Corrections

I. Calculation of the stress intensity factor in Griffith's and penny shape cracks

1. Using the principle of superposition of elastic solutions, the stress intensity factor of a through crack of length 2a (aka Griffith's crack) that undergoes a symmetric stress distribution σ(x) where x is the distance to the y axis is given by:

$$K = 2\sqrt{\frac{a}{\pi}} \int_0^a \frac{\sigma(x)}{\sqrt{a^2 - x^2}} dx$$

With the help of this result, calculate the stress intensity factor of a through crack of length 2a in an infinite body under a remote stress σ.

$$K_I = 2\frac{\sqrt{a}}{\sqrt{\pi}} \int_0^a \frac{\sigma(x)dx}{\sqrt{a^2-x^2}} = 2\frac{\sigma\sqrt{a}}{\sqrt{\pi}} \int_0^a \frac{dx}{\sqrt{a^2-x^2}} = 2\frac{\sigma\sqrt{a}}{\sqrt{\pi}} \int_0^1 \frac{adu}{a\sqrt{1-u^2}} \text{ posing } u = x/a$$

$$K_I = 2\frac{\sigma\sqrt{a}}{\sqrt{\pi}} \int_0^1 \frac{du}{\sqrt{1-u^2}} = 2\frac{\sigma\sqrt{a}}{\sqrt{\pi}} \int_0^{\pi/2} \frac{\cos\theta d\theta}{\sqrt{1-\cos\theta^2}} = 2\frac{\sigma\sqrt{a}}{\sqrt{\pi}} \int_0^{\pi/2} \frac{\cos\theta d\theta}{\sqrt{1-\sin\theta^2}} = 2\frac{\sigma\sqrt{a}}{\sqrt{\pi}} \int_0^{\pi/2} d\theta = 2\frac{\sigma\sqrt{a}}{\sqrt{\pi}}\frac{\pi}{2} = \sigma\sqrt{\pi a}$$

posing u = sinθ

2. In axisymmetric conditions, cylindrical coordinates are used with r, the distance to the symmetry axis and the stress intensity factor becomes:

$$K = \frac{2}{\sqrt{\pi a}} \int_0^a \frac{r\sigma(r)}{\sqrt{a^2 - r^2}} dr$$

Show that the stress intensity factor of a penny shape crack of length 2a in an infinite body under a remote stress σ writes $K_I = \frac{2\sigma}{\pi}\sqrt{\pi a}$.

$$K_I = \frac{2}{\sqrt{a\pi}} \int_0^a \frac{r\sigma(r)dr}{\sqrt{a^2-r^2}} = \frac{\sigma}{\sqrt{a\pi}} \int_0^a \frac{2rdr}{\sqrt{a^2-r^2}} = \frac{2\sigma}{\sqrt{a\pi}} \left[-\sqrt{a^2-r^2}\right]_0^a$$

$$K_I = \frac{2\sigma}{\sqrt{a\pi}}(0+a) = \frac{2\sigma a}{\sqrt{a\pi}} = \frac{2\sigma\sqrt{a}}{\sqrt{\pi}} == \frac{2\sigma\sqrt{\pi a}}{\pi}$$

2. Fatigue lifetime predictions

1.- Derive an expression for the number of stress cycles required to grow a semi-circular surface crack in a semi-infinite plate from an initial radius a_0 of 4 mm to a final size a_f, assuming the Paris equation describes the growth rate with an exponent equal to 4. Assume that the crack maintains its semi-circular shape, and that the stress amplitude Δσ is constant.

For an elliptical crack, K_I writes with C = 1.12 for short cracks:

$$K_I = C\frac{\sigma\sqrt{\pi a}}{\frac{3}{8}\pi + \frac{a^2}{8c^2}\pi}\left(\sin^2\varphi + \frac{a^2}{c^2}\cos^2\varphi\right)^{1/4}$$

Answer:

$$K_I = 2.24\sigma\sqrt{\frac{a}{\pi}}$$

$$\Delta K = 2.24\Delta\sigma\sqrt{\frac{a}{\pi}}$$

$$\frac{da}{dN} = C\Delta K^m = C\left(2.24\Delta\sigma\sqrt{\frac{a}{\pi}}\right)^m = C\left(\frac{2.24\Delta\sigma}{\sqrt{\pi}}\right)^m a^{m/2} \equiv k a^{m/2}$$

$$a^{-m/2}da = kdN$$

$$\left[\frac{a^{1-m/2}}{1-m/2}\right]_{a_0}^{a} = kN$$

$$N = \frac{1}{C}\left(\frac{2.24\Delta\sigma}{\sqrt{\pi}}\right)^{-m}\frac{a^{1-m/2} - a_0^{1-m/2}}{1-m/2}$$

2.- If it takes 1000 cycles for the crack length to increase from 2 to 2.5 mm, how many cycles does it take for the crack to advance from 2 to 4 mm if the Paris exponent, m, is equal to 4?

