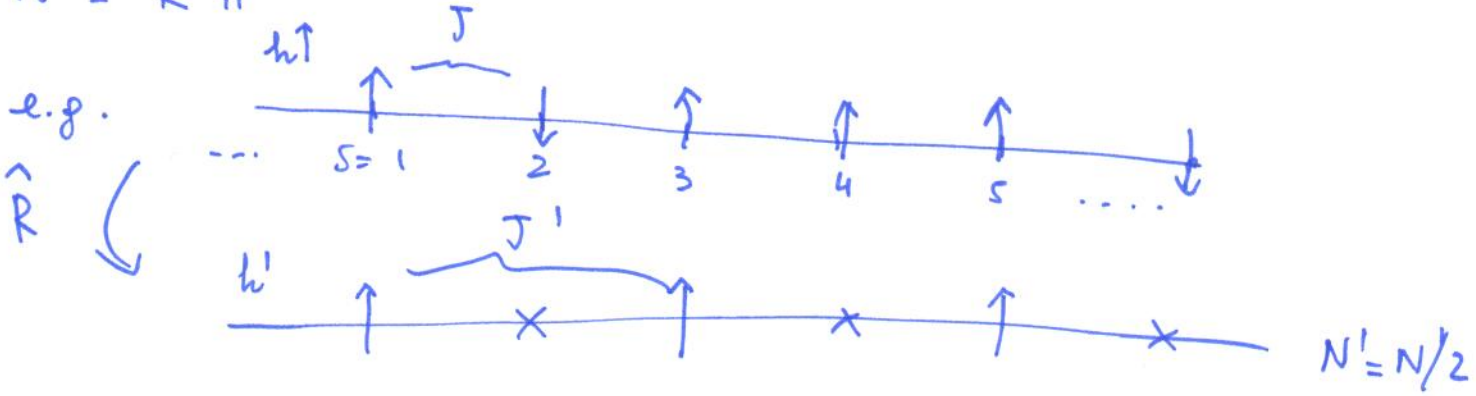
RNG THEORY

The idea is to start from a description of a system at a certain scale, encoded in a Hamiltonian $\tilde{H} = H/kT$ (scaled wrt temperature for convenience) and transform in into a more coarse-grained representation

$$\tilde{H}' = \hat{R}\,\tilde{H}$$



* this rescales all physical quantities. The scale factor is defined as: ~~e.g. length~~

$$b^d = N/N'$$

* $\hat{R}$ is said to be a RNG operation if

$$Z_{N'}(\tilde{H}') = Z_N(\tilde{H})$$

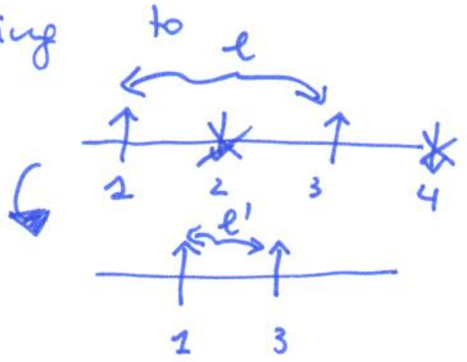
This imposes scaling laws! Since

$$\tilde{A}' = -\ln Z_{N'}(\tilde{H}') = -\ln Z_N(\tilde{H}) = \tilde{A}$$

$$\tilde{f}' = \frac{\tilde{A}'}{N'} = \tilde{A}\,\frac{\cancel{N}}{N'} = \tilde{f}\,\frac{N}{N'} = \tilde{f}\,b^d$$

* What does this imply for correlation lengths?
Length scales vary according to

$$l' = b^{-1} l \qquad \text{e.g.}$$



So if $s' = c\, s$

$$\Gamma' = \langle s'(r', \tilde{H}')\, s'(0, \tilde{H}') \rangle$$

$$= c^2 \langle s(b^{-1} r, \tilde{H})\, s(0, \tilde{H}) \rangle = c^2\, \Gamma(r, H)$$

meaning that the correlation length

$$\xi' = b^{-1} \xi$$

$\Rightarrow$ The only way you can get scale invariance (critical behavior) is if

