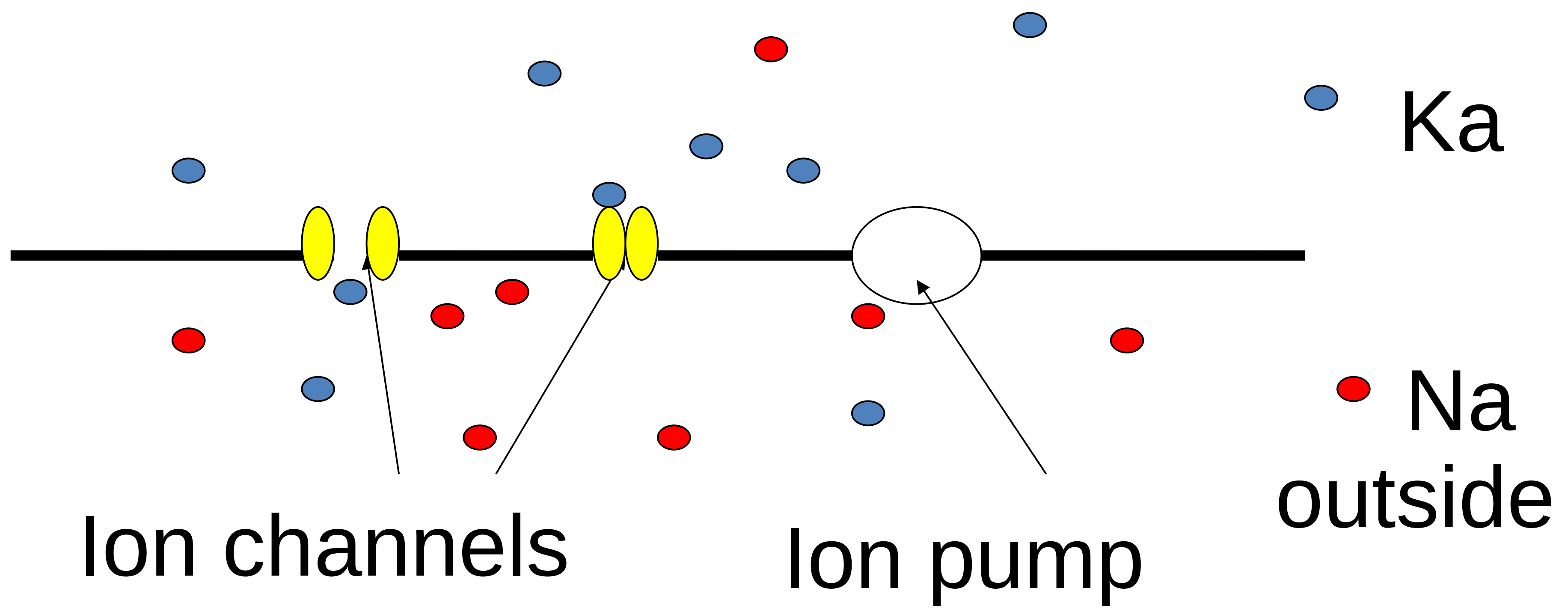
Reversal potential

**ion pumps →concentration difference ⇔ voltage difference**

# Neuronal Dynamics – 4.1. Review: Hodgkin-Huxley Model

*Hodgkin and Huxley, 1952*



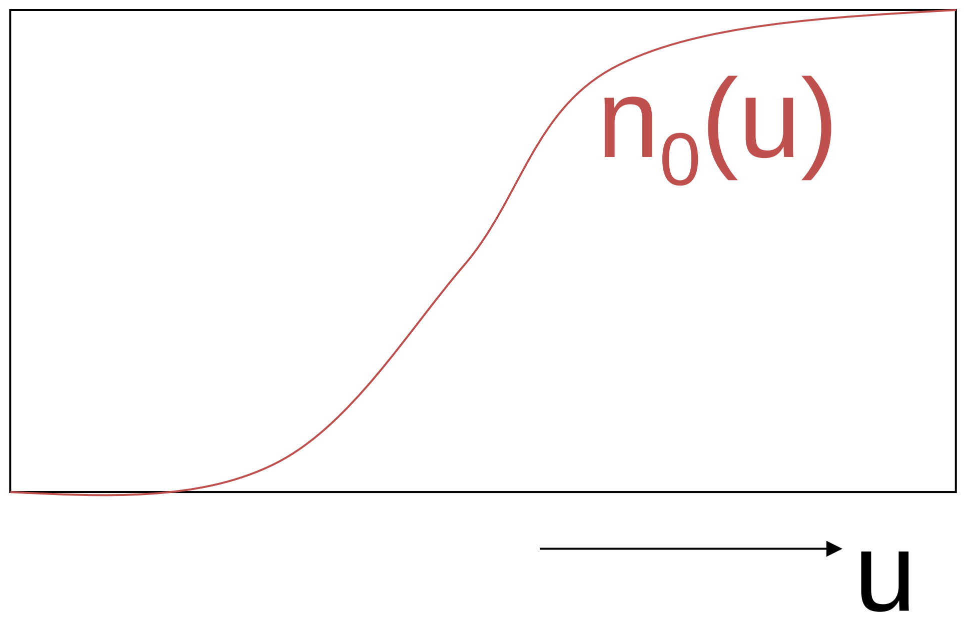

stimulus

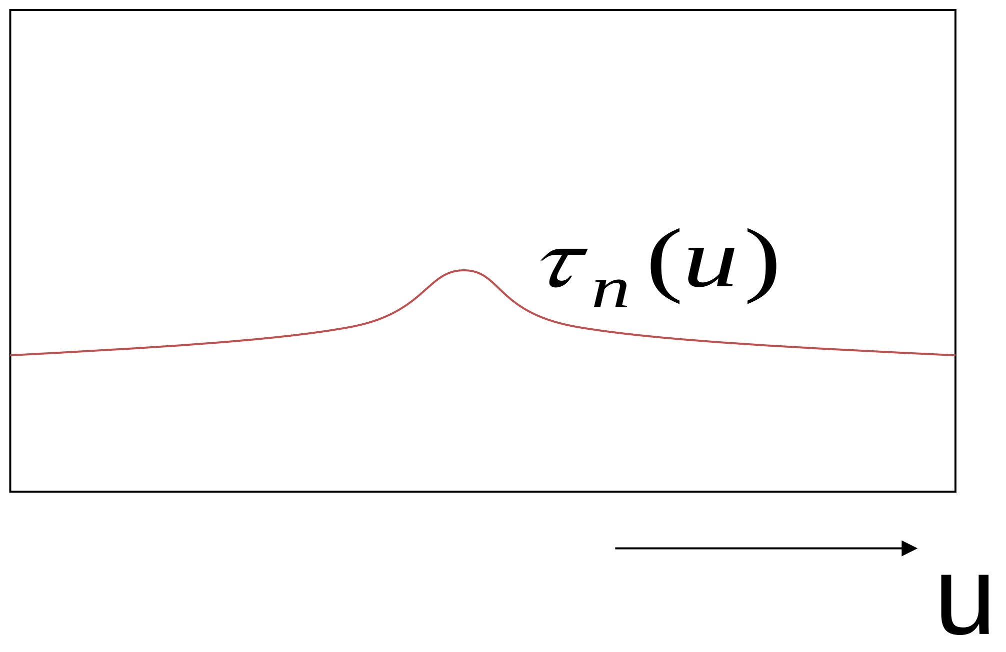
$$C\frac{du}{dt} = \underbrace{-g_{Na}m^3h(u-E_{Na})}_{I_{Na}}\underbrace{-g_K n^4(u-E_K)}_{I_K}\underbrace{-g_l(u-E_l)}_{I_{leak}}+I(t)$$

4 equations
= 4D system

$$\frac{dh}{dt} = -\frac{h-h_0(u)}{\tau_h(u)}$$

$$\frac{dm}{dt} = -\frac{m-m_0(u)}{\tau_m(u)}$$



# Neuronal Dynamics – 4.1. Overview and aims

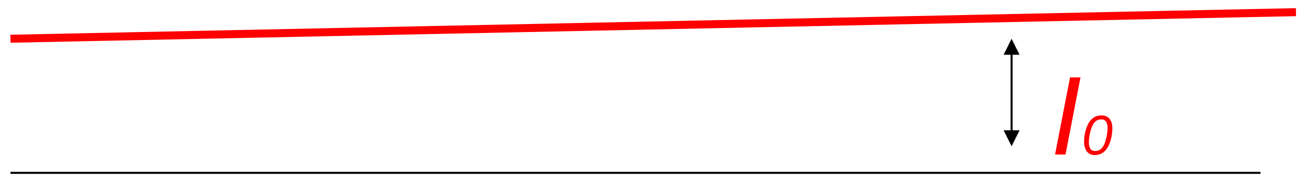
**Can we understand the dynamics of the HH model?**

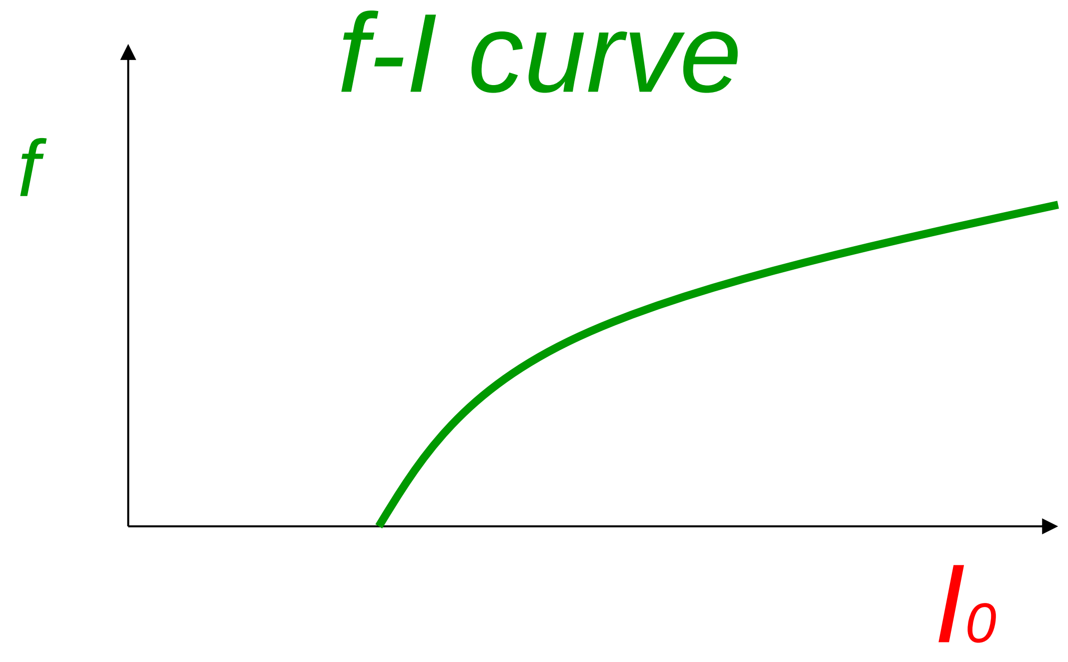
- mathematical principle of Action Potential generation?
- Types of neuron model (type I and II)?
- threshold behavior?

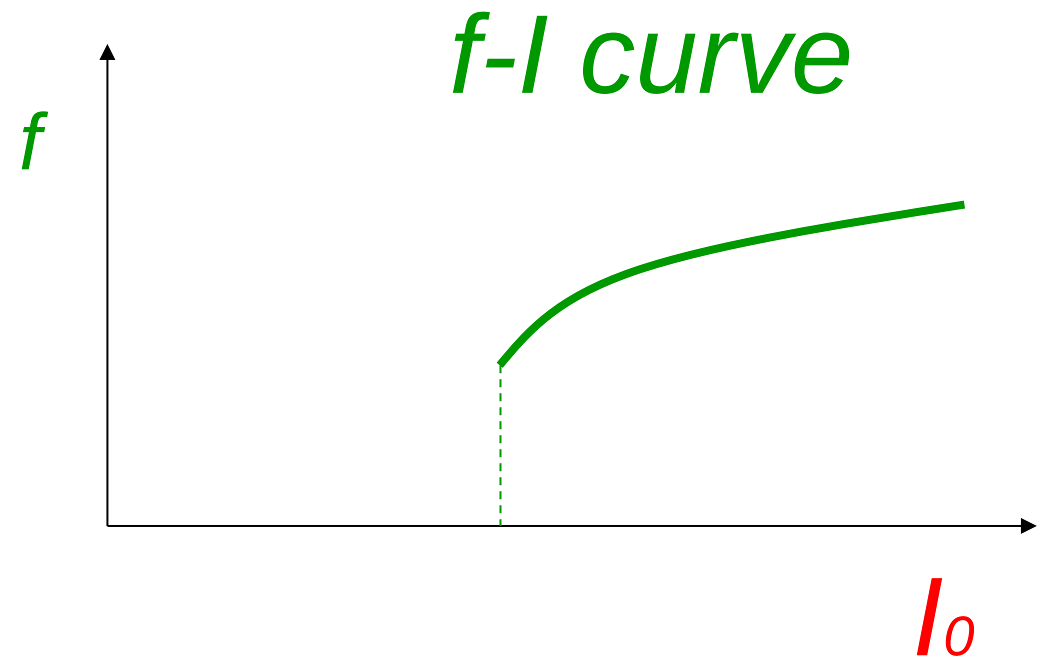
→ Reduce from 4 to 2 equations

Type I and type II models

ramp input/
constant input



# Neuronal Dynamics – 4.1. Overview and aims

## Can we understand the dynamics of the HH model?

→ Reduce from 4 to 2 equations

Type I and type II models

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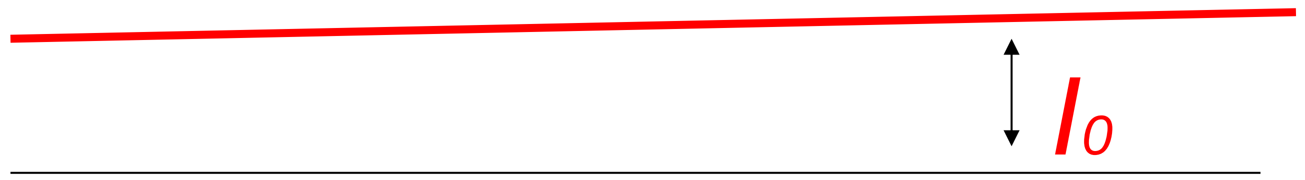
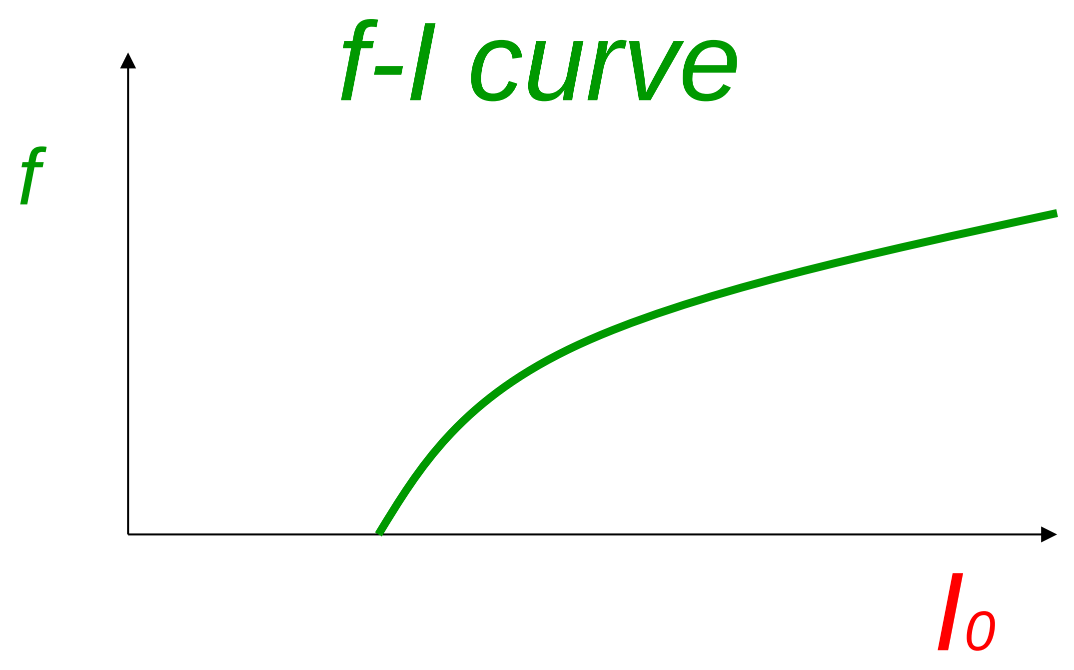
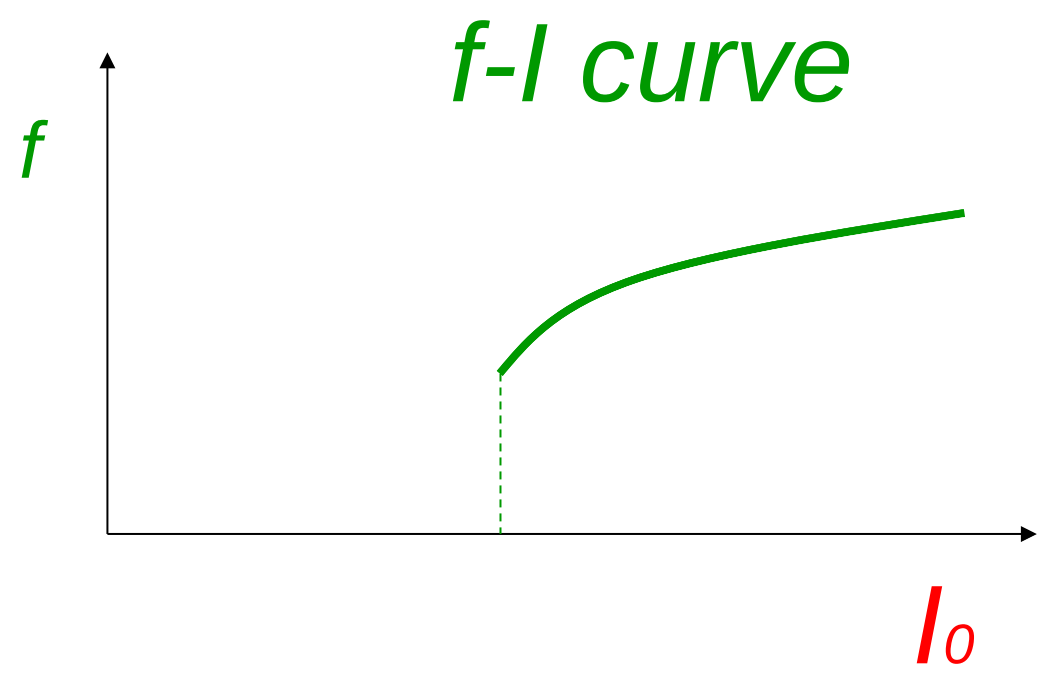


# Toward a two-dimensional neuron model

-**Reduction of Hodgkin-Huxley to 2 dimension**

-step 1: separation of time scales

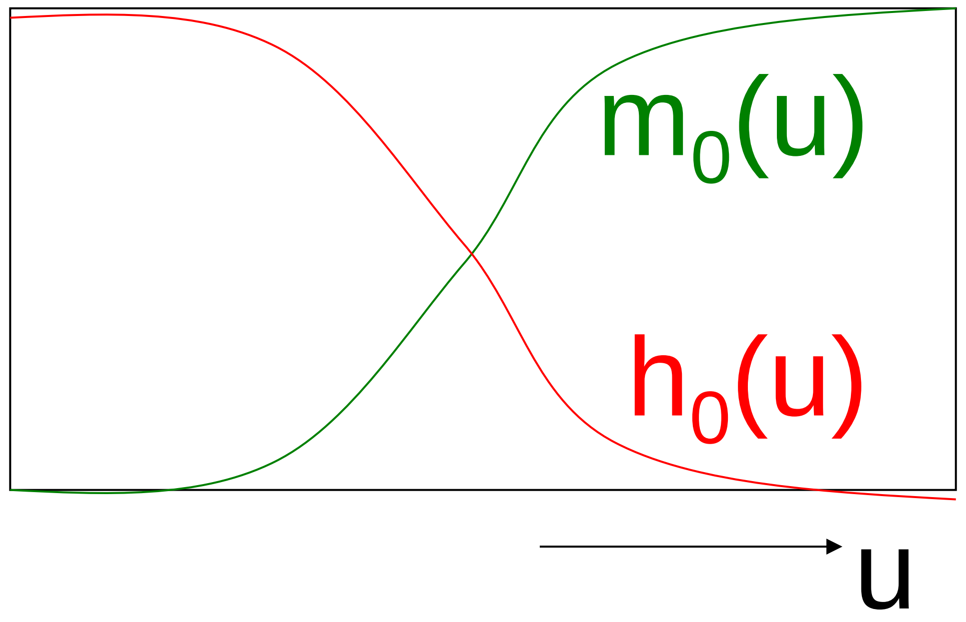
-step 2: exploit similarities/correlations

