I. Irwin plastic zone size in plane strain

1. Calculate the three principal stress components at the tip of a mode I crack in plane strain conditions using the Von Mises plasticity criterion. The yield stress writes σ_{YS}.

The 3 principal stresses are given as

$$\sigma_1 = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1+\sin\left(\frac{\theta}{2}\right)\right]$$

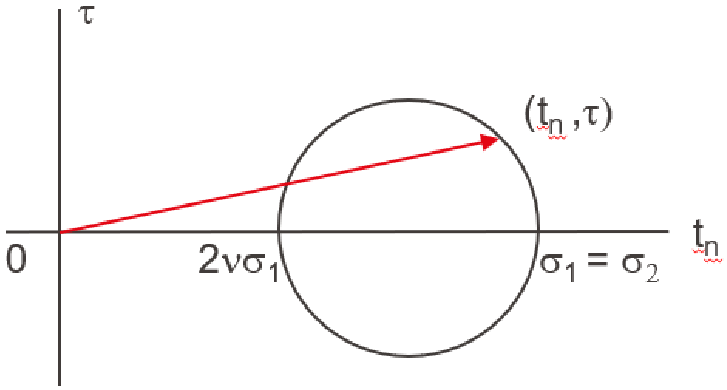
$$\sigma_2 = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[1-\sin\left(\frac{\theta}{2}\right)\right]$$

$$\sigma_3 = 0 \quad \text{(plane stress)}$$
$$= \frac{2\nu K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right) \quad \text{(plane strain)}$$

So in plane strain, we have: $\sigma = \frac{K_I}{\sqrt{2\pi r}}\cos\theta/2 \begin{pmatrix} 1+\sin\theta/2 \\ 1-\sin\theta/2 \\ 2\nu \end{pmatrix}$ so along $\theta = 0$, $\sigma = \frac{K_I}{\sqrt{2\pi r}} \begin{pmatrix} 1 \\ 1 \\ 2\nu \end{pmatrix}$

2. Draw the Mohr's circles and recall its meaning.

As two principal stresses are equal, the Mohr's circles are degenerated in one circle sowing all the possible values for traction and shear for all directions in space.



3. Deduce the first order estimate, r_y, of the plastic zone size under plain strain conditions. Calculate r_y assuming $\nu = 1/3$ and compare this value with the estimate in plane stress conditions.

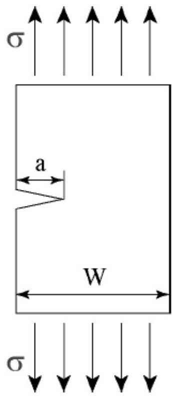
$\sigma_I = \sigma_{II}$ and $\sigma_{III} = 2\nu\sigma_I$ and $\sigma_{Mises} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_I-\sigma_{II})^2+(\sigma_I-2\nu\sigma_I)^2+(\sigma_I-2\nu\sigma_I)^2} = \frac{\sigma_I}{\sqrt{2}}\sqrt{2(1-2\nu)^2}$

$\sigma_{Mises} = (1-2\nu)\sigma_I = (1-2/3)\sigma_I = \frac{\sigma_I}{3} = \sigma_{YS}$ at onset of yielding, thus $\sigma_I = 3\sigma_{YS}$

Using truncation tocalulate r_y, one get $r_y^{PE} = \frac{(1-2\nu)^2}{2\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2 = \frac{1}{18\pi}\left(\frac{K_I}{\sigma_{YS}}\right)^2 = \frac{1}{9}r_y^{PS}$

2. Iterative determination of a second order estimate of the mode I plastic zone size in a 2D plate in plane stress.

Consider a 5 mm thick, 80 mm wide plate made of 6070 heat-treated aluminium (quenched and artificially aged) that contains an edge crack of length a = 20 mm loaded in Mode I with a far field stress of 100 MPa. As the crack is rather long compared to the width, one has to use the formulas below to calculate K_I.
Here $\alpha = a/W = 20/80 = 0.25$ so we need to use a series approximation for the geometrical factor in the K calculation.



Single edge-notched plate

$$K_I = C\sigma\sqrt{\pi a}$$

C = 1.12 for short cracks

$$C = 1.122 - 0.231\left(\frac{a}{W}\right) + 10.55\left(\frac{a}{W}\right)^2 - 21.71\left(\frac{a}{W}\right)^3 + 30.382\left(\frac{a}{W}\right)^4$$

valid for a/W ≤ 0.6

(Janssen, Zuidema and Wanhill p. 53)

Assuming plane-stress yielding conditions and that σ_{YS} = 352 MPa (see www.matweb.com):

1) Give a first order estimate of the plastic zone size using truncation, which we shall call ry0.