$$\xi' = \xi \equiv \xi^* \rightarrow \infty \quad \text{or} \quad \phi$$

CRITICAL BEHAVIOR HAPPENS WHEN

$$\tilde{H}' = \hat{R}\tilde{H} = \tilde{H} = H^*$$

Flows in parameter space

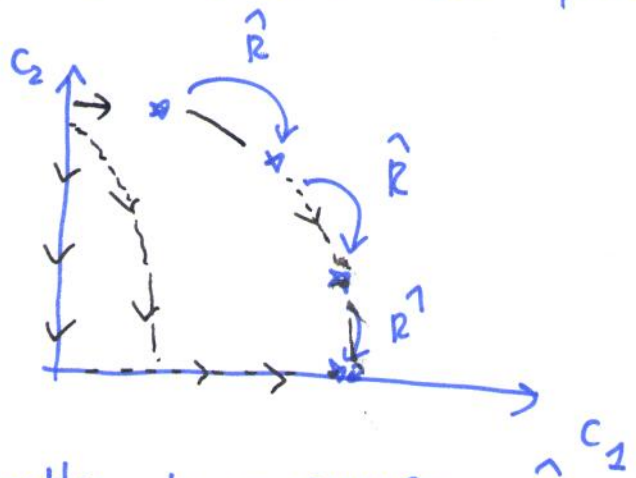
Let's make the RNG operator concrete. Say the Hamiltonian can be written as a sum over functions of the state variables

$$\tilde{H} = \sum c_i f_i[s]$$

The RNG can be seen as a transformation of the parameters

$$\underline{c}' = \hat{R}\,\underline{c}$$

- The application of RNG operation generates a "flow" in Hamiltonian parameter space



- No matter how complex, $\hat{R}$ can be linearised close to a fixed point for which

$$* \quad c' \equiv c \equiv c^*$$

if $c \sim c^* = c^* + \delta c$

linear operator $\downarrow$ $\equiv$ A MATRIX!

$$c' = \hat{R}\, c \sim c^* + A\,\delta c$$

* Repeat application of ~~A~~ $\hat{R}$ ≡ repeat multiplication by A! but $A^n = U^{\cancel{*}} \, \text{diag}(a^n) \, U^T$

↳ the eigenvalues of A determine the behavior close to the critical point

we write $a_i = b^{\enclose{circle}{y_i}}$ ← critical exponents!

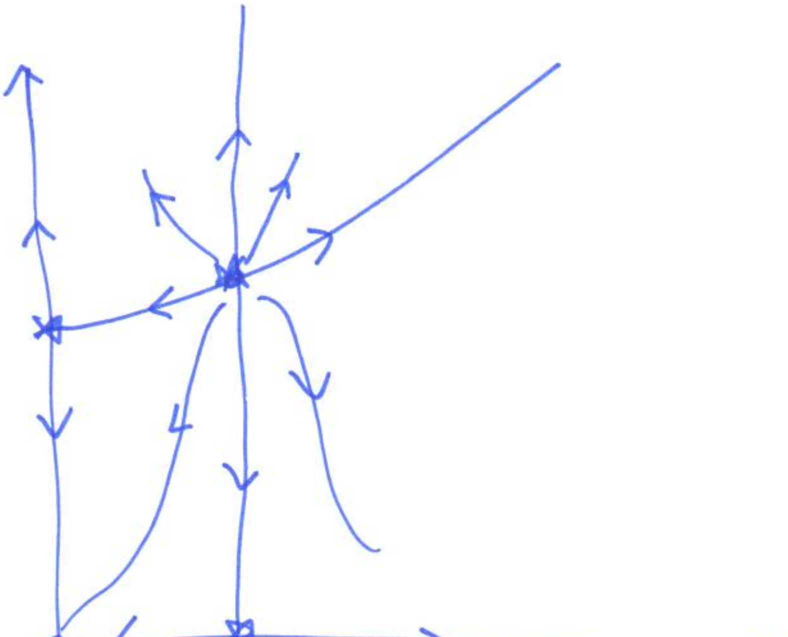
if $y_i > 0$ $a_i > 1$ $a_i^n \to \infty$ divergence!