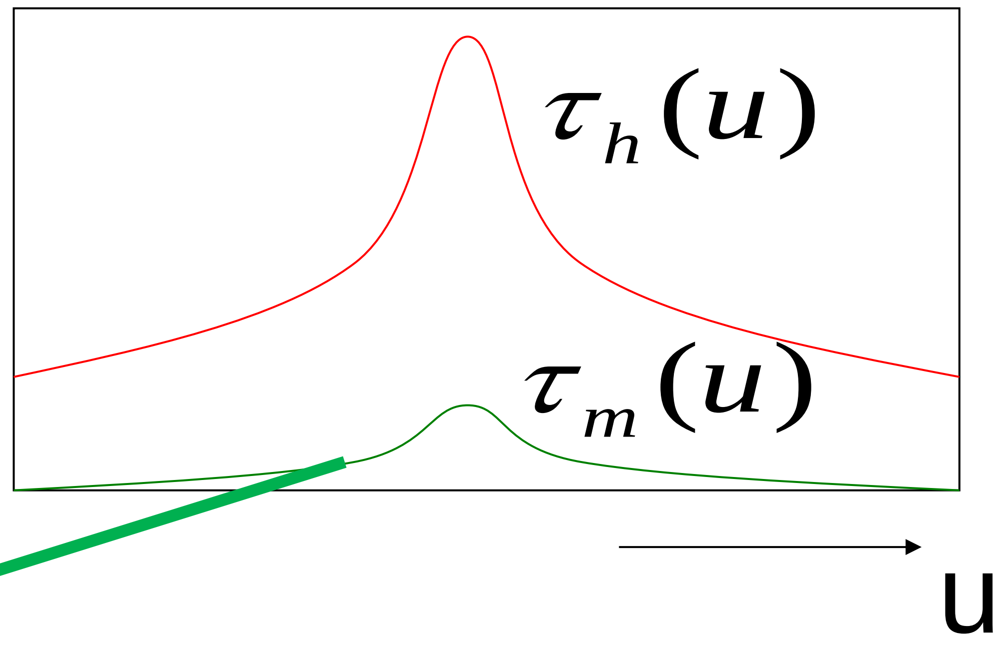
$$C\frac{du}{dt} = \underbrace{-g_{Na}m^3h(u-E_{Na})}_{I_{Na}} \underbrace{-g_K n^4(u-E_K)}_{I_K} \underbrace{-g_l(u-E_l)}_{I_{leak}} + \underbrace{I(t)}_{\text{stimulus}}$$

$$\frac{dm}{dt} = -\frac{m-m_0(u)}{\tau_m(u)}$$

$$\frac{dh}{dt} = -\frac{h-h_0(u)}{\tau_h(u)}$$

$$\frac{dn}{dt} = -\frac{n-n_0(u)}{\tau_n(u)}$$




MathDetour 4.1

1) dynamics of *m* is fast $\longrightarrow$ $m(t) = m_0(u(t))$

# Neuronal Dynamics – 4.1. Reduction of Hodgkin-Huxley model

$$C\frac{du}{dt} = \underbrace{-g_{Na}m^3h(u-E_{Na})}_{I_{Na}} \underbrace{-g_K n^4(u-E_K)}_{I_K} \underbrace{-g_l(u-E_l)}_{I_{leak}} + \underbrace{I(t)}_{\text{stimulus}}$$

$$\frac{dm}{dt} = -\frac{m - m_0(u)}{\tau_m(u)}$$

$$\frac{dh}{dt} = -\frac{h - h_0(u)}{\tau_h(u)}$$

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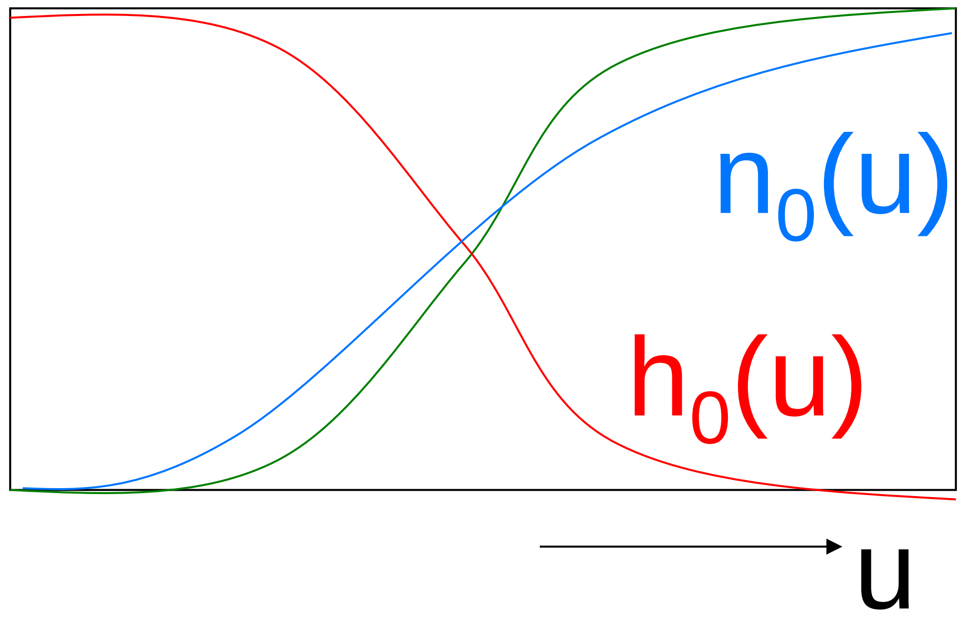
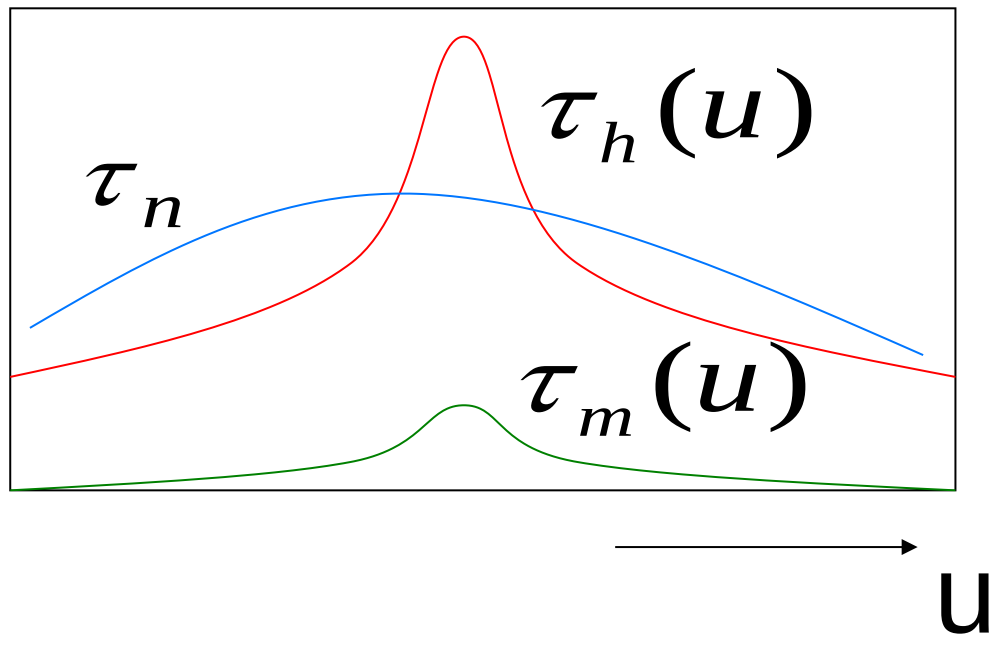



1) dynamics of *m* are fast → $m(t) = m_0(u(t))$

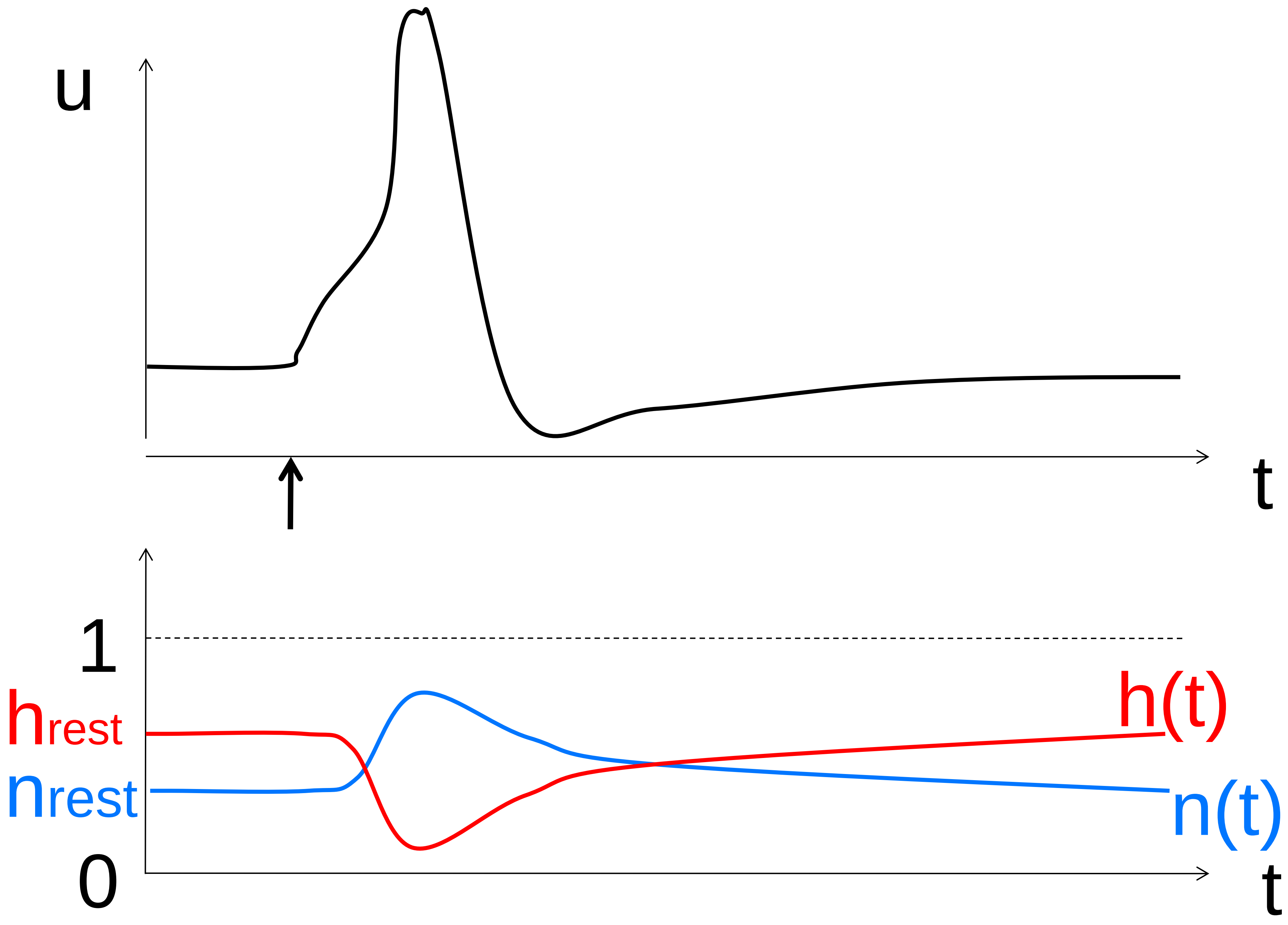
2) dynamics of *h* and *n* are similar

# Neuronal Dynamics – 4.1. Reduction of Hodgkin-Huxley model

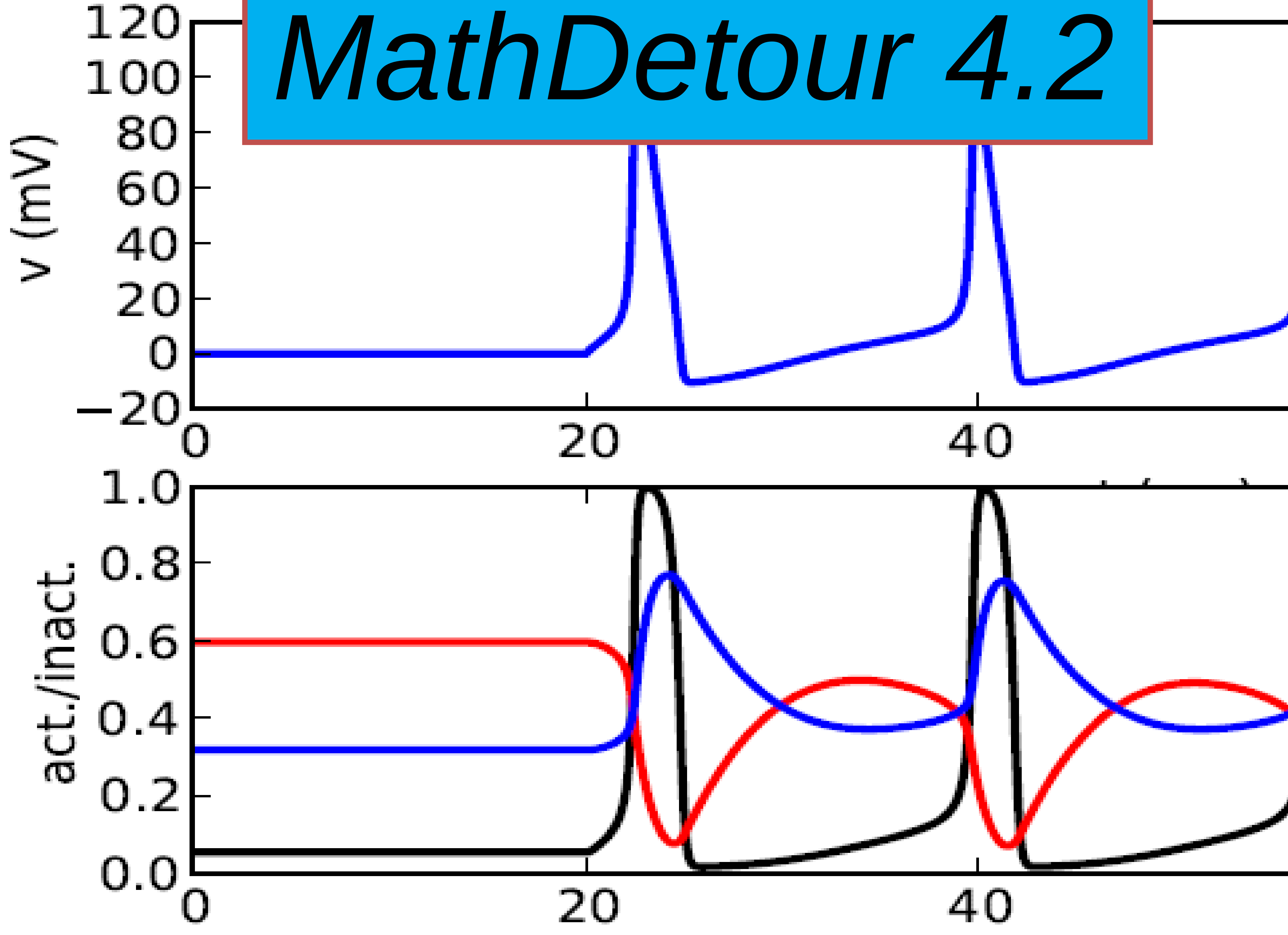
stimulus

$$C\frac{du}{dt} = -g_{Na}m^3h(u-E_{Na}) - g_K n^4(u-E_K) - g_l(u-E_l) + I(t)$$

2) dynamics of *h* and *n* are similar → $1-h(t) = a\,n(t)$




**MathDetour 4.2**



# Neuronal Dynamics – 4.1. Reduction of Hodgkin-Huxley model

$$C\frac{du}{dt} = \overbrace{-g_{Na}[m(t)]^3 h(t)(u(t)-E_{Na})}^{I_{Na}} \overbrace{-g_K[n(t)]^4(u(t)-E_K)}^{I_K} \overbrace{-g_l(u(t)-E_l)}^{I_{leak}} + I(t)$$

$$C\frac{du}{dt} = -g_{Na}\, m_0(u)^3(1-w)(u-E_{Na}) - g_K[\frac{w}{a}]^4(u-E_K) - g_l(u-E_l) + I(t)$$

1) dynamics of *m* are fast → $m(t) = m_0(u(t))$

2) dynamics of *h* and *n* are similar → $\underbrace{1-h(t)}_{w(t)} = \underbrace{a\,n(t)}_{w(t)}$

$$C\frac{du}{dt} = \underbrace{-g_{Na}\, m_0(u)^3(1-w)(u-E_{Na})}_{I_{Na}} \underbrace{- g_K(\frac{w}{a})^4(u-E_K)}_{I_K} \underbrace{- g_l(u-E_l)}_{I_{leak}} + I(t)$$

$$\frac{dw}{dt} = -\frac{w - w_0(u)}{\tau_{eff}(u)}$$

$$C\frac{du}{dt} = f(u(t), w(t)) + I(t)$$

$$\frac{dw}{dt} = g(u(t), w(t))$$

# Neuronal Dynamics – 4.1. Reduction to 2 dimensions

2-dimensional equation

$$C\frac{du}{dt} = f(u(t), w(t)) + I(t)$$

$$\frac{dw}{dt} = g(u(t), w(t))$$

Enables graphical analysis!

-Discussion of threshold

-Type I and II

- Repetitive firing

# Neuronal Dynamics – Quiz 4.1.

**A- Assumptions**: In order to reduce a detailed compartmental neuron model to two dimensions we have to assume that

[ ] dendrites can be approximated as passive
[ ] the neuron model has no dendrite
[ ] the neuron model has at most 2 types of ion channels
[ ] all gating variables are fast
[ ] no gating variable is fast
[ ] gating variables fall in two groups:
those that are fast and those that are slow
[ ] at least one of the ion channels is inactivating
[ ] the neuron does not generate spikes

**B - A biophysical point model** with 3 ion channels, each with activation and inactivation, has a total number of equations equal to

[ ] 3 or [ ] 4 or [ ] 6 or [ ] 7 ; [ ] 8 or more

**C- Separation of time scales**:

We start with two equations

$$\tau_1 \frac{dx}{dt} = -x + I(t)$$

$$\tau_2 \frac{dy}{dt} = -y + x^2 + A$$

We assume that $\tau_1 \ll \tau_2$

In this case a reduction of dimensionality

[ ] is not possible

[ ] is possible and the result is

$$\tau_2 \frac{dy}{dt} = -y + [I(t)]^2 + A$$

[ ] is possible and the result is

$$\tau_1 \frac{dx}{dt} = -x + x^2 + A$$

# Week 4 – part 1 : Separation of time scales

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# Neuronal Dynamics: Computational Neuroscience of Single Neurons

## Week 4 – Reducing detail: Two-dimensional neuron models

Wulfram Gerstner

EPFL, Lausanne, Switzerland

4.1 **From Hodgkin-Huxley to 2D**
- Overview: From 4 to 2 dimensions
- MathDetour 1: Separation of time scales
- MathDetour 2: Exploiting similarities

4.2 **Phase Plane Analysis**
- role of nullclines

4.3 **Analysis of a 2D Neuron Model**
- MathDetour 3: Stability of fixed points

4.4 **Type I and II Neuron Models**
- where is the firing threshold?
- separation of time scales

4.5. **Nonlinear Integrate-and-fire**
- from two to one dimension

# Neuronal Dynamics – MathDetour 4.1: Separation of time scales

Two coupled differential equations

$$\tau_1 \frac{dx}{dt} = -x + c(t)$$

$$\tau_2 \frac{dy}{dt} = f(y) + g(x)$$

Separation of time scales

$$\tau_1 \ll \tau_2$$

Reduced 1-dimensional system

$$\tau_2 \frac{dy}{dt} = f(y) + g(c(t))$$

# Neuronal Dynamics – MathDetour 4.1: Separation of time scales

Two coupled differential equations

$$\tau_1 \frac{dx}{dt} = -x + c(t)$$

$$\tau_2 \frac{dy}{dt} = f(y) + g(x)$$

Separation of time scales

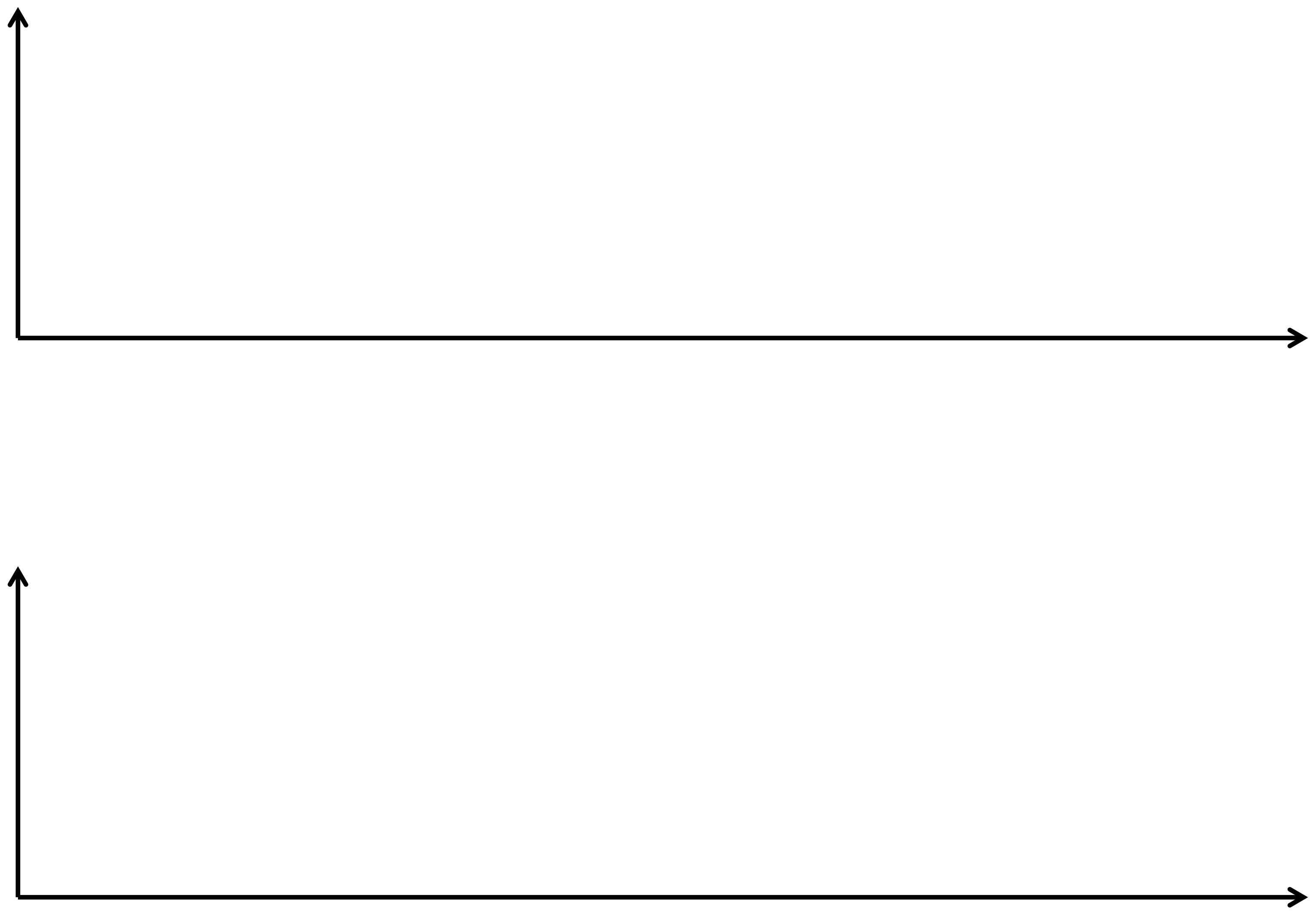
$$\tau_1 \ll \tau_2$$

## Reduced 1-dimensional system

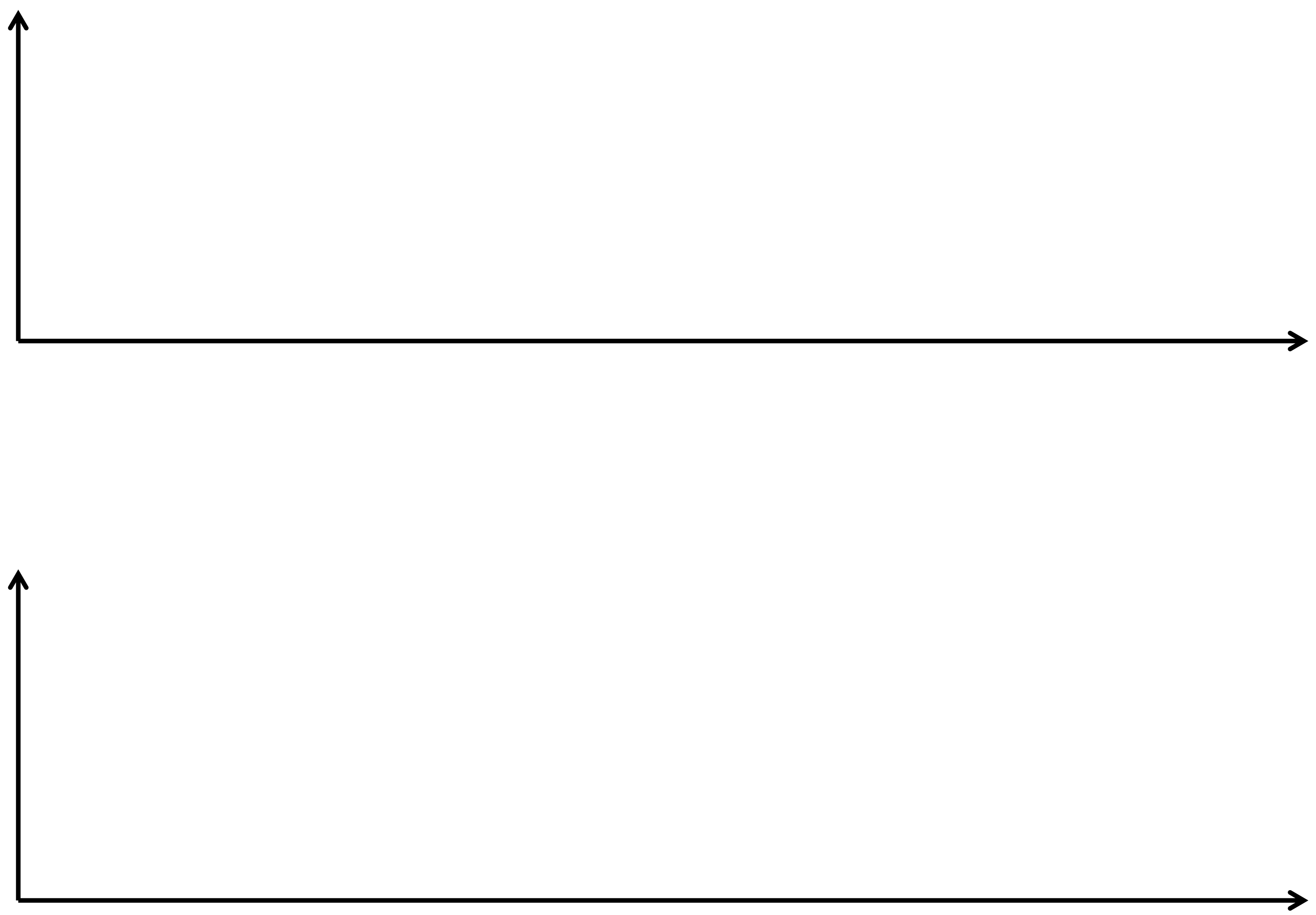
$$\tau_2 \frac{dy}{dt} = f(y) + g(c(t))$$