For a fixed stress amplitude,

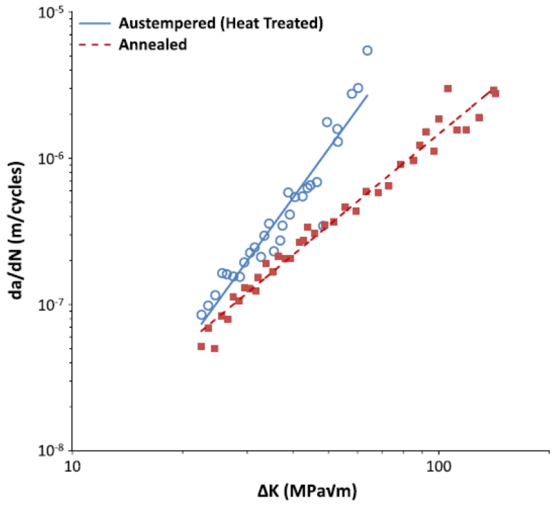
$$N = A\left(a^{1-\frac{m}{2}} - a_o^{1-\frac{m}{2}}\right), \text{where A is a constant;}$$

$$1000 = A\left(\frac{1}{2.5} - \frac{1}{2}\right) = -0.1A \Rightarrow A = -10'000;$$

$$N(4[mm]) = -10'000 \times \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{10'000}{4} = 2'500 \text{ cycles}$$

3. Dynamic fatigue crack growth under load-control

The figure below shows fatigue data from (i) cold-rolled and annealed and (ii) austempered compact tension (CT) specimens of a high carbon steel with a/W = 0.3, W = 50.8 mm and B = 4 mm, tested under load control, that is, with fixed values of the maximum and minimum load during each cycle.



Comparison of the dynamic fatigue crack growth behaviour of annealed and austempered compact tension (CT) specimens of AISI 4140 steel tested at 5 Hz under load control, with R = 0.1 and a 2 mm long starter crack (V. Ramasagara Nagarajan, S.K. Putatunda, International Journal of Fatigue 62 (2014) 236–248.)

1.- Estimate the Paris law constants for the two data sets in the figure.

You can get the Paris law constants from the slope (m) and intercept of a linear fit to the data on this log-log plot with ΔK = 1 (i.e. log ΔK = 0). The authors of the paper the data came from give the following values:

Material condition	C	m
Batch A – Cold rolled and annealed	1×10^{-10}	2.08
Batch B – Heat treated (austempered)	2×10^{-12}	3.45

2.- What are the initial values of Kmax and Kmin noticing that crack growth starts roughly at ΔK = 20 MPam$^{1/2}$.

Answer:

$$\left.\begin{aligned} \frac{K_{\min}}{K_{\max}} &= 0.1 \\ K_{\max} - K_{\min} &= 20\ \mathrm{MPa\cdot m^{1/2}} \end{aligned}\right\}$$

$$0.9K_{\max} = 20\ \mathrm{MPa\cdot m^{1/2}}$$

$$K_{\max} = 22.2\ \mathrm{MPa\cdot m^{1/2}}$$

$$K_{\min} = 2.22\ \mathrm{MPa\cdot m^{1/2}}$$

3.- What is the maximum load during each loading cycle? The stress intensity factor K_{max} of a CT specimen loaded under P_{max} writes:

$$K_{\max} = \frac{P_{\max}}{BW^{1/2}} f\left(\frac{a}{W}\right)$$

$$f\left(\frac{a}{W}\right) = \frac{\left(2+\frac{a}{W}\right)}{\left(1-\frac{a}{W}\right)^{3/2}}\left[0.886+4.64\left(\frac{a}{W}\right)-13.32\left(\frac{a}{W}\right)^2+14.72\left(\frac{a}{W}\right)^3-5.6\left(\frac{a}{W}\right)^4\right]$$

Answer:

$$K_{\max} = \frac{P_{\max}}{BW^{1/2}} f\left(\frac{a}{W}\right)$$

$$f\left(\frac{a}{W}\right) = \frac{\left(2+\frac{a}{W}\right)}{\left(1-\frac{a}{W}\right)^{3/2}}\left[0.886+4.64\left(\frac{a}{W}\right)-13.32\left(\frac{a}{W}\right)^2+14.72\left(\frac{a}{W}\right)^3-5.6\left(\frac{a}{W}\right)^4\right]$$

$$f(0.3) = \frac{(2+0.3)}{(1-0.3)^{3/2}}\left[0.886+4.64(0.3)-13.32(0.3)^2+14.72(0.3)^3-5.6(0.3)^4\right] = 5.62$$

$$\frac{BW^{1/2}K_{\max}}{f\left(\frac{a}{W}\right)} = P_{\max} = \frac{4\times10^{-3}\times\sqrt{50.8\times10^{-3}}\times22.2\times10^{6}}{5.62} = 3.56\ \mathrm{kN}$$

3.- If both specimens have the same static Kc, which specimen is likely to fail first?

Kmax depends on the crack length and the magnitude of the maximum stress. In this kind of experiment the stress amplitude is constant, so Kmax depends only on the crack length. The crack length increases more rapidly with the number of cycles for the austempered specimen (blue data points in the figure). Kmax will therefore reach Kc sooner, i.e. after fewer cycles, in the austempered specimen.