<u>RELEVANT VARIABLE</u>

if $y_i < 0$ $a_i < 1$ $a_i^n \to 0$ convergence

<u>IRRELEVANT VARIABLE</u>

if $y_i = 0$ we cannot determine behavior based on linear description : <u>MARGINAL VARIABLE</u>



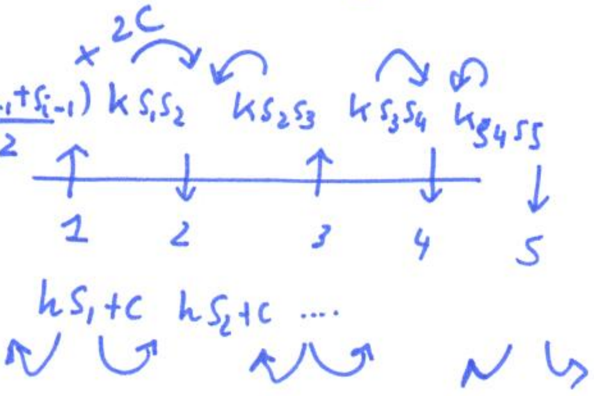
1D ISING MODEL example

$$\tilde{H} = -\overset{\overset{J/k_B}{\uparrow}}{k} \sum_{\{ij\} \nwarrow \text{pairs}} S_i S_J + \sum_i \overset{\overset{H\mu/kT}{\uparrow}}{h} S_i - \sum_i \overset{\Rsh \text{ initially zero ....}}{C}$$

let's write the partition fuction

$$Z = \sum_{\{s\}} e^{\tilde{H}[\{s\}]} = \sum_{\{s\}} e^{k \sum_{\{ij\}} S_i S_j + h \sum_i S_i + C \sum_i 1}$$

$$= \sum_{\{s\}} \prod_{i \in \text{even}} e^{k S_i (S_{i+1} + S_{i-1}) + h\left(S_i + \frac{S_{i+1}+S_{i-1}}{2}\right)}$$



→ DECIMATION ≡ PARTIAL TRACE! Sum explicitly over the even spins which we can do because even spins are decoupled

$$= \sum_{\{S \in \text{ODD}\}} \sum_{S_\text{even}} \prod_{i \in \text{even}} \text{\_\_\_\_}$$

$$\Vert$$

$$\prod_{i \in \text{even}} \sum_{S_i = \pm 1} \text{\_\_\_\_}$$

$$= \sum_{\{S_\text{ODD}\}} \prod_{i \in \text{even}} \left[e^{k(S_{i+1}+S_{i-1}) + h + 2C + \frac{h}{2}(S_{i+1}+S_{i-1})} + e^{-k(S_{i+1}+S_{i-1}) + h + 2C + \frac{h}{2}(S_{i+1}+S_{i-1})} \right]$$

* can we express this as a flow through the k, h, c parameter space ?

1. we renumber the spins

$$\mathcal{Z}_{N'}(\mathcal{H}') = \sum_{s} \prod_{i} \Big[e^{k(s_i + s_{i+1}) + h + 2c + \frac{h}{2}(s_i + s_{i+1})} + e^{-k(s_i + s_{i+1}) - h + 2c + \frac{h}{2}(s_i + s_{i+1})} \Big]$$

2. try to write this as

$$\sum_{s} \prod_{i} e^{k' s_i s_{i+1} + \underset{\substack{\downarrow \\ h'(s_i + s_{i+1})}}{h' s_i} + c'} =$$