# Neuronal Dynamics – MathDetour 4.1: Separation of time scales

Linear differential equation $$\tau_1 \frac{dx}{dt} = -x + c(t)$$



step



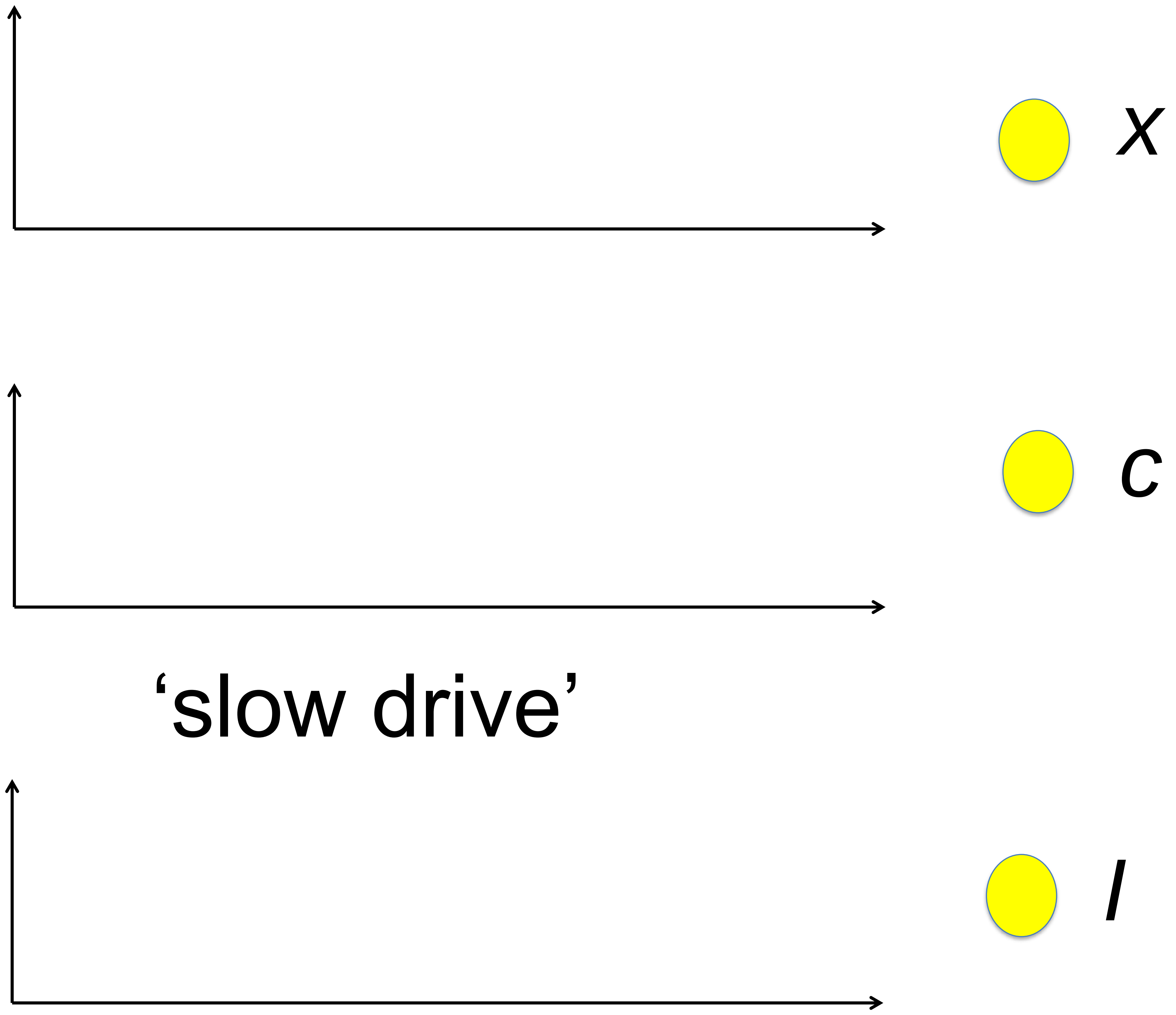
‘slow drive’

Two differential equations

$$\tau_1 \frac{dx}{dt} = -x + c(t)$$

$$\tau_2 \frac{dc}{dt} = -c + I(t)$$

$$\tau_1 \ll \tau_2$$



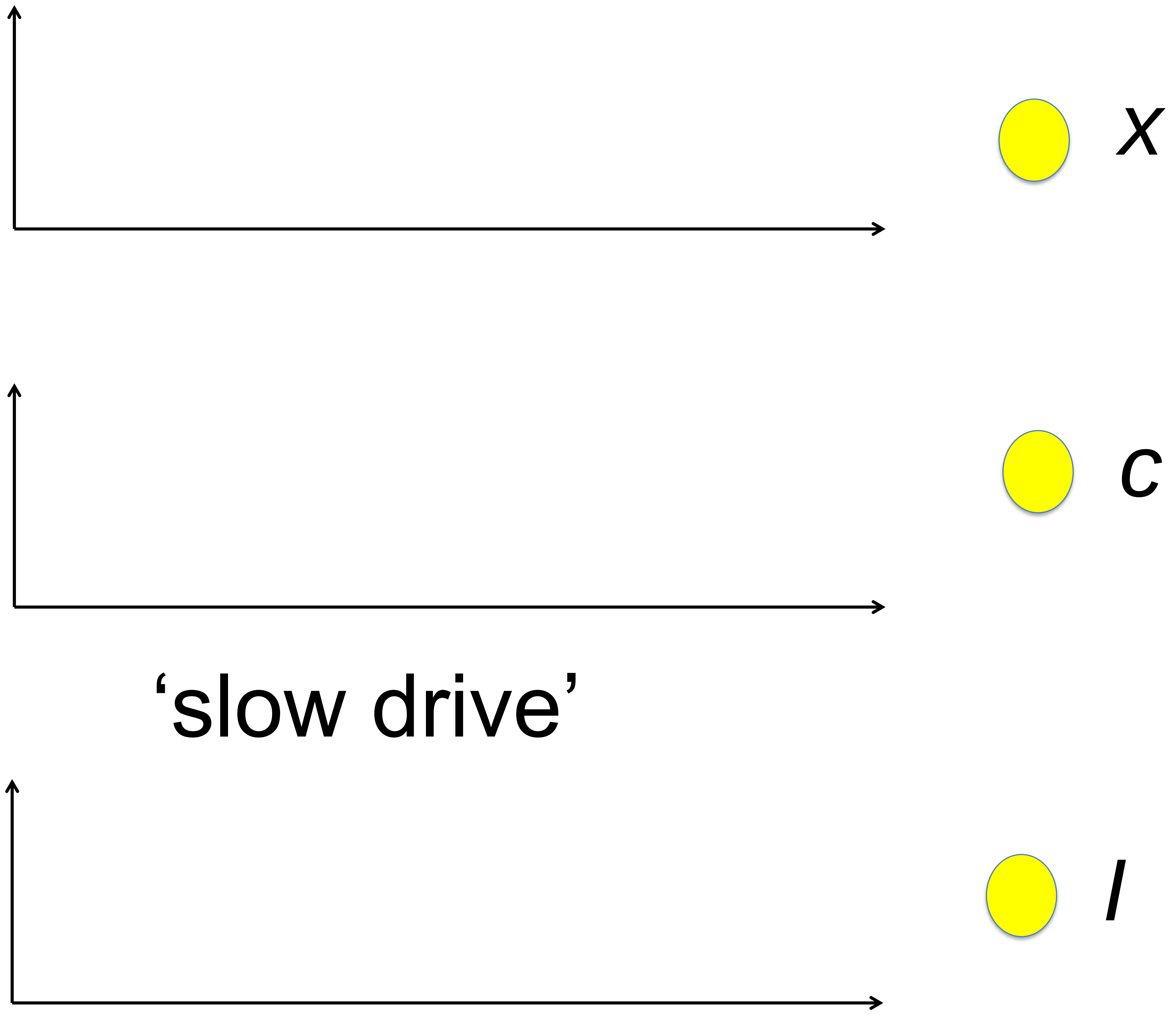
# Neuronal Dynamics – MathDetour 4.1: Separation of time scales

Two coupled differential equations

$$\tau_1 \frac{dx}{dt} = -x + c(t)$$

$$\tau_2 \frac{dc}{dt} = -c + f(x) \quad + I(t)$$

$$\tau_1 \ll \tau_2$$



# Neuronal Dynamics – Reduction of Hodgkin-Huxley model

stimulus

$$C\frac{du}{dt} = \underbrace{-g_{Na}m^3h(u-E_{Na})}_{I_{Na}} \underbrace{-g_K n^4(u-E_K)}_{I_K} \underbrace{-g_l(u-E_l)}_{I_{leak}} + I(t)$$

$$\frac{dm}{dt} = -\frac{m-m_0(u)}{\tau_m(u)}$$

$$\frac{dh}{dt} = -\frac{h-h_0(u)}{\tau_h(u)}$$

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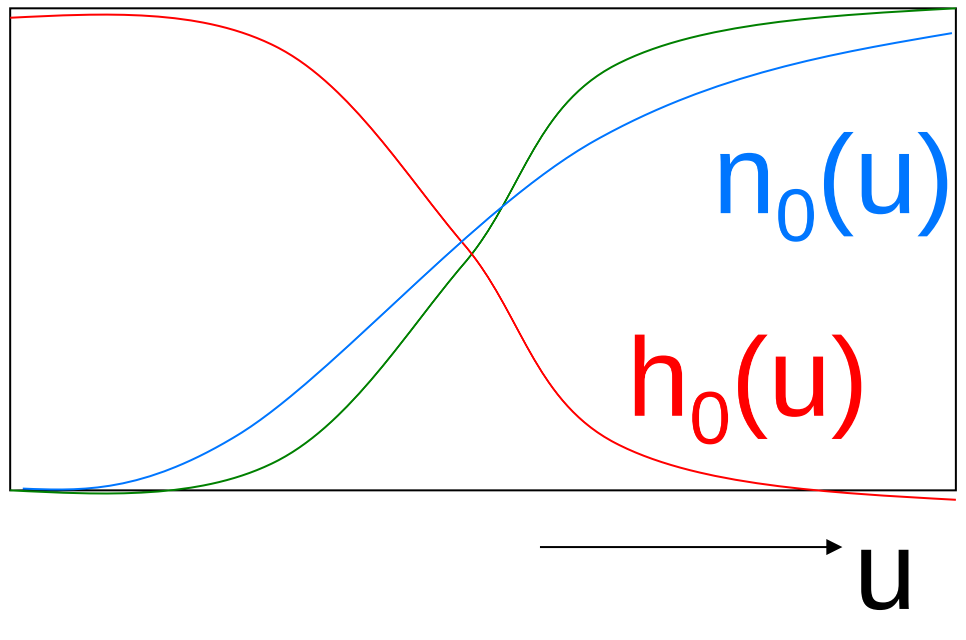


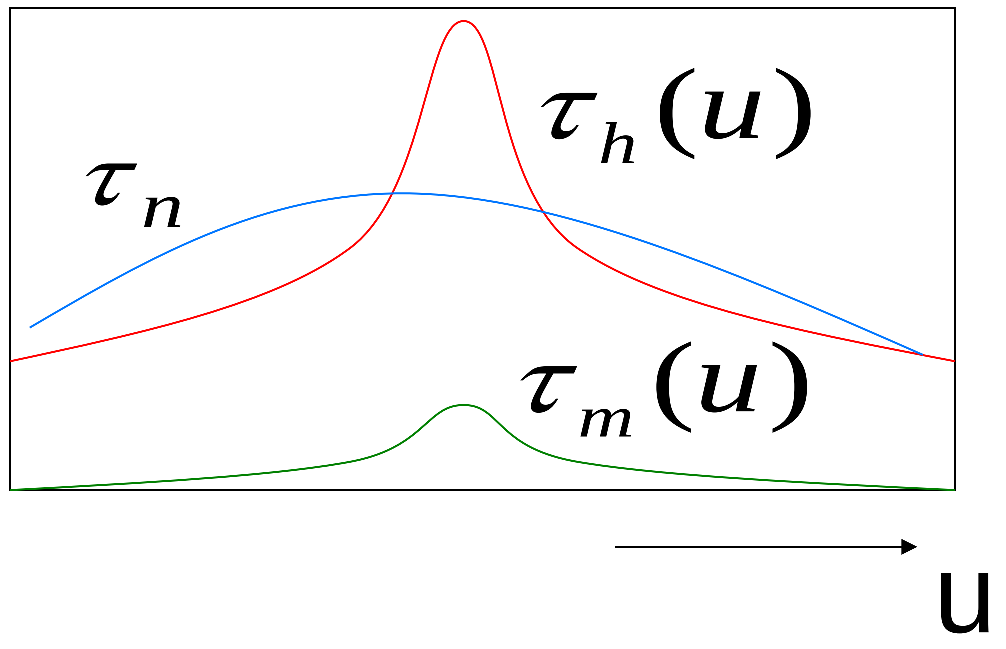


dynamics of *m* is fast → $m(t) = m_0(u(t))$

*Fast compared to what?*

# Neuronal Dynamics – MathDetour 4.1: Separation of time scales

Two coupled differential equations

$$\tau_1 \frac{dx}{dt} = -x + h(y)$$

$$\tau_2 \frac{dy}{dt} = f(y) + g(x)$$

Separation of time scales

$$\tau_1 \ll \tau_2 \quad \rightarrow x = h(y)$$

Reduced 1-dimensional system

$$\tau_2 \frac{dy}{dt} = f(y) + g(h(y))$$

# Neuronal Dynamics – Quiz 4.2.

**A- Separation of time scales**:

We start with two equations

$$\tau_1 \frac{dx}{dt} = -x + y + I(t)$$

$$\tau_2 \frac{dy}{dt} = -y + x^2 + A$$

[ ] If $\tau_1 \square \tau_2$ then the system can be reduded to

$$\tau_2 \frac{dy}{dt} = -y + [y + I(t)]^2 + A$$

[ ] If $\tau_2 \square \tau_1$ then the system can be reduded to

$$\tau_1 \frac{dx}{dt} = -x + x^2 + A + I(t)$$

[ ] None of the above is correct.

**B- Separation of time scales**:

A channel with gating variable *r*, given by

$$\tau_1 \frac{dr}{dt} = -r + r_0(u)$$

influences the voltage

$$\tau_2 \frac{du}{dt} = -(u - u_0) + r^2 A$$

We assume that $\tau_1 \square \tau_2$

In this case a reduction of dimensionality

[ ] is not possible

[ ] is possible and the result is

$$\tau_2 \frac{du}{dt} = -u + u_0 + [r_0(u)]^2 A$$

[ ] is possible and the result is

$$\tau_1 \frac{dr}{dt} = -r + r_0(u_0 + r^2 A)$$

# Week 4 – MathDetour 2: Exploiting similarities

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

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## Week 4 – Reducing detail: Two-dimensional neuron models

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- Overview: From 4 to 2 dimensions
- MathDetour 1: Separation of time scales
- MathDetour 2: Exploiting similarities

**4.2 Phase Plane Analysis**
- role of nullclines
- MathDetour 3: Stability of fixed points

**4.3 Analysis of a 2D Neuron Model**

**4.4 Type I and II Neuron Models**
- where is the firing threshold?
- separation of time scales

**4.5. Nonlinear Integrate-and-fire**
- from two to one dimension

## Reduction of Hodgkin-Huxley Model to 2 Dimension

-step 1:
separation of time scales
(→ 4.1 and 4-Detour1)

-step 2:
exploit similarities/correlations

*Now !*

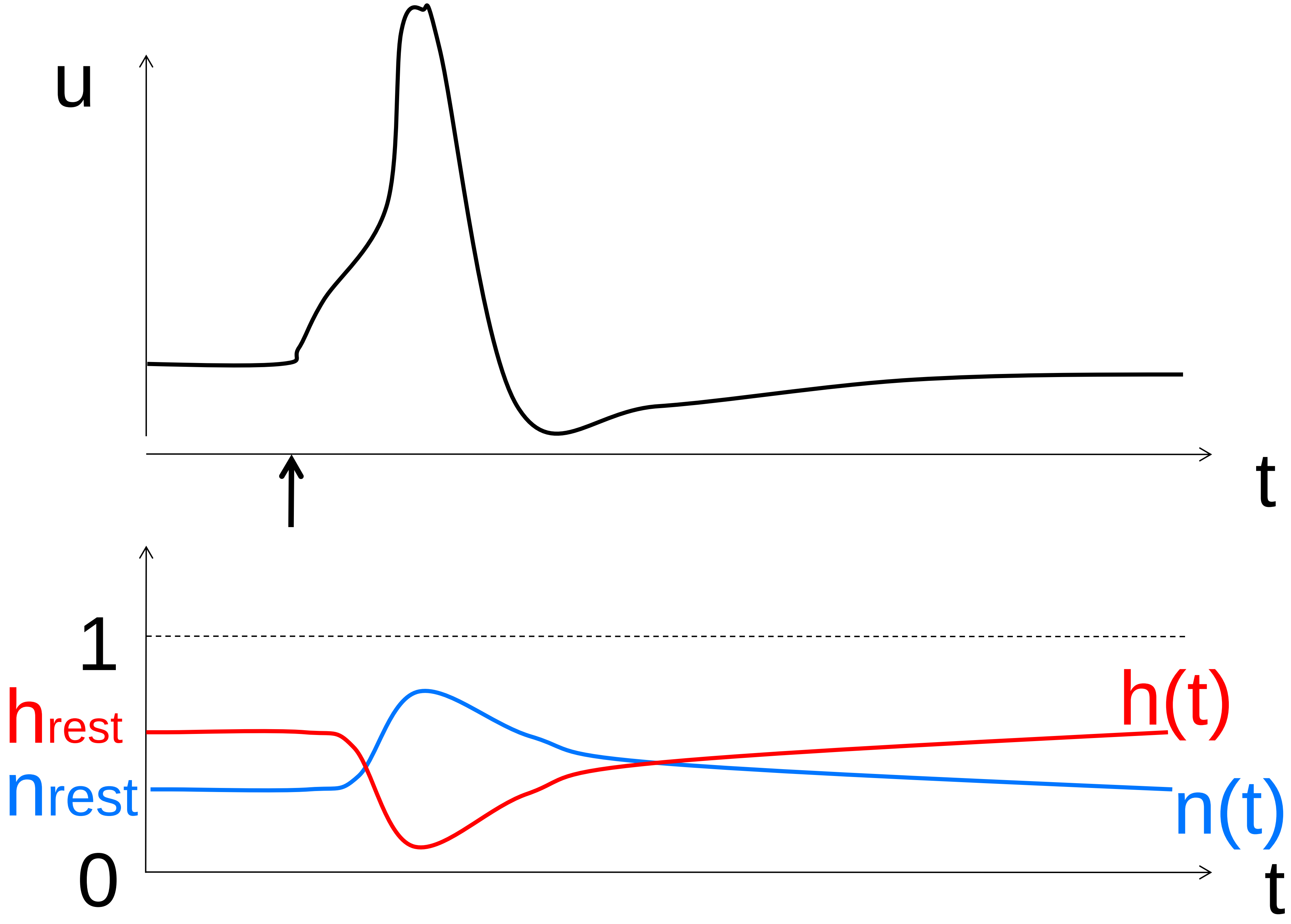
# Neuronal Dynamics – Detour 4.2. Exploit similarities/correlations