$$C(\alpha) = 1.12 - 0.23\alpha + 10.55\alpha^2 - 21.72\alpha^3 + 30.39\alpha^4 = 1.501;$$

$$K_1 = 100\sqrt{\pi \cdot 20 \cdot 10^{-3}} \cdot 1.501 = 37.63 \text{ MPa}\sqrt{\text{m}};$$

$$r_{y0} = \frac{1}{2\pi}\left(\frac{K_1}{\sigma_y}\right)^2 = \frac{37.63^2}{2\pi \cdot 352^2} = 1.82 \text{ mm};$$

2) KI0 for a crack of length a + ry0 and give a corresponding first order estimate for the plastic zone size, ry1.

$$K_{10} = 100\sqrt{\pi \cdot 21.82 \cdot 10^{-3}} \cdot 1.569 = 41.1 \text{ MPa}\sqrt{\text{m}};$$

$$r_{y1} = \frac{41.1^2}{2\pi \cdot 352^2} = 2.17 \text{ mm};$$

3) KI1 for a crack of length a + ry1 and give a corresponding first order estimate for the plastic zone size, ry2.

$$r_{y2} = 2.24 \text{ mm}; K_{12} = 41.93 \text{ MPa}\sqrt{\text{m}};$$

4) Iterate steps (ii) and (iii) a few times.

$$r_{y3} = 2.26 \text{ mm}; K_{13} = 41.96 \text{ MPa}\sqrt{\text{m}};$$

5) estimation of KI∞ and ry∞ and give a second order estimate of the plastic zone size (considering energy redistribution).
We get convergence after say 4 iterations with ry =2,26 mm and thus a process zone of 4,52 mm which approaches one fourth of the crack length of 20 mm.

iteration	a	a/w	C	K1	ry
0	20	0,25	1,501210938	37,6292229	1,81885515
1	21,82	0,27275	1,569583783	41,0941884	2,16924412
2	22,1692441	0,27711555	1,583431468	41,7871976	2,24302496
3	22,243025	0,27803781	1,58638839	41,934839	2,25890295
4	22,2589029	0,27823629	1,587026184	41,9666693	2,26233345
5	22,2623335	0,27827917	1,58716405	41,973549	2,26307526
6	22,2630753	0,27828844	1,587193865	41,9750368	2,2632357
7	22,2632357	0,27829045	1,587200313	41,9753586	2,2632704
8	22,2632704	0,27829088	1,587201708	41,9754282	2,2632779
9	22,2632779	0,27829097	1,58720201	41,9754432	2,26327952
10	22,2632795	0,27829099	1,587202075	41,9754465	2,26327987
11	22,2632799	0,278291	1,587202089	41,9754472	2,26327995
12	22,26328	0,278291	1,587202092	41,9754473	2,26327997

6) Show explicitly that this estimate satisfies the force balance conditions.

Indeed, of KI∞ and ry∞ satisfy the force balance using a virtual crack of length a + ry∞:

$$r_{y\infty}\sigma_y = \int_0^{r_{y\infty}} \frac{K_I[a+r_{y\infty}]}{\sqrt{2\pi r}}dr - r_{y\infty}\sigma_y;$$

$$2r_{y\infty}\sigma_y = \frac{K_I[a+r_{y\infty}]}{\sqrt{\pi}}\sqrt{2r_{y\infty}};$$

$$r_{y\infty} = \frac{1}{2\pi}\left(\frac{K_I[a+r_{y\infty}]}{\sigma_y}\right)^2$$

2. Fracture of a pressure vessel

A cylindrical steel pressure vessel, with a diameter of 6.1 m and a wall thickness of 25.4 mm, failed when the internal pressure reached 175 bar. The yield stress and modulus of the steel are σ_{YS} = 2450 MPa and E = 210 GPa, respectively, and Gc is estimated to be 131 kJ/m2 in plane stress.

The stress field in a pressure vessel under pressure p writes in cylindrical coordinates:

$$\sigma = K\begin{pmatrix} 1-(R_e/r)^2 & 0 & 0 \\ 0 & 1+(R_e/r)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } R_i \le r \le R_e \text{ with } K = \frac{pR_i^2}{R_e^2 - R_i^2}, R_i \text{ and } R_e \text{ inner and outer radii.}$$

1. Write down the stress tensor in the case of a thin walled pressure vessel denoting t its thickness and R its inner radius.