there are 3 cases : $s_i = s_{i+1} = 1$ Ⓐ

$s_i = s_{i+1} = -1$ Ⓑ

$s_i = -s_{i+1} = \pm 1$ Ⓒ

Ⓐ $$e^{2k + 2h + 2c} + e^{-2k + 2c} = e^{k' + h' + c'}$$

Ⓑ $$e^{-2k + 2c} + e^{2k - 2h + 2c} = e^{k' - h + c'}$$

Ⓒ $$e^{h + 2c} + e^{-h + 2c} = e^{-k' + c'}$$

can be solved with change of variables

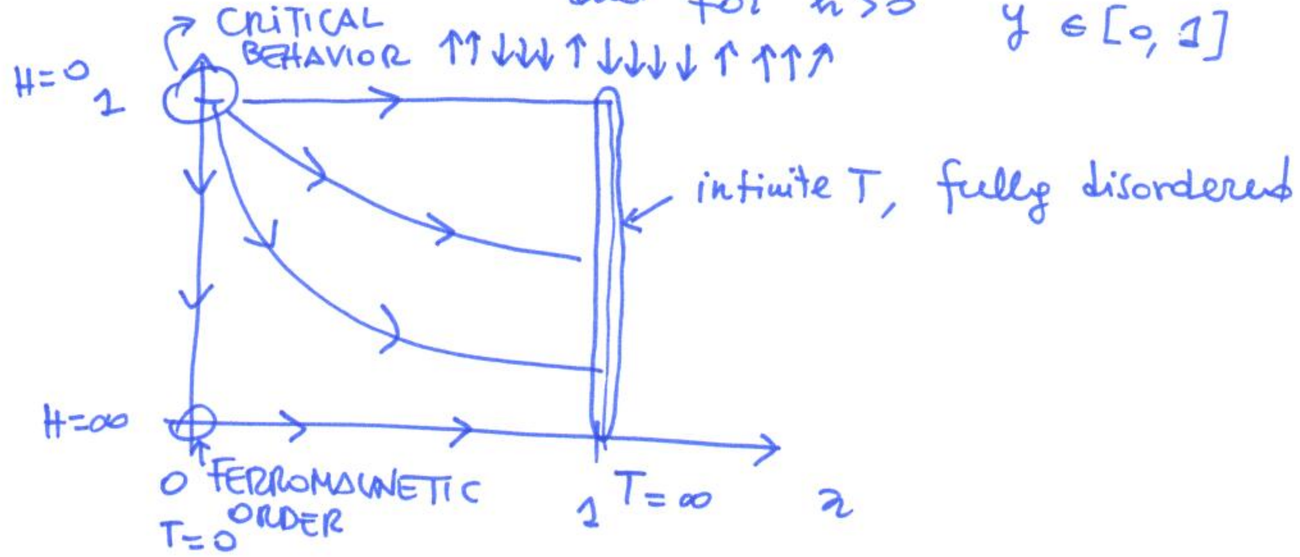
$w = e^{-4c}$ $\quad x = e^{-4k}$ $\quad y = e^{-2h}$

$$w' = w^2 x y^2 (1+y)^2 / (x+y)(1+xy)$$

$$x' = x(1+y)^2 / (x+y)(1+xy)$$

$$y' = y(x+y) / (1+xy)$$

ω describes the energy scale — we can ignore it for the moment. for $K>0$ $x \in [0, 1]$ and for $h>0$ $y \in [0, 1]$



fixed points :

$x=0,\ y=0$

$x=1,\ y=0$

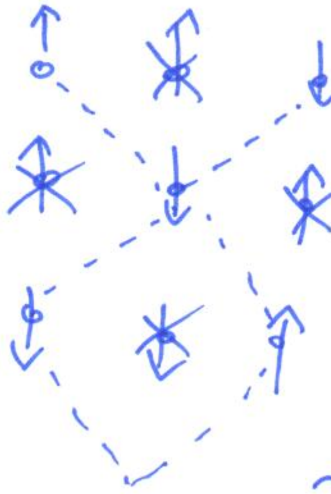
$x=1,\ y=1$

$x=0,\ y=1$

N.B.: the 1D ising model can be solved analytically

$$Z_N(\tilde{H}) = \lambda^N \qquad \lambda = e^K\left[\cosh h + \sqrt{\sinh^2 h + e^{-4K}}\right]$$

What happens in many dimensions?



every application of $\hat{R}$ introduces longer-range interactions (the C in the 1D model!)

so it is not possible to map EXACTLY $R\,C \rightarrow C'$

⇝ one must truncate the RNG expression setting longer-range interactions to zero, or by approximating them in a mean-field way.

TAKE-HOME MESSAGE

RNG provides physical justification to the observed universality of critical behavior! All is traced to

1. invariance of scale of critical fluctuations
2. symmetry of the interactions and dimensionality of the problem.

In all but the simplest cases, an approximate treatment is still needed, but the approximations can be made on a high order of interactions.