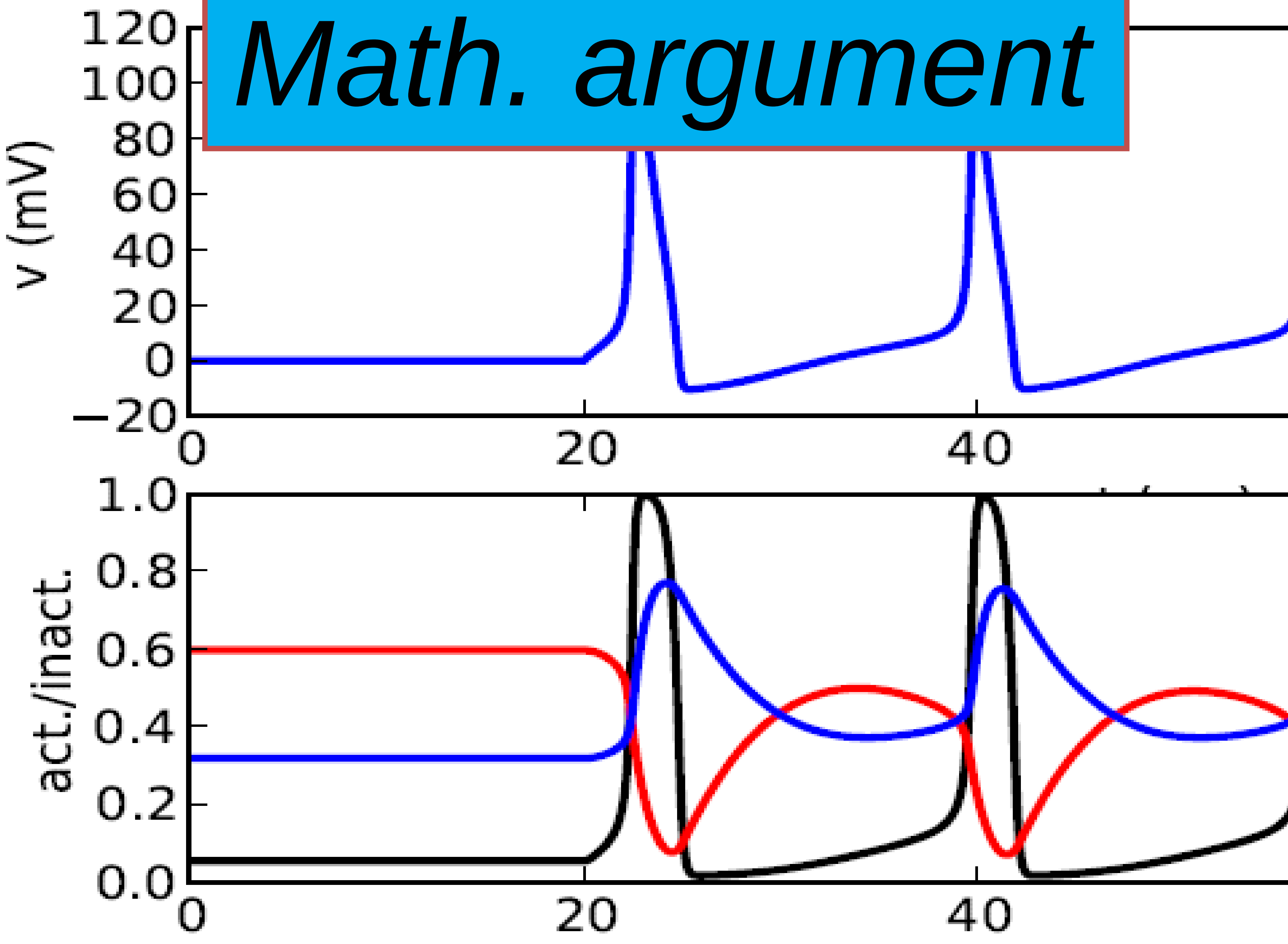
stimulus

$$C\frac{du}{dt} = -g_{Na}m^3h(u-E_{Na}) - g_K n^4(u-E_K) - g_l(u-E_l) + I(t)$$

dynamics of *h* and *n* is similar → $1-h(t) = a\,n(t)$

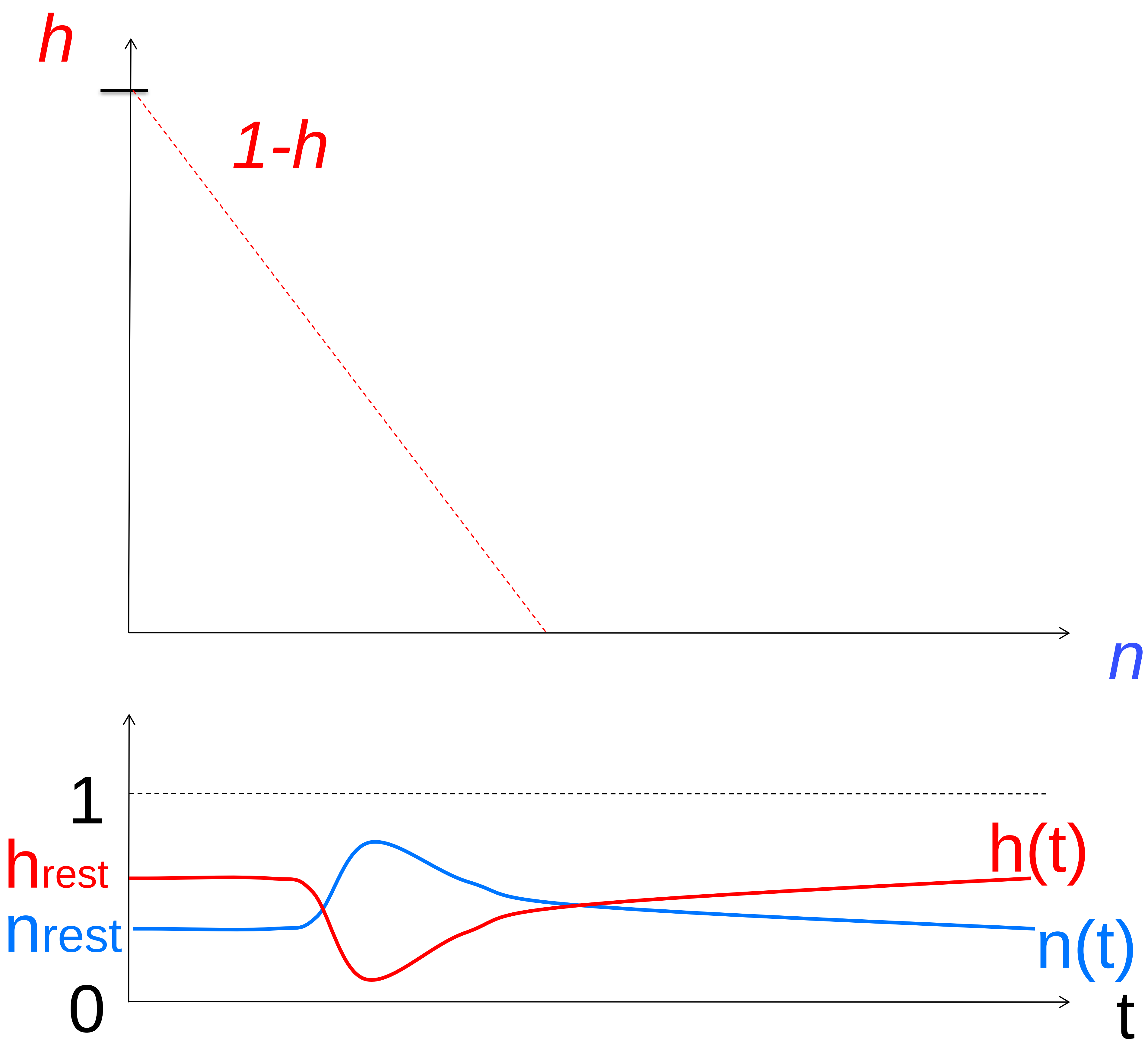
**Math. argument**

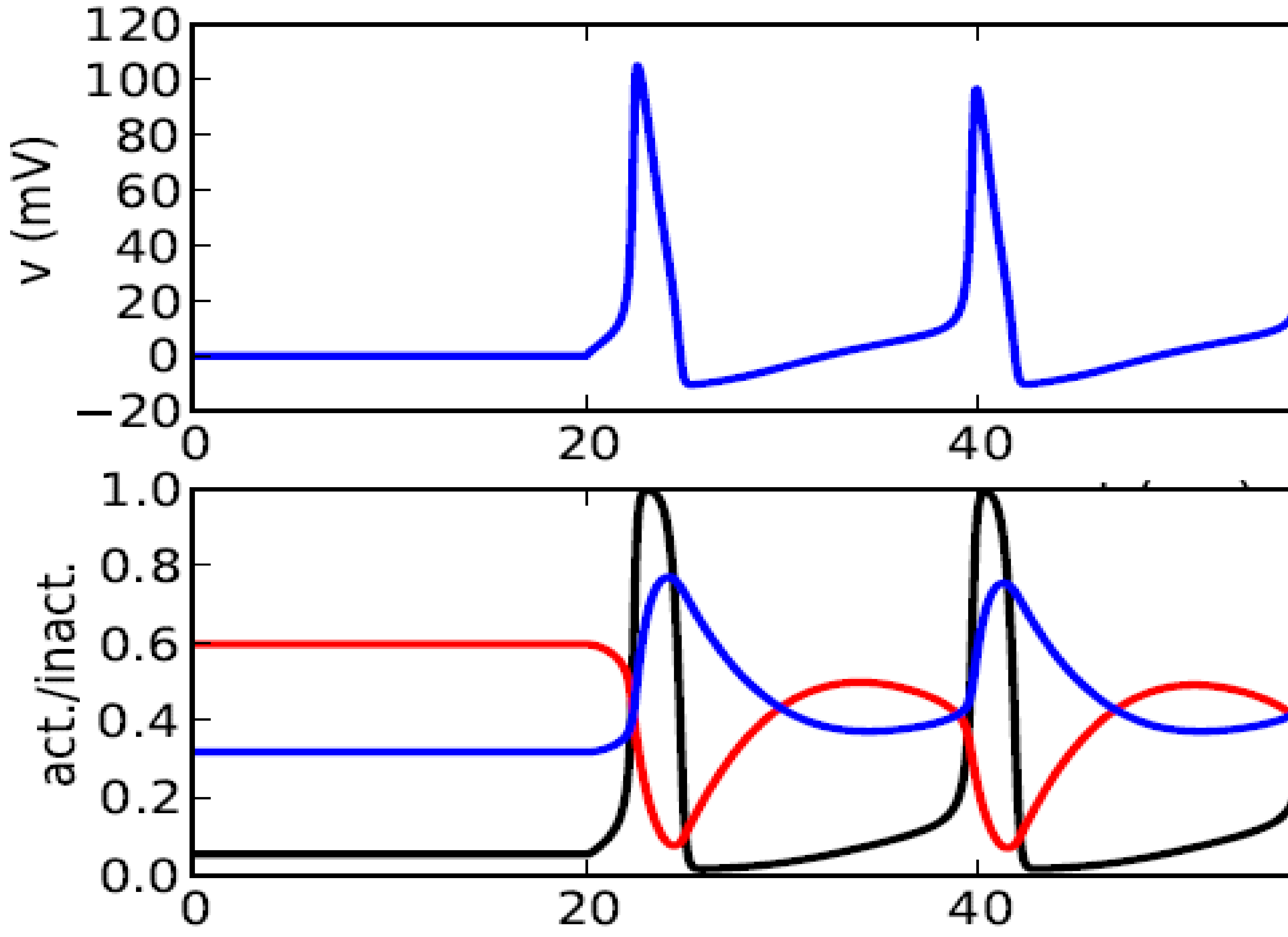


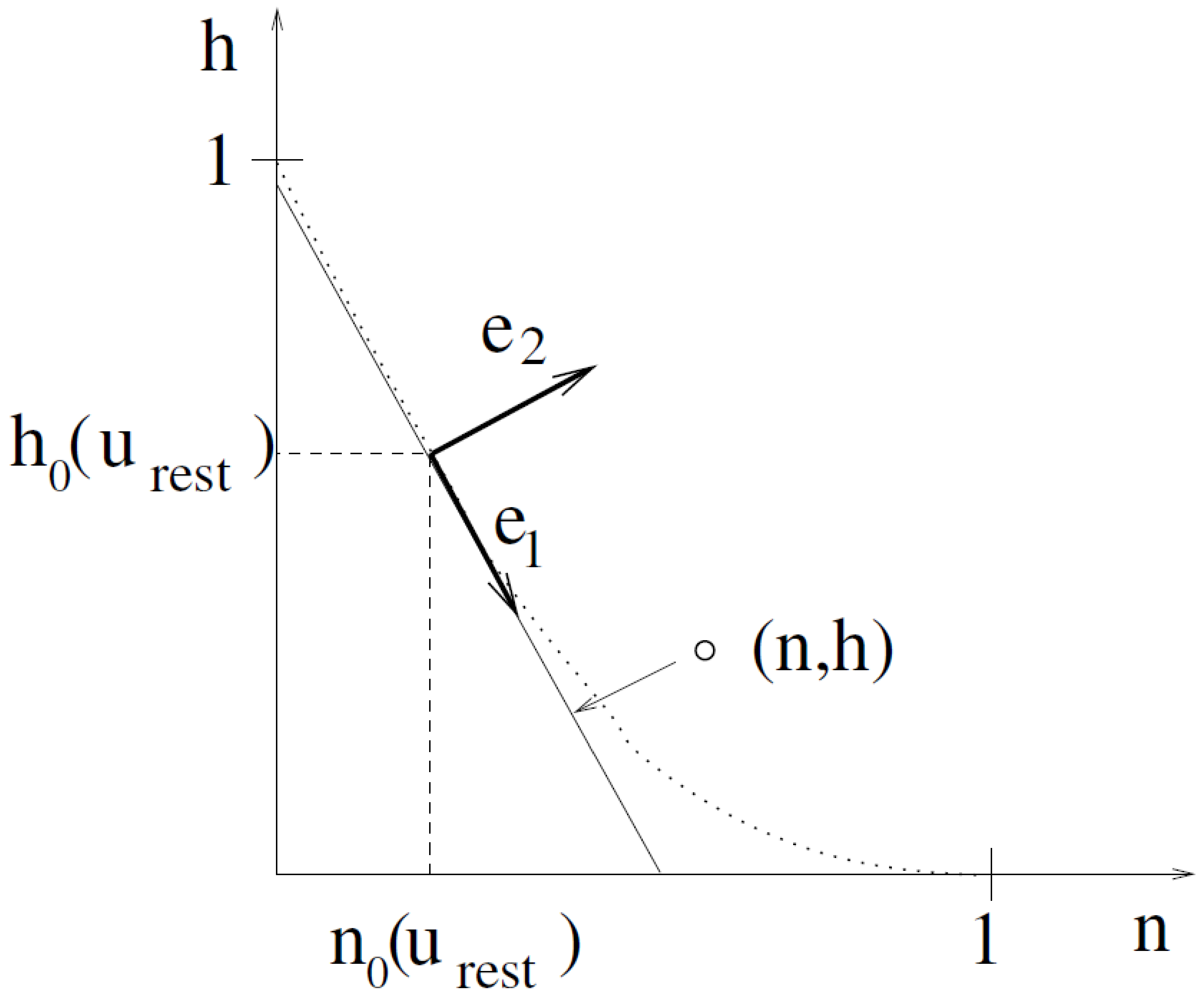



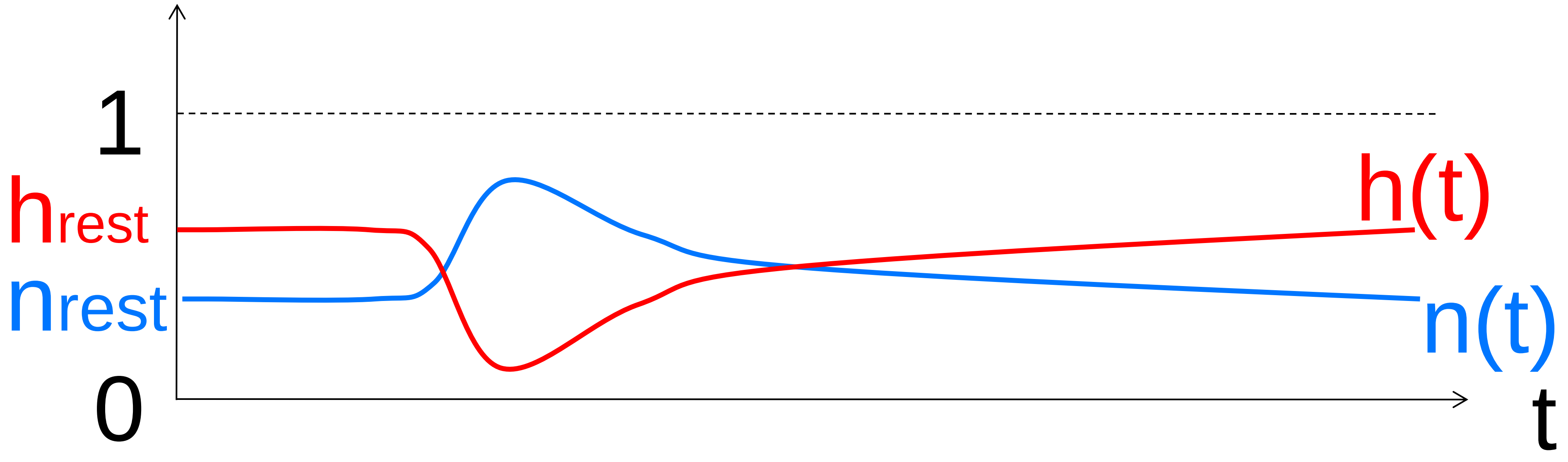
dynamics of *h* and *n* are similar

$$1 - h(t) = a\, n(t)$$






dynamics of *h* and *n* are similar

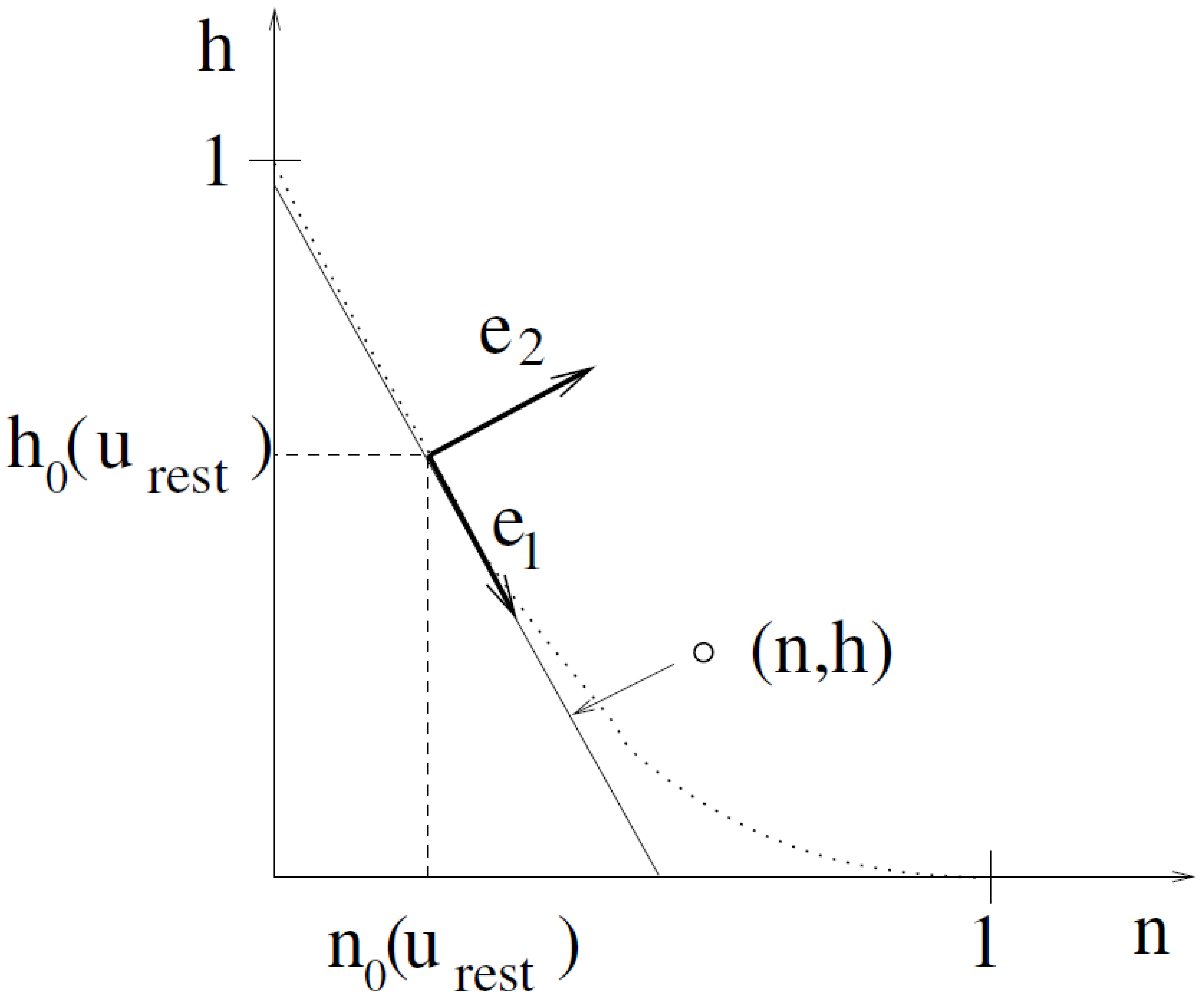
$$1 - h(t) = a\, n(t)$$

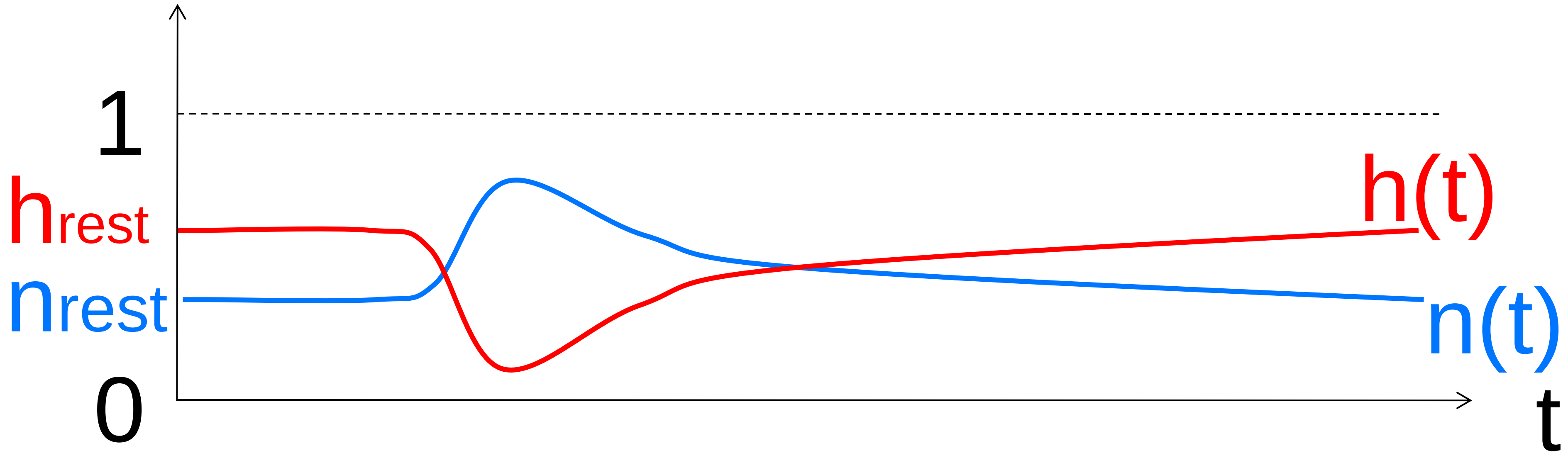
*at rest*

$$\frac{dh}{dt} = -\frac{h - h_0(u)}{\tau_h(u)}$$

$$\frac{dn}{dt} = -\frac{n - n_0(u)}{\tau_n(u)}$$

# Neuronal Dynamics – Detour 4.2. Exploit similarities/correlations




dynamics of *h* and *n* are similar

(i) Rotate coordinate system
(ii) Suppress one coordinate
(iii) Express dynamics in new variable

$$1 - h(t) = a\, n(t) = w(t)$$

$$\frac{dh}{dt} = -\frac{h - h_0(u)}{\tau_h(u)}$$

$$\frac{dn}{dt} = -\frac{n - n_0(u)}{\tau_n(u)}$$

$$\frac{dw}{dt} = -\frac{w - w_0(u)}{\tau_{eff}(u)}$$

# Neuronal Dynamics – 4.1. Reduction of Hodgkin-Huxley model

$$C\frac{du}{dt} = \underbrace{-g_{Na}[m(t)]^3 h(t)(u(t)-E_{Na})}_{I_{Na}} \underbrace{-g_K[n(t)]^4(u(t)-E_K)}_{I_K} \underbrace{-g_l(u(t)-E_l)}_{I_{leak}} + I(t)$$

$$C\frac{du}{dt} = -g_{Na}\, m_0(u)^3(1-w)(u-E_{Na}) - g_K[\frac{w}{a}]^4(u-E_K) - g_l(u-E_l) + I(t)$$

1) dynamics of *m* are fast $\longrightarrow$ $m(t) = m_0(u(t))$

2) dynamics of *h* and *n* are similar $\longrightarrow$ $\underbrace{1-h(t)}_{w(t)} = \underbrace{a\,n(t)}_{w(t)}$

$$\frac{dh}{dt} = -\frac{h-h_0(u)}{\tau_h(u)}$$

$$\frac{dn}{dt} = -\frac{n-n_0(u)}{\tau_n(u)}$$

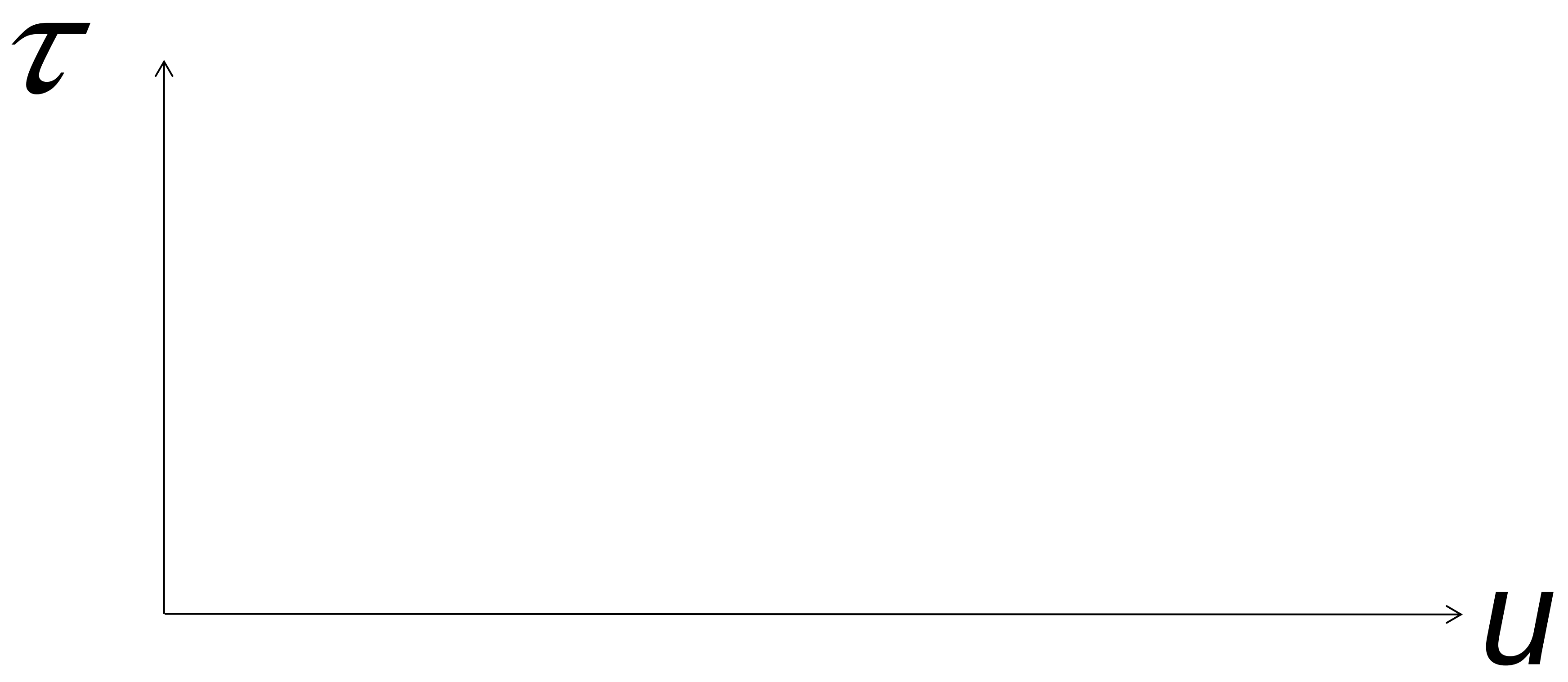
$\longrightarrow$

$$\frac{dw}{dt} = -\frac{w-w_0(u)}{\tau_{eff}(u)}$$

# Neuronal Dynamics – 4.1. Reduction of Hodgkin-Huxley model

$$C\frac{du}{dt} = \overbrace{-g_{Na}\, m_0(u)^3(1-w)(u-E_{Na})}^{I_{Na}} \overbrace{-\, g_K(\frac{w}{a})^4(u-E_K)}^{I_K} \overbrace{-\, g_l(u-E_l)}^{I_{leak}} + I(t)$$

$$\frac{dw}{dt} = -\frac{w - w_0(u)}{\tau_{eff}(u)}$$




$$\tau\frac{du}{dt} = F(u(t), w(t)) + R\,I(t)$$

$$\tau_w\frac{dw}{dt} = G(u(t), w(t))$$

2-dimensional equation

$$\tau \frac{du}{dt} = F(u,w) + R\,I(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

Enables graphical analysis!