With $R_i = R$, $R \le r \le R + t$ and $R_e = R_i + t$, $K \approx pR_i^2/2tR_i = pR/2t$

$$\sigma_{rr} = \frac{pR}{2t}\left[1-\left(\frac{R_e}{r}\right)^2\right] \approx \frac{pR}{2t}\left[1-\left(\frac{R+t}{R}\right)^2\right] \approx \frac{pR}{2t}\left[\left(\frac{-2Rt}{R^2}\right)\right] = -p$$

NB: $\sigma_{rr} \approx \frac{pR}{2t}\left[\left(\frac{-2Rt+t^2}{R^2}\right)\right] \approx -p + \frac{pR}{2t}\left(\frac{t^2}{R^2}\right) = -p + \frac{pt}{2R}$ to the 1st order.

$$\sigma_{\theta\theta} = \frac{pR}{2t}\left[1+\left(\frac{R_e}{r}\right)^2\right] \approx \frac{pR}{2t}\left[1+\left(\frac{R+t}{R}\right)^2\right] \approx \frac{pR}{2t}\left(\frac{2R^2+2Rt}{R^2}\right) \approx \frac{pR}{t}$$

$$\sigma = p\begin{pmatrix} -1 & 0 & 0 \\ 0 & R/t & 0 \\ 0 & 0 & R/2t \end{pmatrix} \text{ with } R_i = R,\ R \le r \le R+t \text{ and } R_e = R_i + t$$

2. Calculate the three principal stresses in the walls of the cylindrical part of the pressure vessel (in this case these are just the longitudinal stress, the tangential stress, and the radial stress).

the tensor σ is diagonal so the 3 principal values are

$$\sigma_I = \frac{pR}{t}, \sigma_{II} = \frac{pR}{2t} \text{ and } \sigma_{III} = -p$$

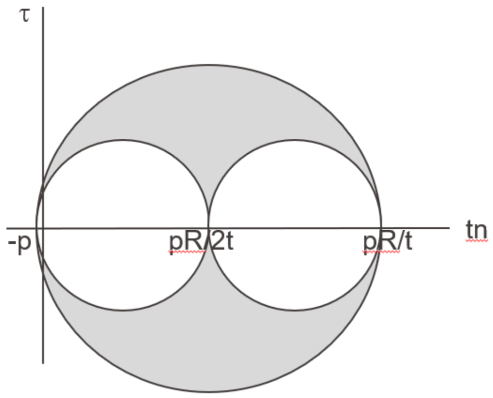
175 bar = 17.5 MPa and R/t = 3050/25.4 = 120.

Thus, σ_I = 2100 MPa, σ_{II} = 1050 MPa and σ_{III} = - 17.5 MPa

3. Draw the Mohr circles at r =R.

The 3 principal values are σ_I = 2100 MPa, σ_{II} = 1050 MPa and σ_{III} = -17.5 MPa at r=R

-17.5 MPa is negligible in front the 2 other principal values.



4 Use the von Mises criterion to show that failure could not be due to general yielding.

σ_I = 2100 MPa, σ_{II} = 1050 MPa and $\sigma_{III} \approx$ 0. so

$$\sigma_{Mises} = \frac{1}{\sqrt{2}}\sqrt{(2100-1050)^2+(2100-0)^2+(1050-0)^2} = \frac{1}{\sqrt{2}}\sqrt{2.1050^2+2100^2} = \frac{2572}{\sqrt{2}} = 1818 \text{ MPa}$$

$\sigma_{Mises} = 1818 \text{ MPa} < \sigma_{YS} = 2450 \text{ MPa}$ so no yielding occurs.

5 Use the Tresca criterion to show that failure could not be due to general yielding.

$$\tau_{max} = \frac{1}{2}(\sigma_I\text{-}\sigma_{III}) = \frac{1}{2}(2100\text{+}p) = \frac{1}{2}(2100\text{+}17.5) = 1059 \text{ MPa} < \sigma_{YS}/2 = 1225 \text{ MPa}$$ so no yielding occurs.

NB: Tresca criterion is more restrictive than Von Mises

6 Which crack orientation in the pressure vessel wall would represent the worst-case scenario with respect to fracture?

The one perpendicular to the maximum principal stress (hoop stress here), so in this case, cracks oriented along the axis of the pressure vessel i.e. in a (r,z) plane.

7 Give a rough estimate of the critical crack size. Hence, explain why the vessel did not leak before failing.

We can just use Kc = $\sigma(\pi a_c)^{1/2}$ to get a first estimate, using σ = 2100 MPa and estimating Kc from $(EGc)^{1/2}$, leading to a value of about 4 mm. Hence surface flaws of about this size will lead to fracture. For the vessel to leak rather than fail, you'd need a critical flaw size of at least 25.4 mm, i.e., greater than or equal to the wall thickness. This is actually quite a common consideration in pressure vessel design - it's easier to fix leaks than a pressure vessel that has exploded.

8 Do you think we are justified in assuming plane stress conditions here?

Given that all the effective specimen dimensions are a lot greater than the critical flaw size, and certainly a lot bigger than the plastic zone size, then probably not.

9. What difference would it have made if you had assumed plane strain conditions?

It makes little difference whether you use the correction factor $(1 - \nu^2)$ or not. The stress state may make a difference to Gc, but usually a plane strain state will lead to less plastic deformation and smaller values of Gc than in plane stress so we are likely to overestimate the critical flaw size if we use the plane stress value for Gc.