-Discussion of threshold
-Repetitive firing
-Type I and II

# Neuronal Dynamics – Quiz 4.3.

Exploiting similarities:

A sufficient condition to replace two gating variables *r*,*s* by a single gating variable *w* is

[ ] Both *r* and *s* have the same time constant (as a function of u)

[ ] Both *r* and *s* have the same activation function

[ ] Both *r* and *s* have the same time constant (as a function of u) AND the same activation function

[ ] Both *r* and *s* have the same time constant (as a function of u) AND activation functions that are identical after some additive rescaling

[ ] Both *r* and *s* have the same time constant (as a function of u) AND activation functions that are identical after some multiplicative rescaling

# Week 4 – part 2: Phase Plane Analysis

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

# Neuronal Dynamics:
## Computational Neuroscience of Single Neurons

## Week 4 – Reducing detail:
## Two-dimensional neuron models

Wulfram Gerstner

EPFL, Lausanne, Switzerland

√ **4.1 From Hodgkin-Huxley to 2D**
- Overview: From 4 to 2 dimensions
- MathDetour 1: Separation of time scales
- MathDetour 2: Exploiting similarities

**4.2 Phase Plane Analysis**
- Role of nullcline

**4.3 Analysis of a 2D Neuron Model**
- MathDetour 3: Stability of fixed points

**4.4 Type I and II Neuron Models**
- where is the firing threshold?
- separation of time scales

**4.5. Nonlinear Integrate-and-fire**
- from two to one dimension

2-dimensional equation

stimulus

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

Enables graphical analysis!

-Discussion of threshold

-Type I and II

$$C\frac{du}{dt} = \underbrace{-g_{Na}\, m_0(u)^3(1-w)(u-E_{Na})}_{I_{Na}} \underbrace{- g_K(\frac{w}{a})^4(u-E_K)}_{I_K} \underbrace{- g_l(u-E_l)}_{I_{leak}} + I(t)$$

$$\frac{dw}{dt} = -\frac{w - w_0(u)}{\tau_w(u)}$$

stimulus

$$\tau\frac{du}{dt} = F(u,w) + RI(t)$$

$$\tau_w\frac{dw}{dt} = G(u,w)$$

# Neuronal Dynamics – 4.2. Nullclines of reduced HH model

stimulus

$$\tau \frac{du}{dt} = F(u,w) + I(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

there is a factor *R* missing here ( assume *R=1*)

u-nullcline

w-nullcline

# Neuronal Dynamics – 4.2. Nullclines of reduced HH model

stimulus

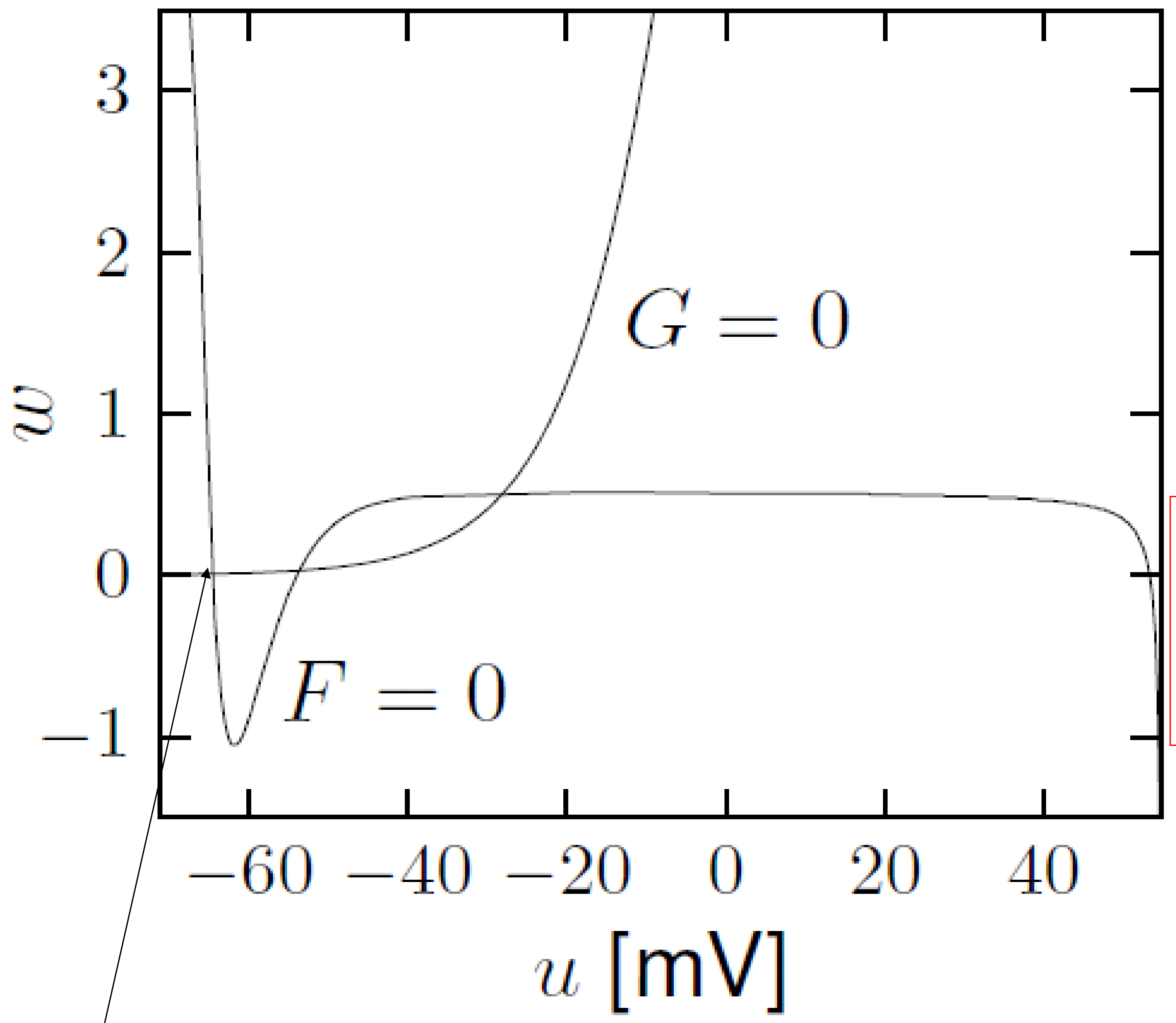
$$\tau \frac{du}{dt} = F(u,w) + I(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

u-nullcline

w-nullcline

$$\frac{dw}{dt} = 0$$




$$\frac{du}{dt} = 0$$

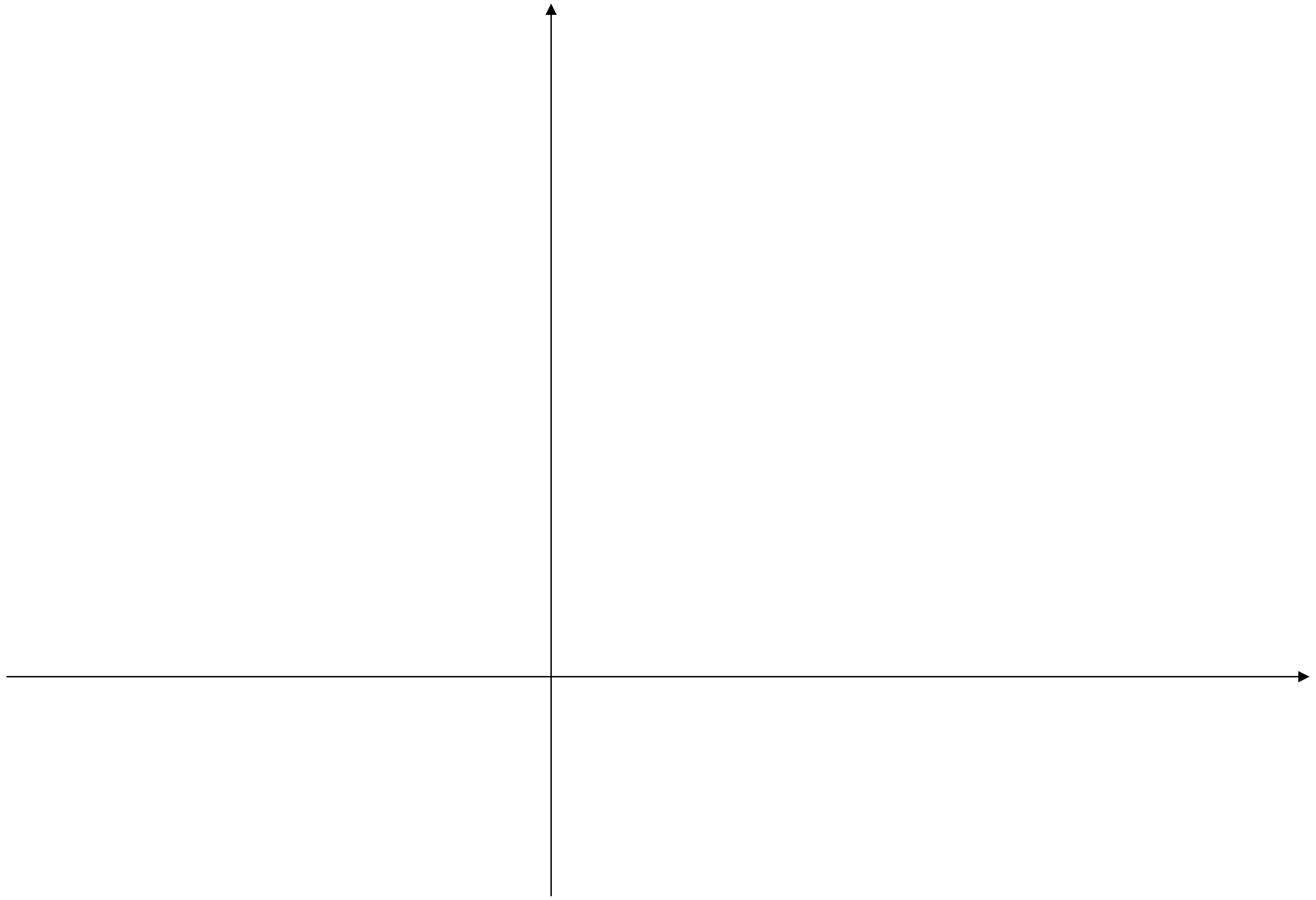
Stable fixed point

# Neuronal Dynamics – 4.2. FitzHugh-Nagumo Model

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$= u - \frac{1}{3}u^3 + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w) = b_0 + b_1 u - w$$



u-nullcline

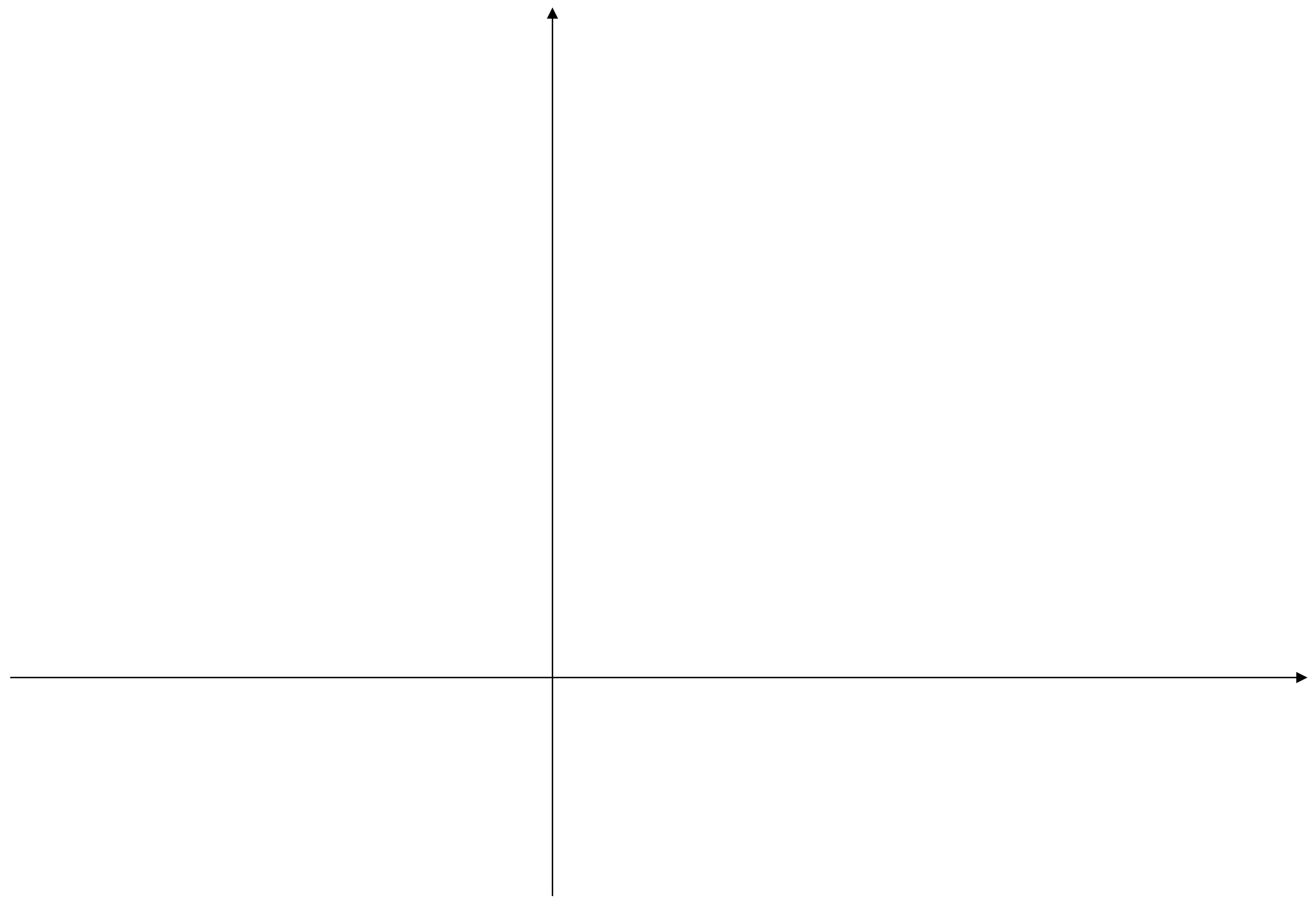
w-nullcline

# Neuronal Dynamics – 4.2. flow arrows

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$= u - \frac{1}{3}u^3 + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w) = b_0 + b_1 u - w$$



u-nullcline

w-nullcline

# Neuronal Dynamics – 4.2. flow arrows

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

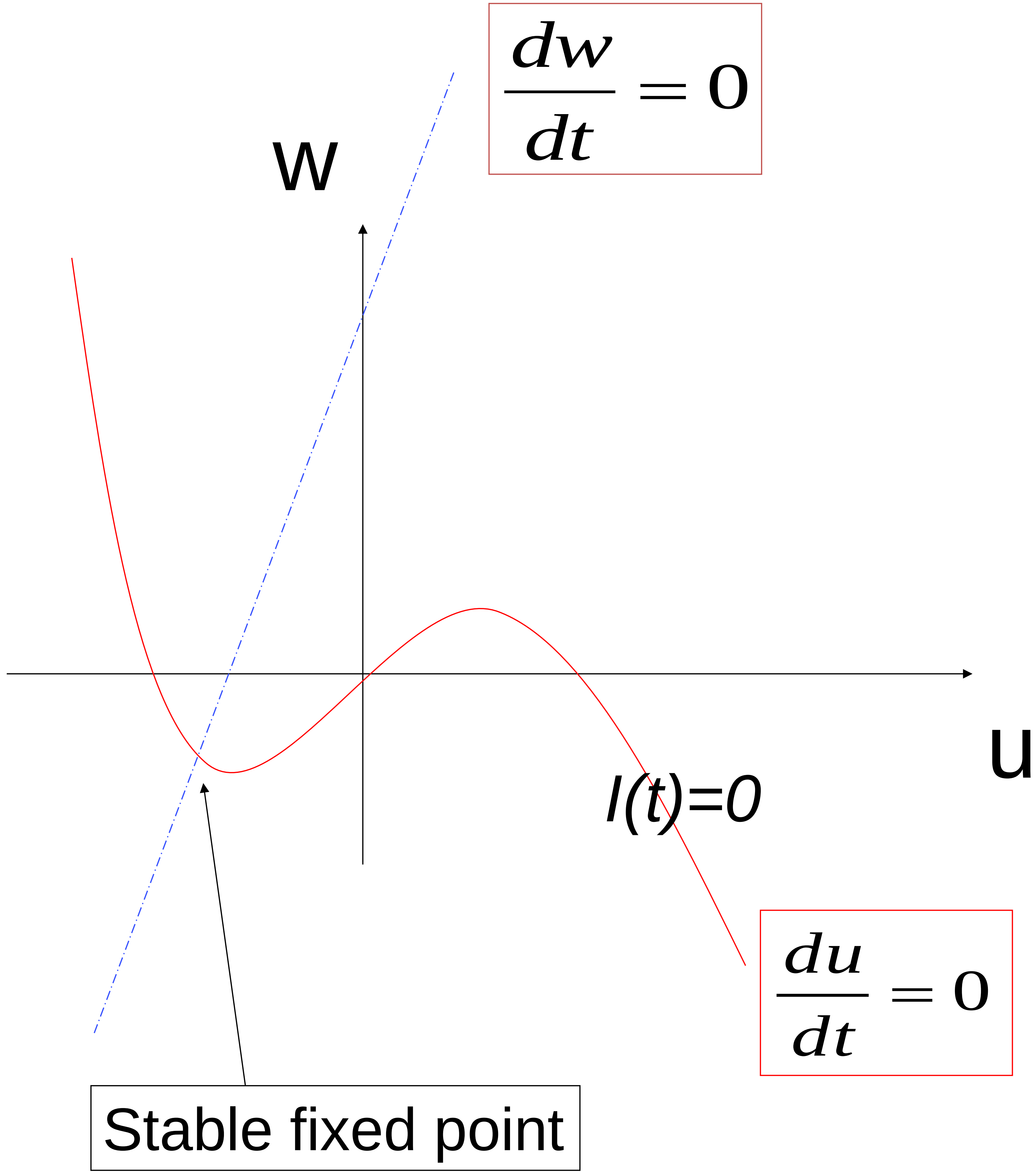
Stimulus I=0

$$\tau_w \frac{dw}{dt} = G(u,w)$$

Consider change in small time step

Flow on nullcline

Flow in reginons between nullclines

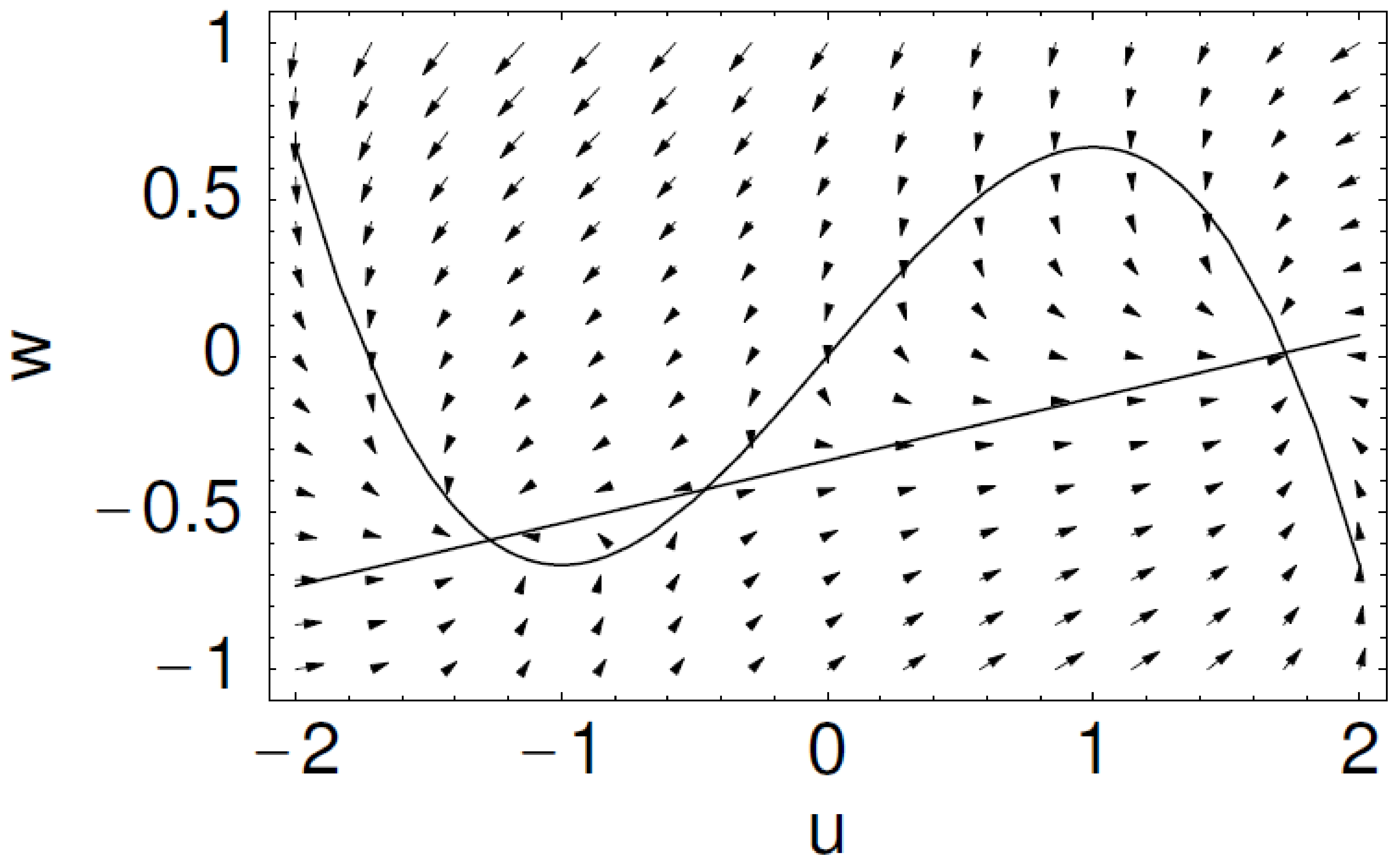


# Neuronal Dynamics – 4.2. FitzHugh-Nagumo Model

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$= u - \frac{1}{3}u^3 + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w) = b_0 + b_1 u - w$$



2-dimensional equation

stimulus ↓

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

Enables graphical analysis!

Important role of

- nullclines
- flow arrows

# Neuronal Dynamics – Quiz 4.4.

**A. u-Nullclines**

[ ] On the u-nullcline, arrows are always vertical
[ ] On the u-nullcline, arrows point always vertically upward
[ ] On the u-nullcline, arrows are always horizontal
[ ] On the u-nullcline, arrows point always to the left
[ ] On the u-nullcline, arrows point always to the right

**B. w-Nullclines**

[ ] On the w-nullcline, arrows are always vertical
[ ] On the w-nullcline, arrows point always vertically upward
[ ] On the w-nullcline, arrows are always horizontal
[ ] On the w-nullcline, arrows point always to the left
[ ] On the w-nullcline, arrows point always to the right
[ ] On the w-nullcline, arrows can point in an arbitrary direction

# Week 4 – part 3A: Analysis of a 2D neuron model – pulse input

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- separation of time scales

4.5. **Nonlinear Integrate-and-fire**
- from two to one dimension

2-dimensional equation

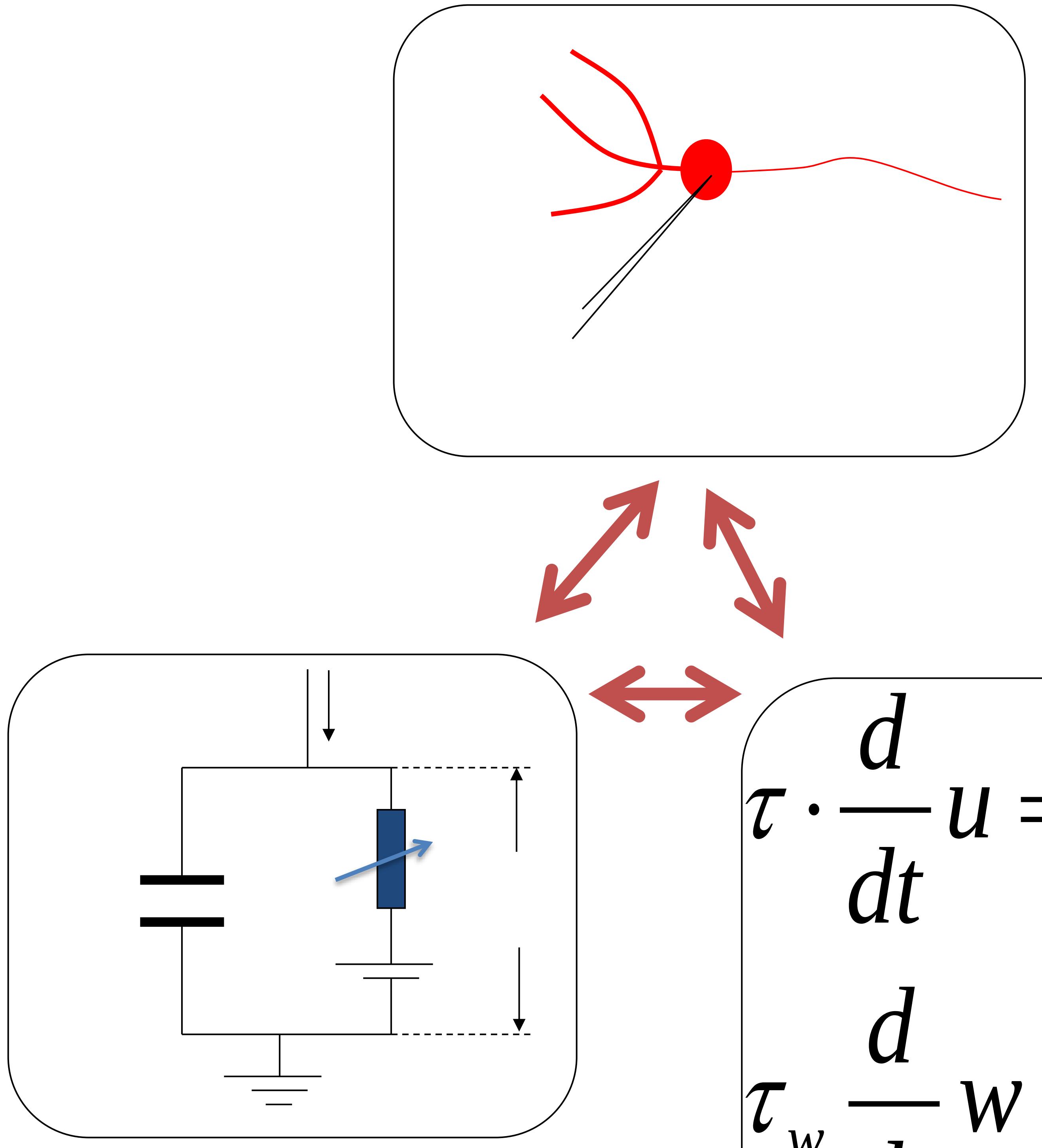
stimulus

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

Enables graphical analysis!
- Pulse input
- Constant input

# Neuronal Dynamics – 4.3. 2D neuron model : Pulse input



$$\tau \cdot \frac{d}{dt} u = F(u,w) + RI$$

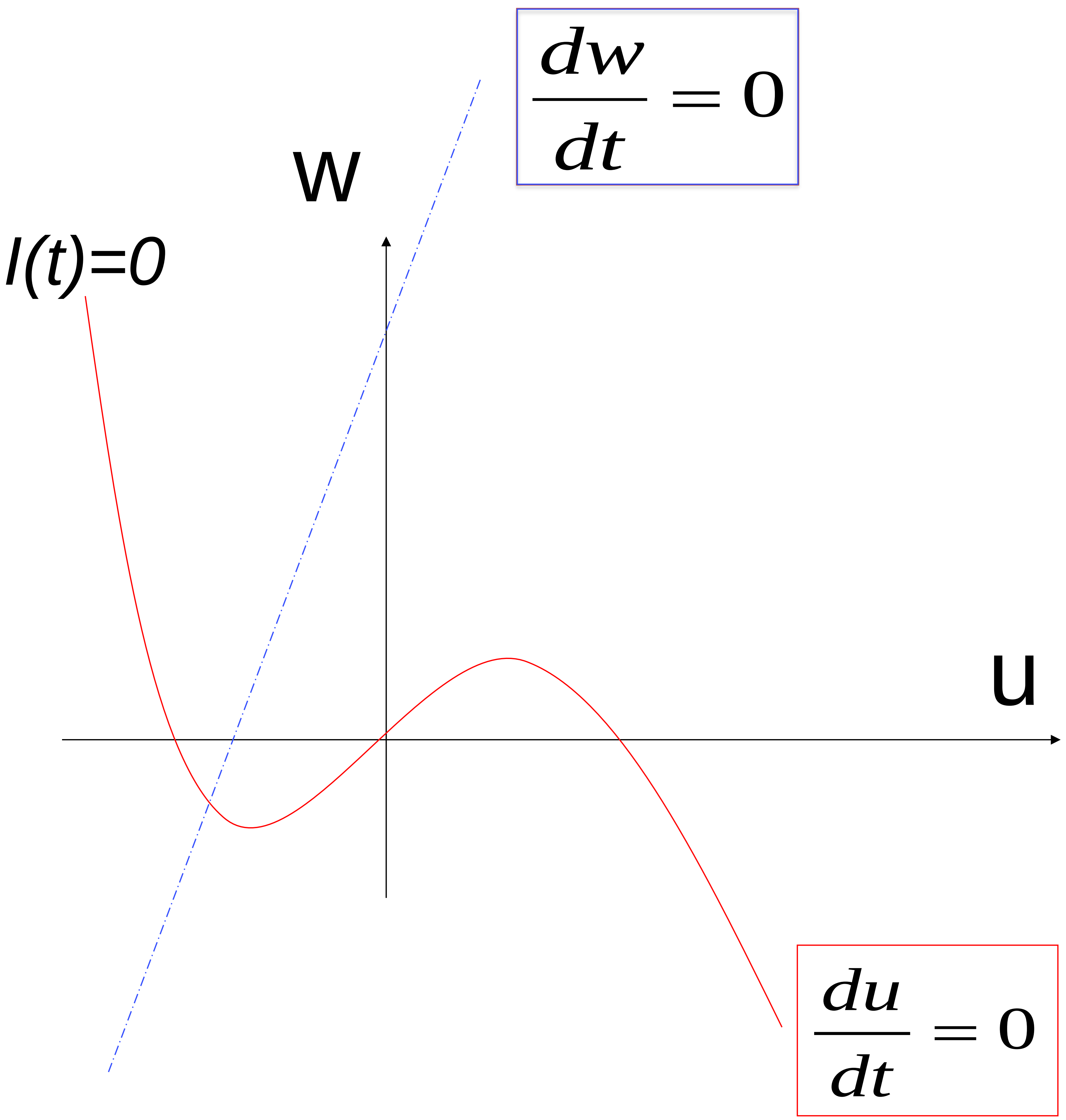
$$\tau_w \frac{d}{dt} w = G(u,w)$$

pulse input

# Neuronal Dynamics – 4.3. FitzHugh-Nagumo Model : Pulse input

$$\tau \frac{du}{dt} = F(u,w) + RI(t) = u - \frac{1}{3}u^3 - w + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w) \qquad = b_0 + b_1 u - w$$

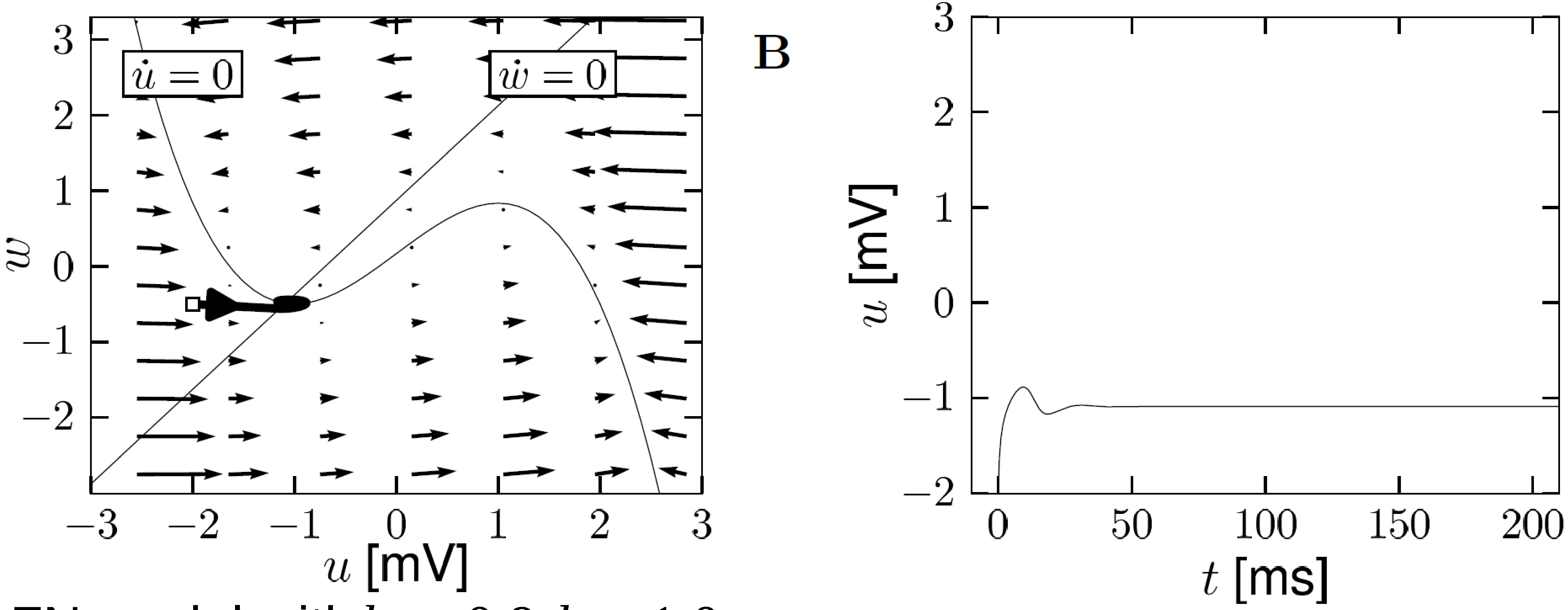



pulse input

I(t)

Pulse input: jump of voltage

# Neuronal Dynamics – 4.3. FitzHugh-Nagumo Model : Pulse input




FN model with $b_0 = 0.9; b_1 = 1.0$

Pulse input: jump of voltage/initial condition

*Image: Neuronal Dynamics, Gerstner et al., Cambridge Univ. Press (2014*

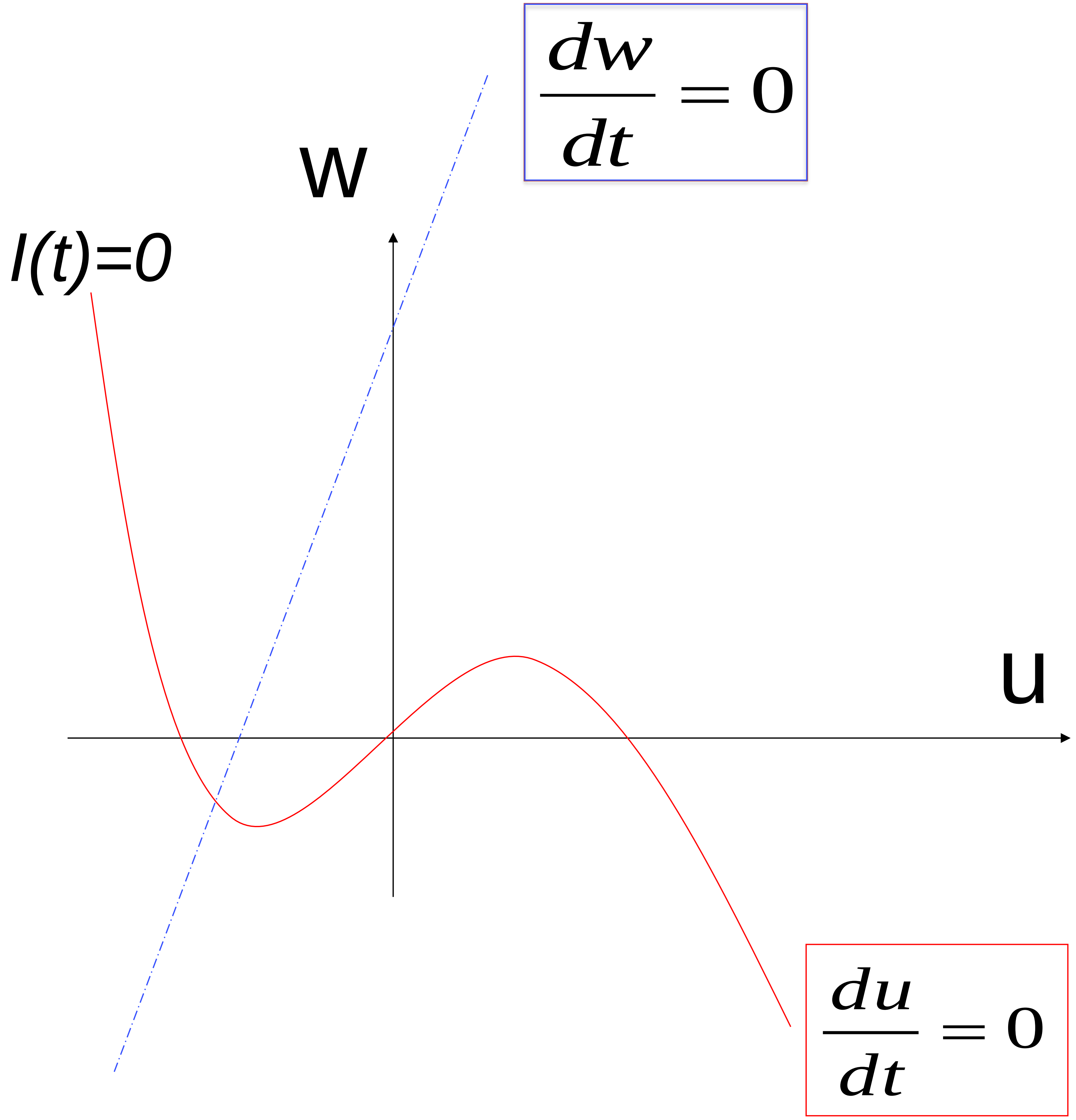
**Pulse input:** DONE!

- jump of voltage
- ‘new initial condition’
- spike generation for large input pulses

**constant input:**

- graphics?
- spikes?
- repetitive firing?

Lesson 4.3B!
…
Comes next



# Week 4 – part 3B: Analysis of a 2D neuron model – constant input

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# Neuronal Dynamics:
## Computational Neuroscience of Single Neurons

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## Two-dimensional neuron models

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√4.2 **Phase Plane Analysis**
- Role of nullcline

4.3 **Analysis of a 2D Neuron Model**
- MathDetour 3: Stability of fixed points

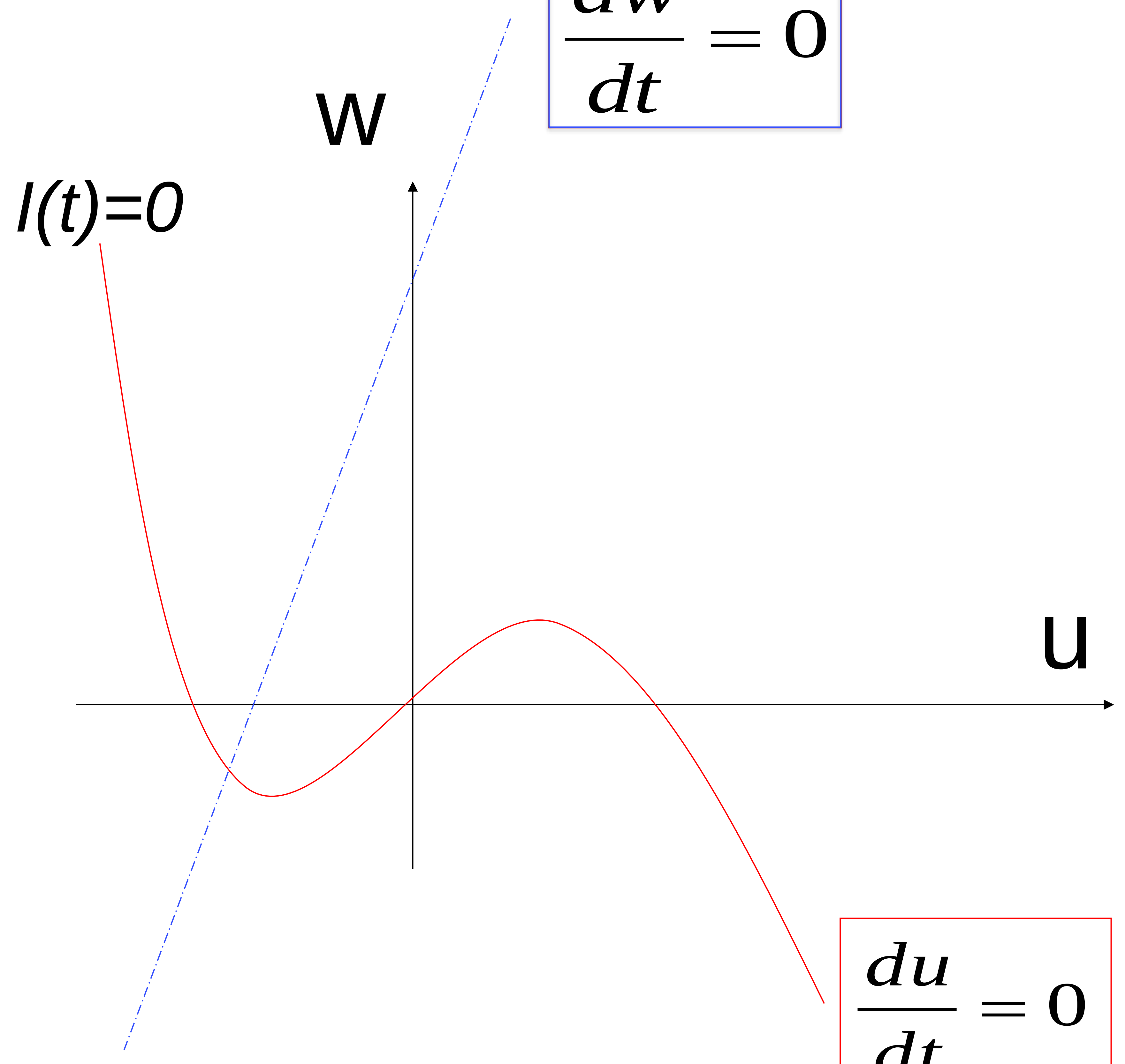
4.4 **Type I and II Neuron Models**
- where is the firing threshold?
- separation of time scales

4.5. **Nonlinear Integrate-and-fire**
- from two to one dimension

**Pulse input:**
- jump of voltage
- 'new initial condition'
- spike generation for large input pulses

**constant input:**
- graphics?
- spikes?
- repetitive firing?

Now!




$$\frac{dw}{dt} = 0$$

$$\frac{du}{dt} = 0$$

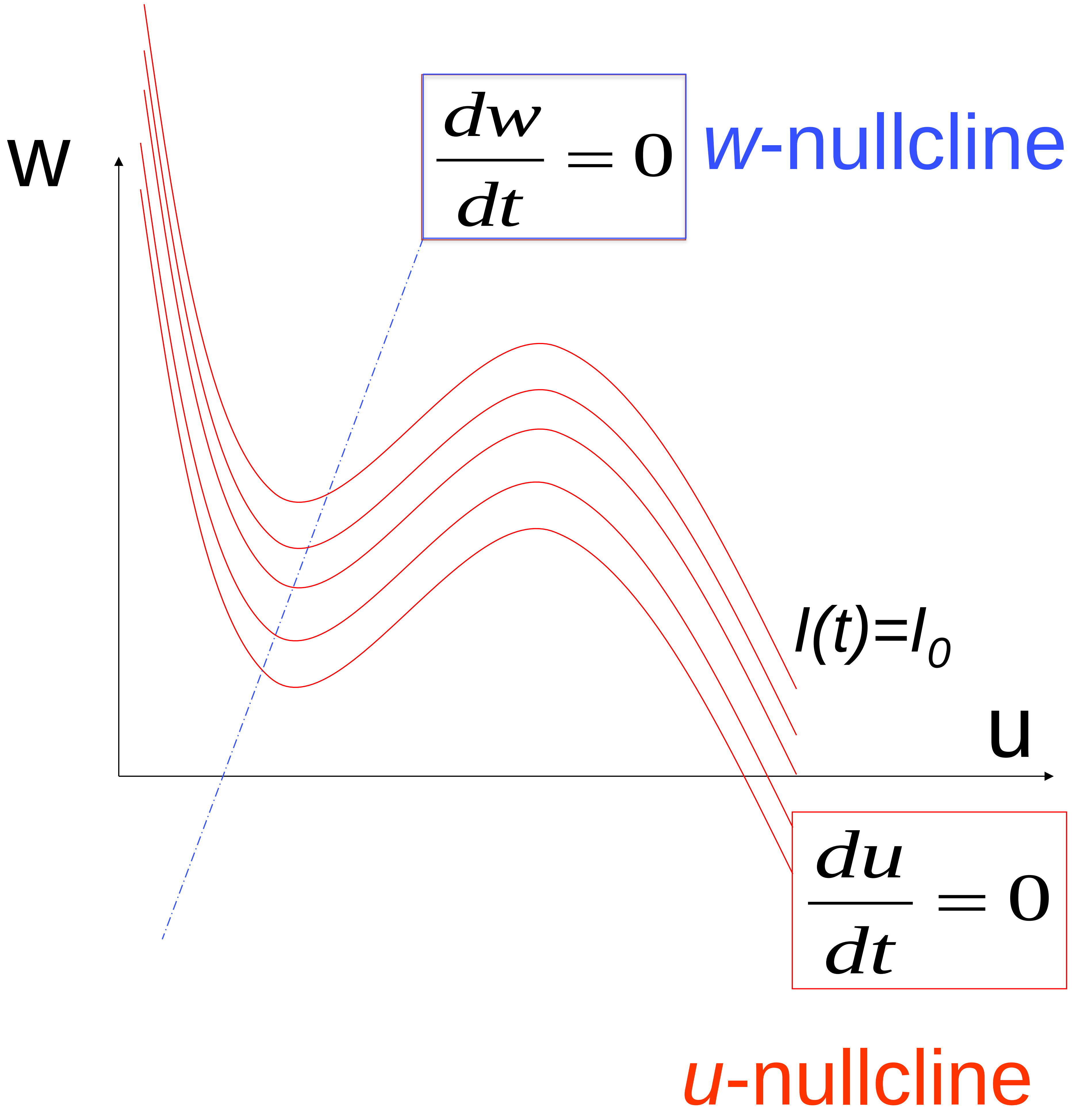
# Neuronal Dynamics – 4.3. FitzHugh-Nagumo Model: Constant input

$$\tau \frac{du}{dt} = F(u,w) + RI_0$$

$$= u - \frac{1}{3}u^3 - w + RI_0$$

$$\tau_w \frac{dw}{dt} = G(u,w) = b_0 + b_1 u - w$$

Intersection point (fixed point)
-moves
-changes Stability



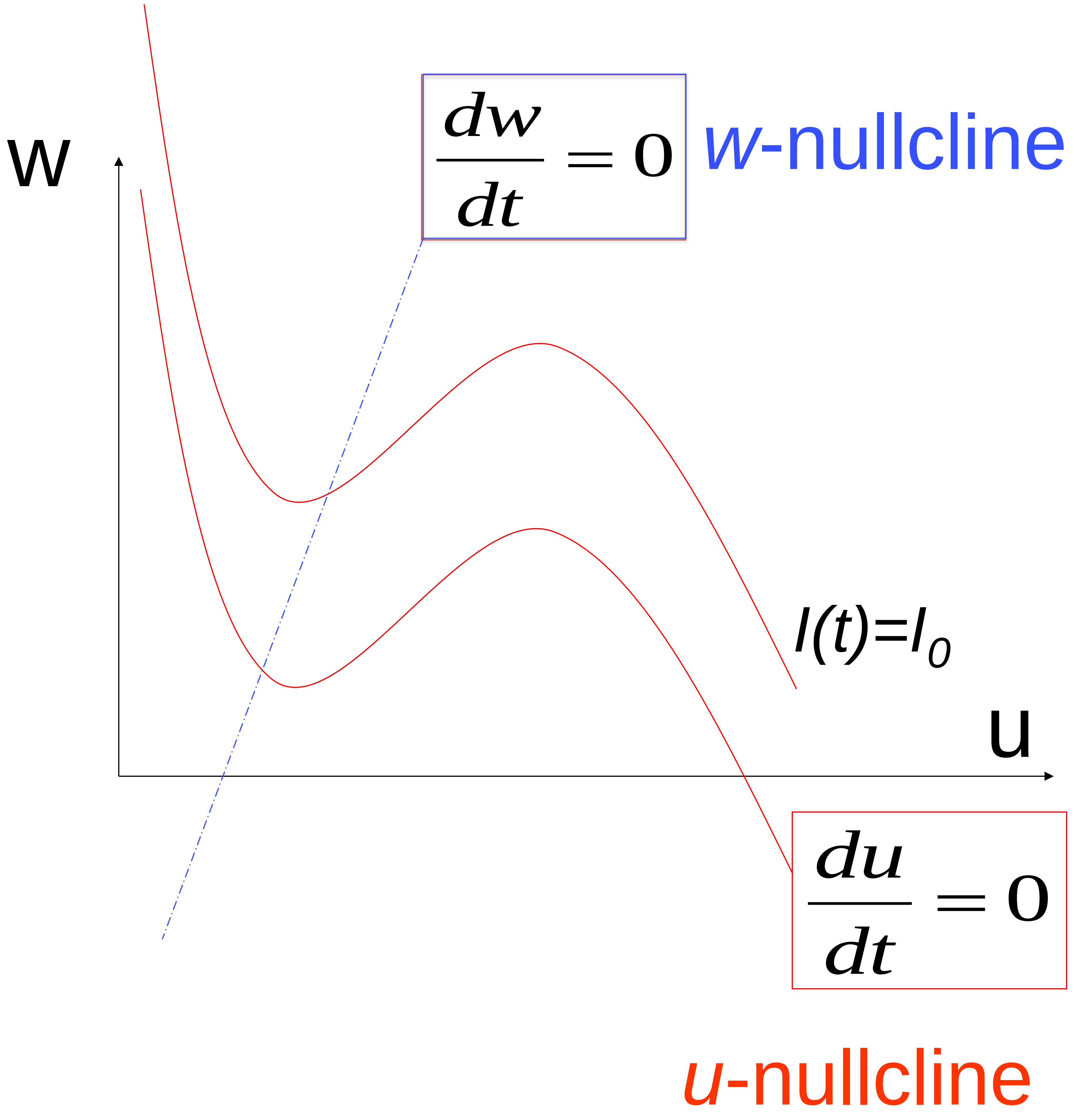
# Neuronal Dynamics – 4.3. FitzHugh-Nagumo Model: Constant input

$$\tau \frac{du}{dt} = F(u,w) + RI_0$$

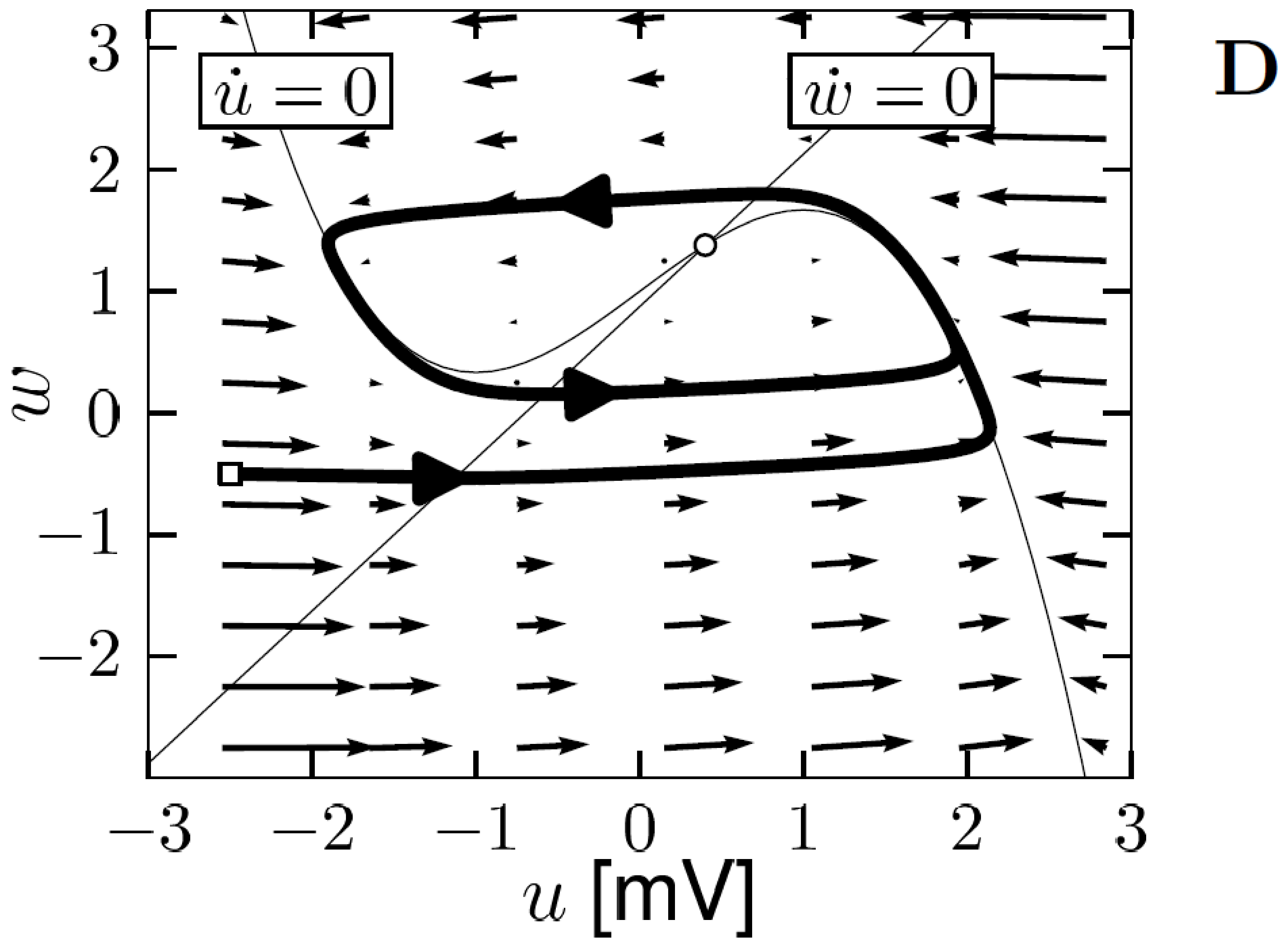
$$= u - \frac{1}{3}u^3 - w + RI_0$$

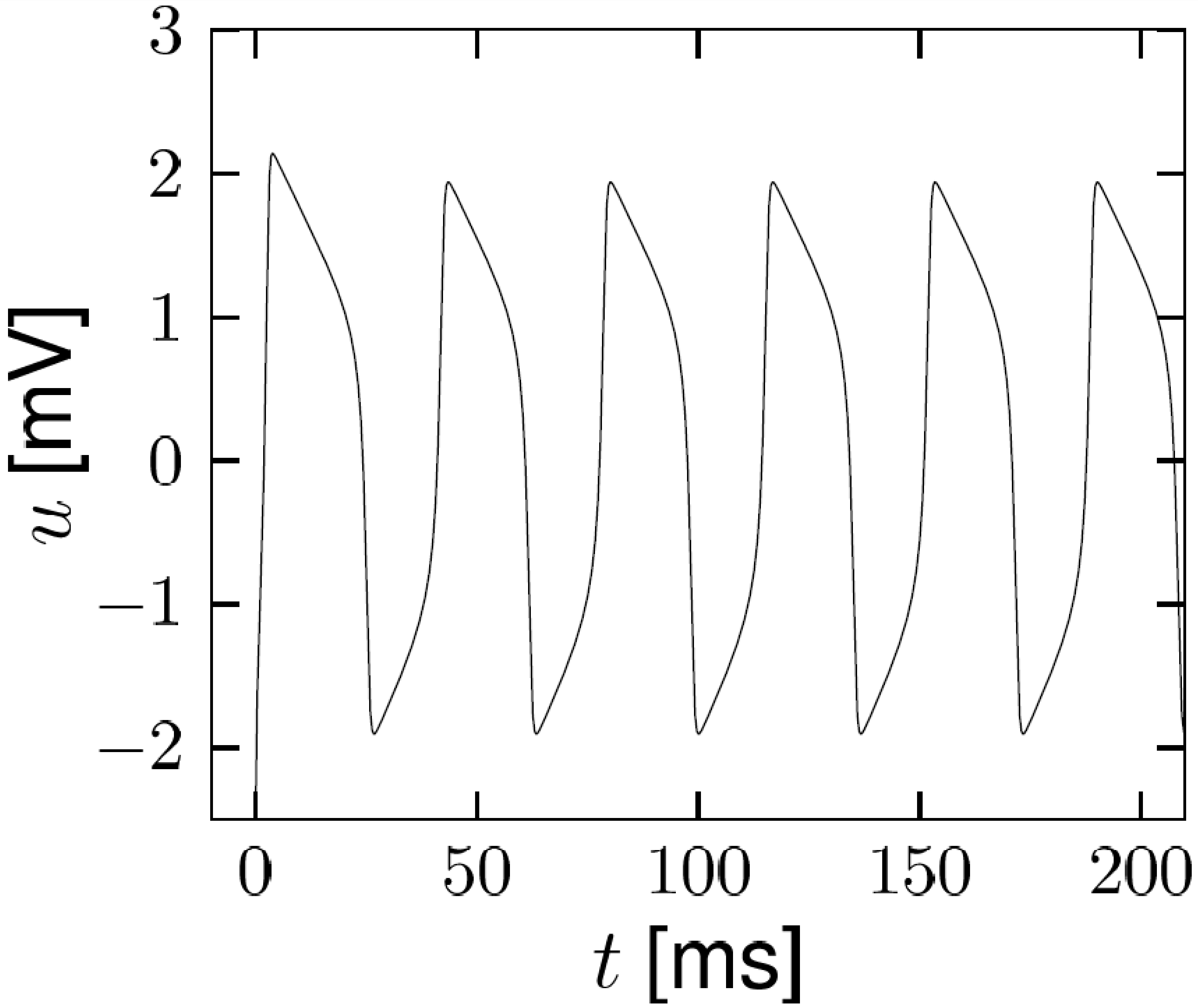
$$\tau_w \frac{dw}{dt} = G(u,w) = b_0 + b_1 u - w$$

Intersection point (fixed point)
-moves
-changes Stability



# Neuronal Dynamics – 4.3. FitzHugh-Nagumo Model : Constant input



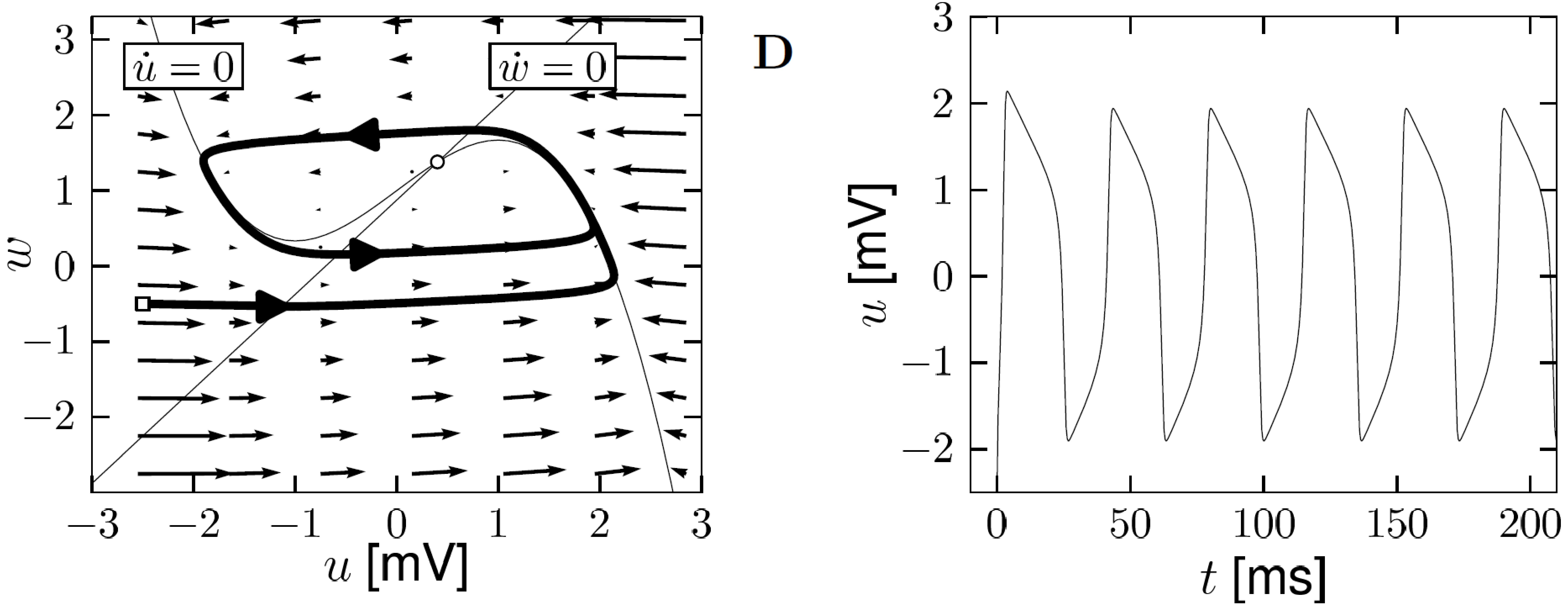

FN model with $b_0 = 0.9; b_1 = 1.0; RI_0 = 2$

constant input: u-nullcline moves

limit cycle

*Image: Neuronal Dynamics, Gerstner et al., Cambridge Univ. Press (2014)*

# Neuronal Dynamics – 4.3. Limit Cycle

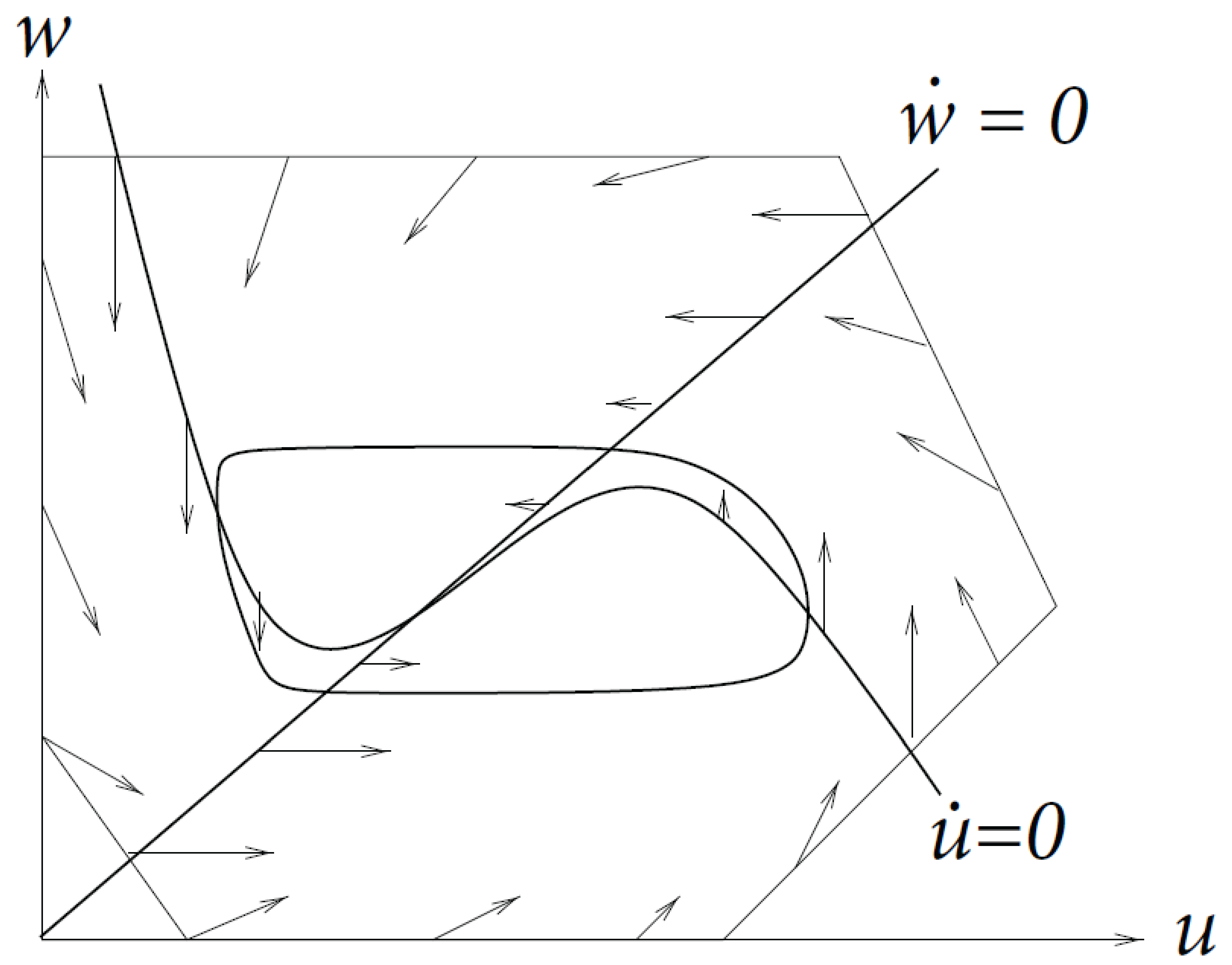



-unstable fixed point in 2D

-bounding box with inward flow

→ limit cycle *(Poincare Bendixson)*

*Image: Neuronal Dynamics, Gerstner et al., Cambridge Univ. Press (2014)*




In 2-dimensional equations,
a limit cycle must exist, if we can
find a surface

-containing one unstable fixed point
-bounding box with inward flow
→ limit cycle *(Poincare Bendixson)*

*Image: Neuronal Dynamics, Gerstner et al., Cambridge Univ. Press (2014)*

2-dimensional equation

stimulus

$$\tau \frac{du}{dt} = F(u,w) + RI(t)$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

**Enables graphical analysis!**

-Pulse input

→ AP firing (or not)

- Constant input

→ repetitive firing (or not)

# Neuronal Dynamics – Quiz 4.5.

**A. Short current pulses.** In a 2-dimensional neuron model, the effect of a delta current pulse can be analyzed

[ ] By moving the u-nullcline vertically upward
[ ] By moving the w-nullcline vertically upward
[ ] As a potential change in the stability or number of the fixed point(s)
[ ] As a new initial condition
[ ] By following the flow of arrows in the appropriate phase plane diagram

**B. Constant current.** In a 2-dimensional neuron model, the effect of a constant current can be analyzed

[ ] By moving the u-nullcline vertically upward
[ ] By moving the w-nullcline vertically upward
[ ] As a potential change in the stability or number of the fixed point(s)
[ ] As a new initial condition
[ ] By following the flow of arrows in the appropriate phase plane diagram

# Week 4 – MathDetour 3: Stability of fixed points

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

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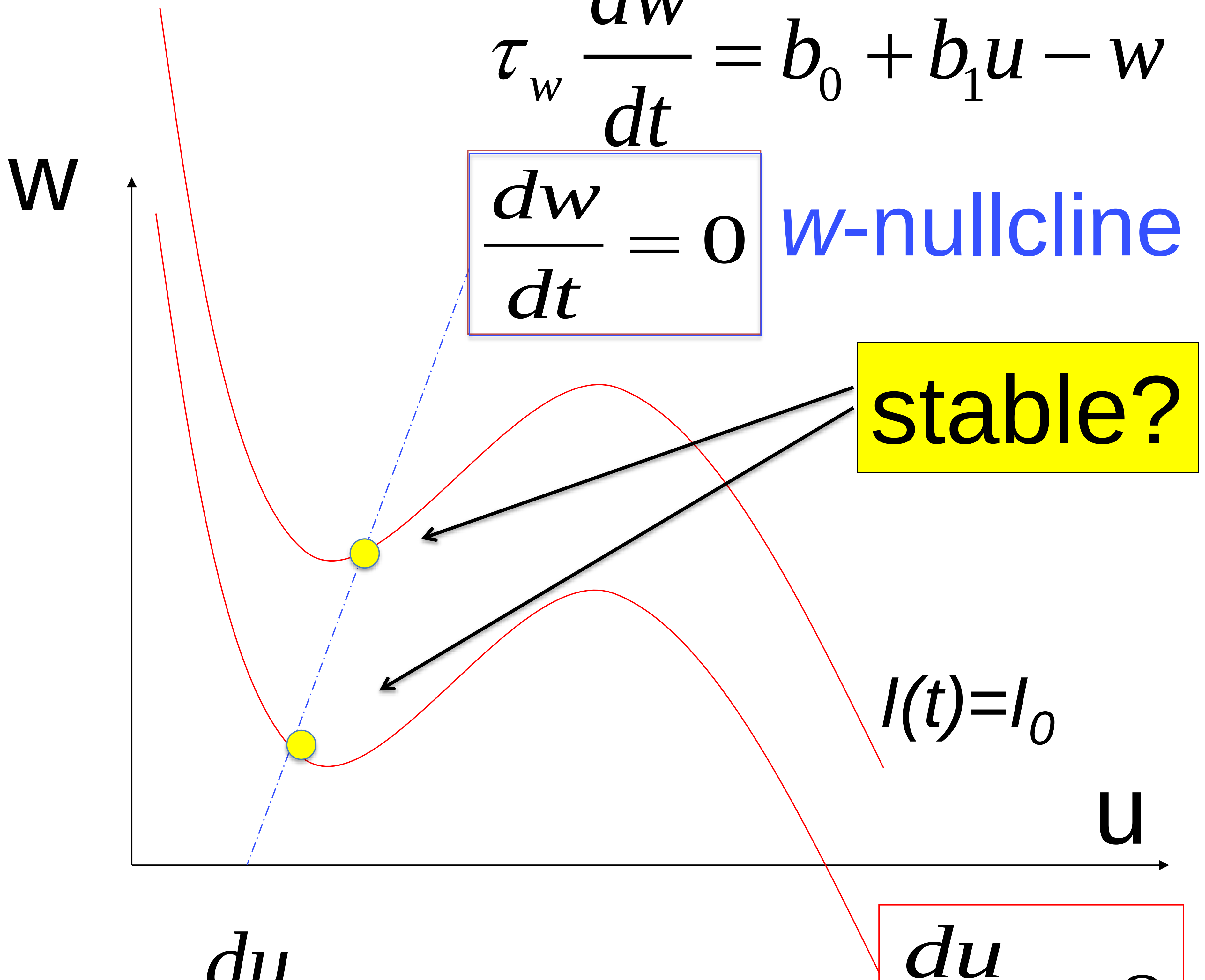
# Neuronal Dynamics – Detour 4.3 : Stability of fixed points.

$$\tau_w \frac{dw}{dt} = b_0 + b_1 u - w$$

$$\frac{dw}{dt} = 0$$ *w*-nullcline




$$\tau \frac{du}{dt} = F(u,w) + RI_0$$

$$\frac{du}{dt} = 0$$

*u*-nullcline

2-dimensional equation

stimulus

$$\tau \frac{du}{dt} = F(u,w) + RI_0$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

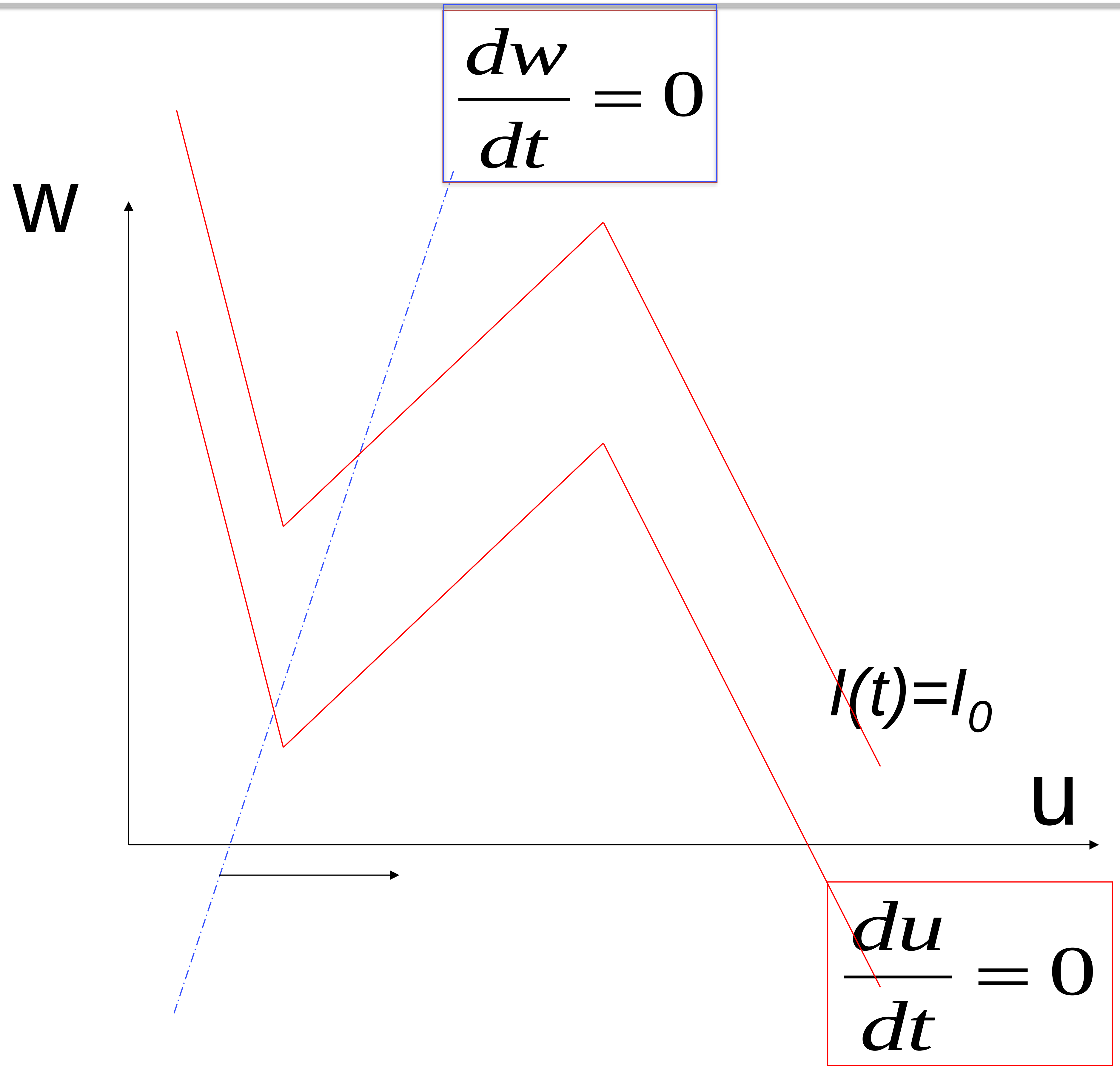
How to determine stability of fixed point?

# Neuronal Dynamics – 4.3 Detour. Stability of fixed points

stimulus

$$\tau \frac{du}{dt} = au - w + I_0$$

$$\tau_w \frac{dw}{dt} = cu - w$$

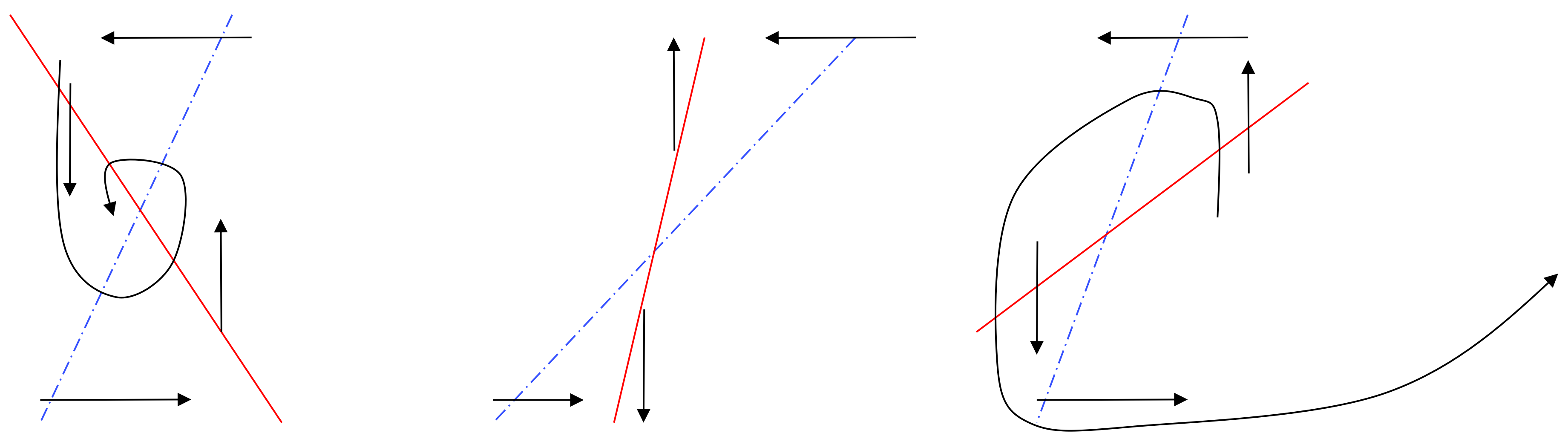


# Neuronal Dynamics – 4.3 Detour. Stability of fixed points

$$\tau \frac{du}{dt} = F(u,w) + RI_0$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

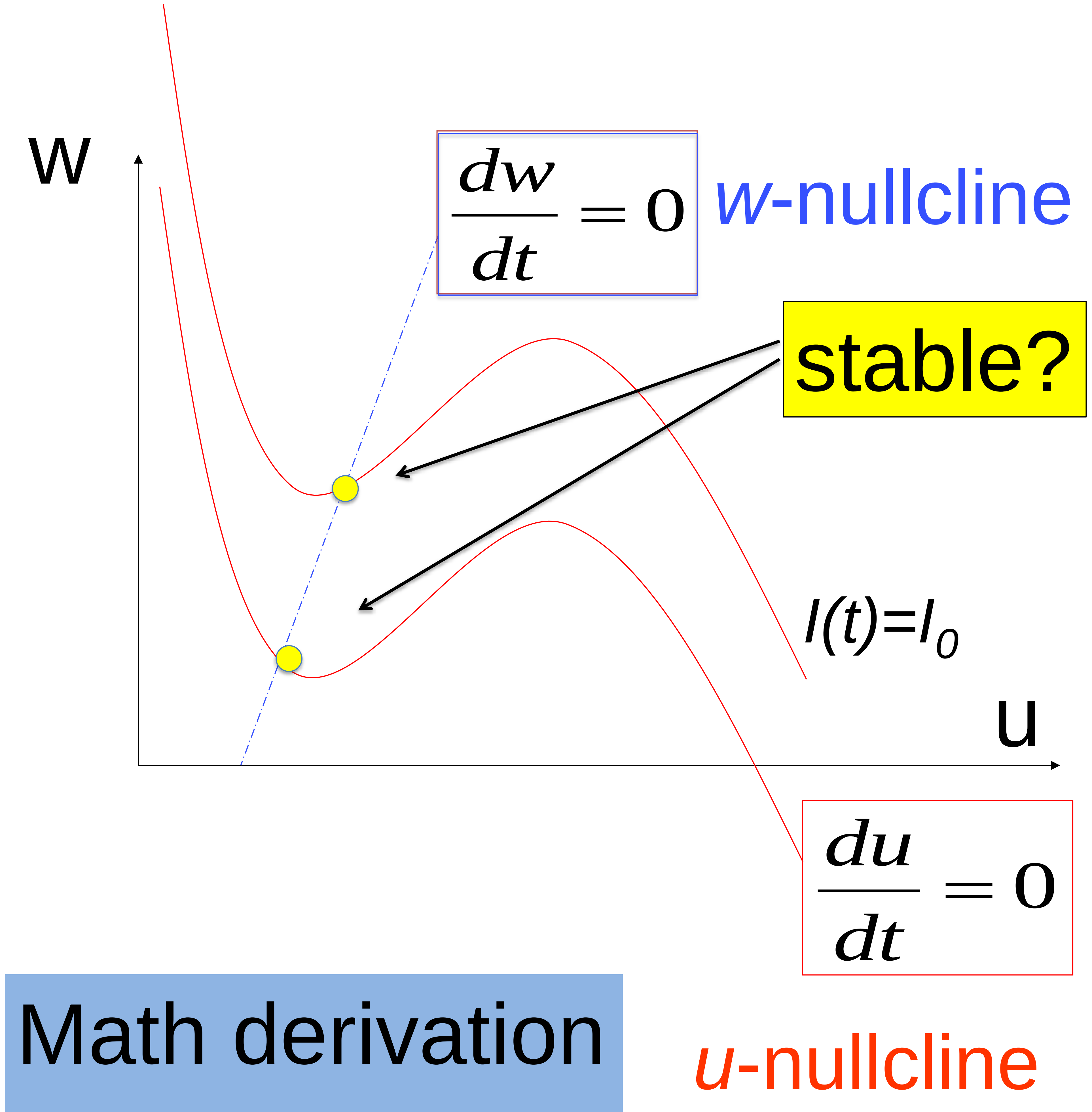
zoom in:



stable

saddle

unstable




Math derivation now

# Neuronal Dynamics – 4.3 Detour. Stability of fixed points

$$\tau \frac{du}{dt} = F(u,w) + RI_0$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

Fixed point at $(u_0, w_0)$

At fixed point

$$0 = F(u_0, w_0) + RI_0$$

$$0 = G(u_0, w_0)$$

zoom in:

$$x = u - u_0$$

$$y = w - w_0$$

# Neuronal Dynamics – 4.3 Detour. Stability of fixed points

$$\tau \frac{du}{dt} = F(u,w) + RI_0$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

Fixed point at $(u_0, w_0)$

At fixed point

$$0 = F(u_0, w_0) + RI_0$$

$$0 = G(u_0, w_0)$$

zoom in:

$$x = u - u_0$$

$$y = w - w_0$$

$$\tau \frac{dx}{dt} = F_u x + F_w y$$

$$\tau_w \frac{dy}{dt} = G_u x + G_w y$$

$$\frac{d}{dt}\boldsymbol{x} = \begin{pmatrix} F_u & F_w \\ G_u & G_w \end{pmatrix} \boldsymbol{x} ;$$

Linear matrix equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \begin{pmatrix} F_u & F_w \\ G_u & G_w \end{pmatrix} \boldsymbol{x} ;$$

Search for solution

$$\boldsymbol{x}(t) = \boldsymbol{e} \exp(\lambda t)$$

Two solution with Eigenvalues $\lambda_+ , \lambda_-$

$$\lambda_+ + \lambda_- = F_u + G_w$$

$$\lambda_+ \lambda_- = F_u G_w - F_w G_u$$

# Neuronal Dynamics – 4.3 Detour. Stability of fixed points

Linear matrix equation

$$\frac{d}{dt}\boldsymbol{x} = \begin{pmatrix} F_u & F_w \\ G_u & G_w \end{pmatrix} \boldsymbol{x}$$

Search for solution

$$\boldsymbol{x}(t) = \boldsymbol{e} \exp(\lambda t)$$

Two solution with Eigenvalues $\lambda_+, \lambda_-$

$$\lambda_+ + \lambda_- = F_u + G_w$$

$$\lambda_+ \lambda_- = F_u G_w - F_w G_u$$

Stability requires:

$$\lambda_+ < 0 \quad and \quad \lambda_- < 0$$

$$F_u + G_w < 0$$

**and**

$$F_u G_w - F_w G_u > 0$$

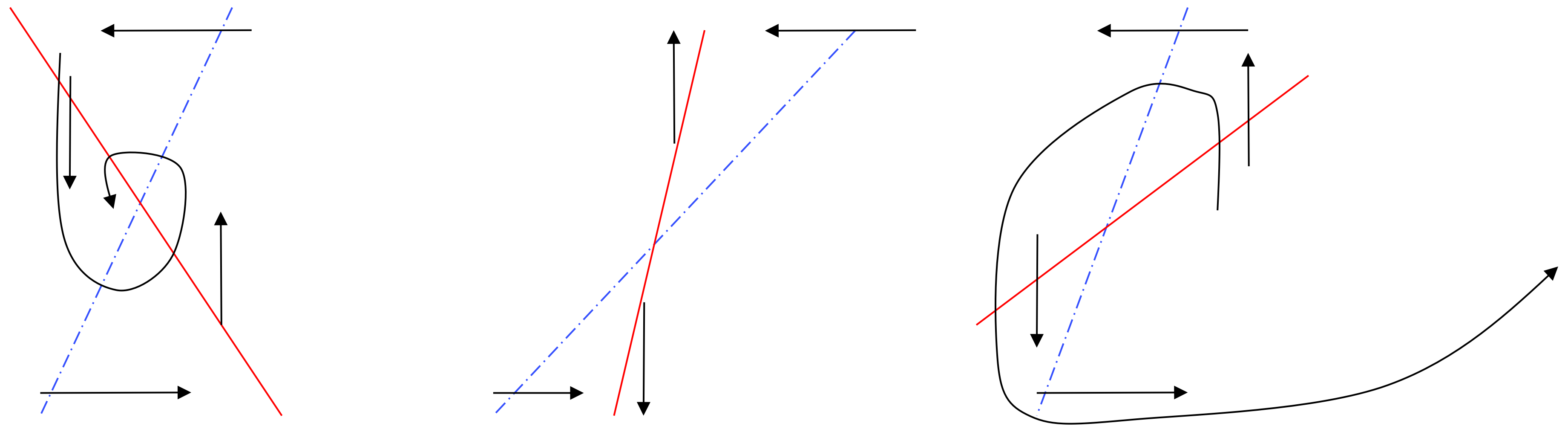
# Neuronal Dynamics – 4.3 Detour. Stability of fixed points

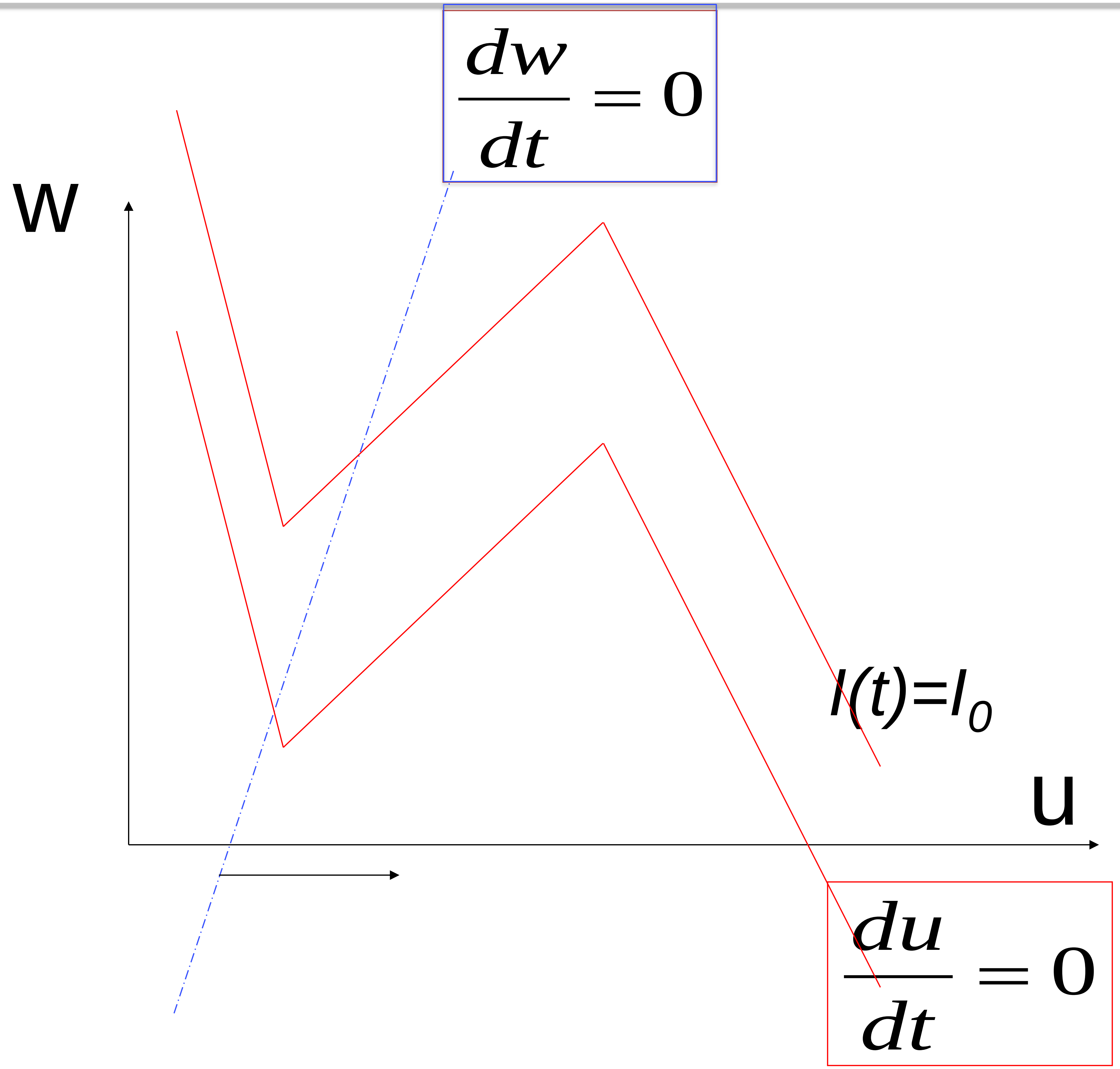
stimulus

$$\tau \frac{du}{dt} = au - w + I_0$$

$$\tau_w \frac{dw}{dt} = cu - w$$

$$\lambda_{+/-} =$$





2-dimensional equation

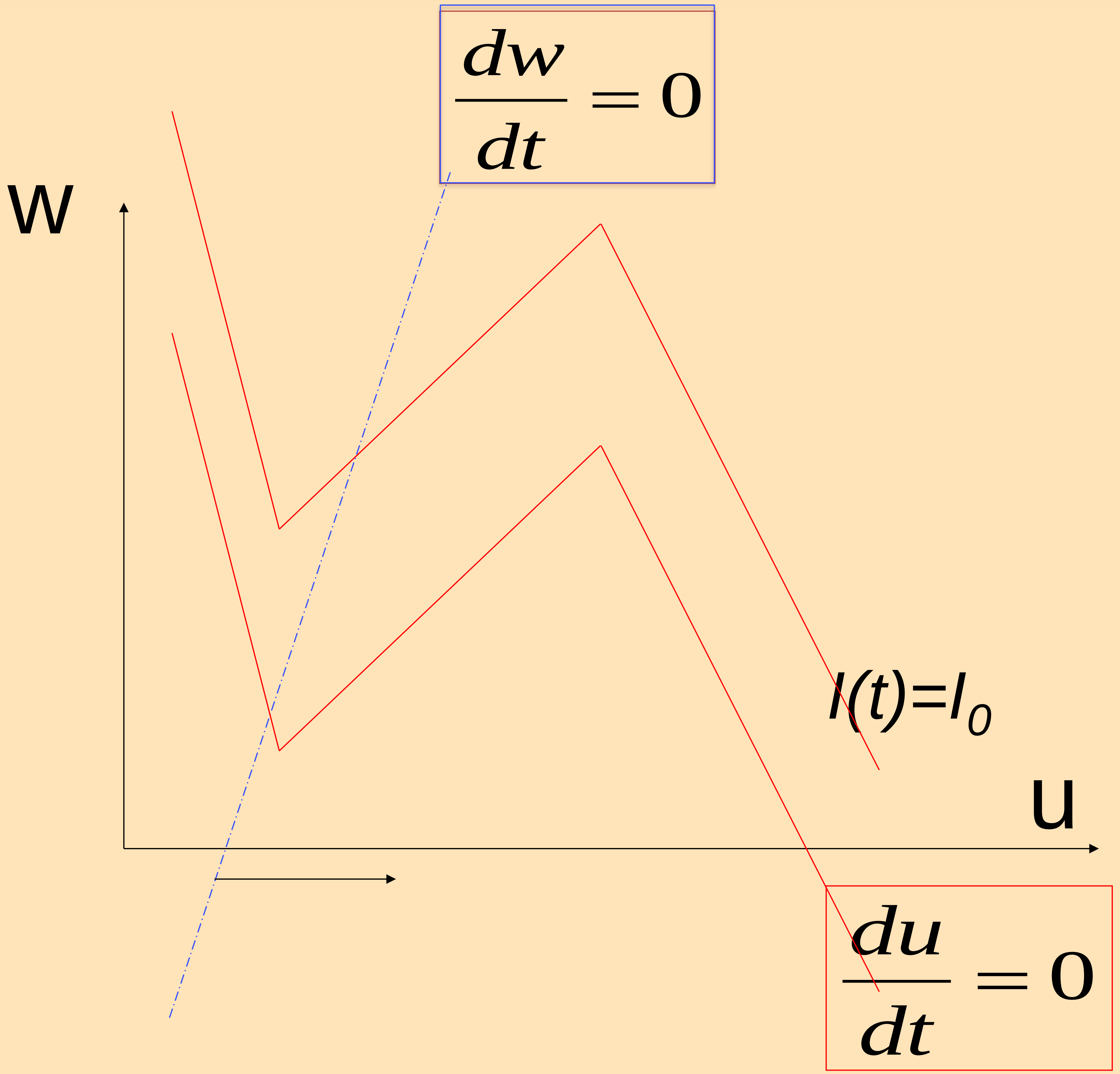
$$\tau \frac{du}{dt} = F(u,w) + RI_0$$

$$\tau_w \frac{dw}{dt} = G(u,w)$$

Stability characterized by Eigenvalues of linearized equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = \begin{pmatrix} F_u & F_w \\ G_u & G_w \end{pmatrix} \boldsymbol{x}$$

# Neuronal Dynamics – Assignment.




Stability analysis of 2-dimensional equations is important for the homework assignment of week 4.