Lecture 3 *03-04.03.2025*

- *Tensors – non-mathematical introduction for tensors in applied physics and engineering of crystalline materials*

- what tensors are

- notations, operations, properties

- transformations of tensors, some useful techniques

- second rank tensors

- tensors and symmetry – Neumann principle

Lecture 2, Exercises

Initial symmetry:
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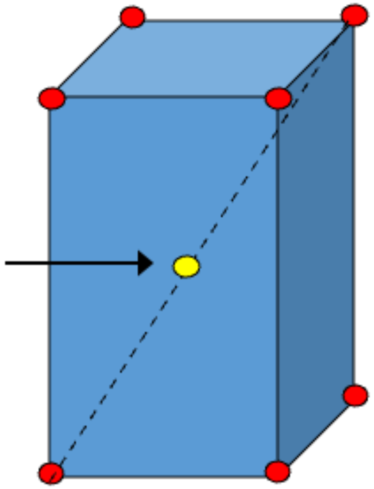
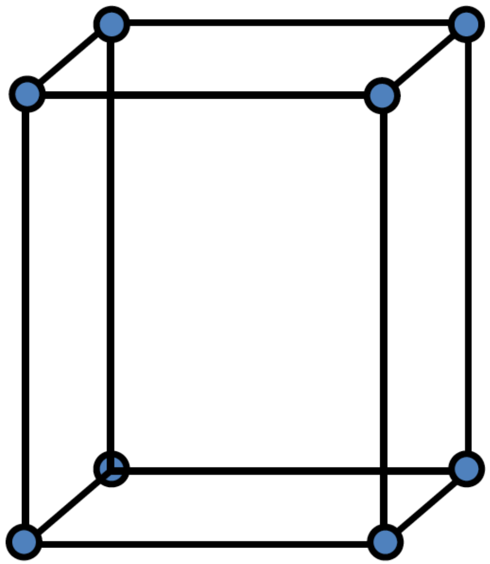


mmm



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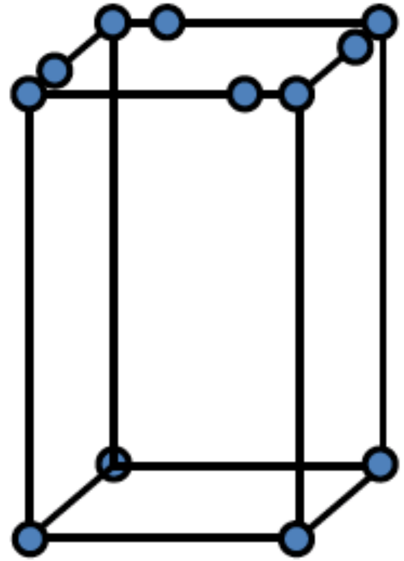

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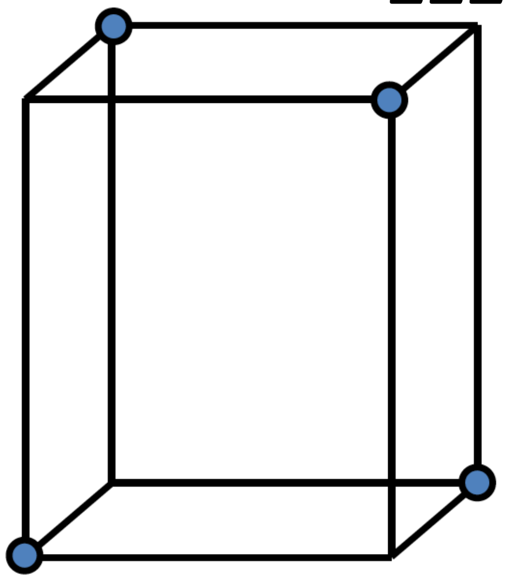

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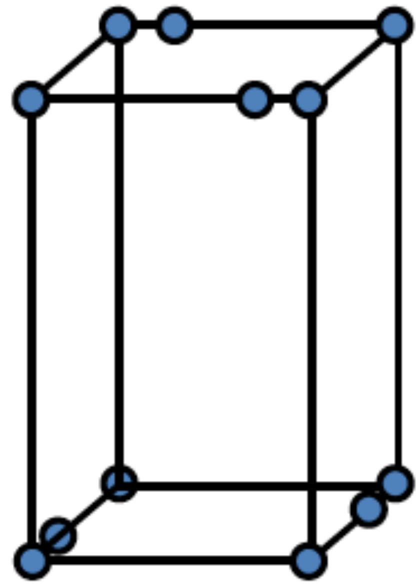
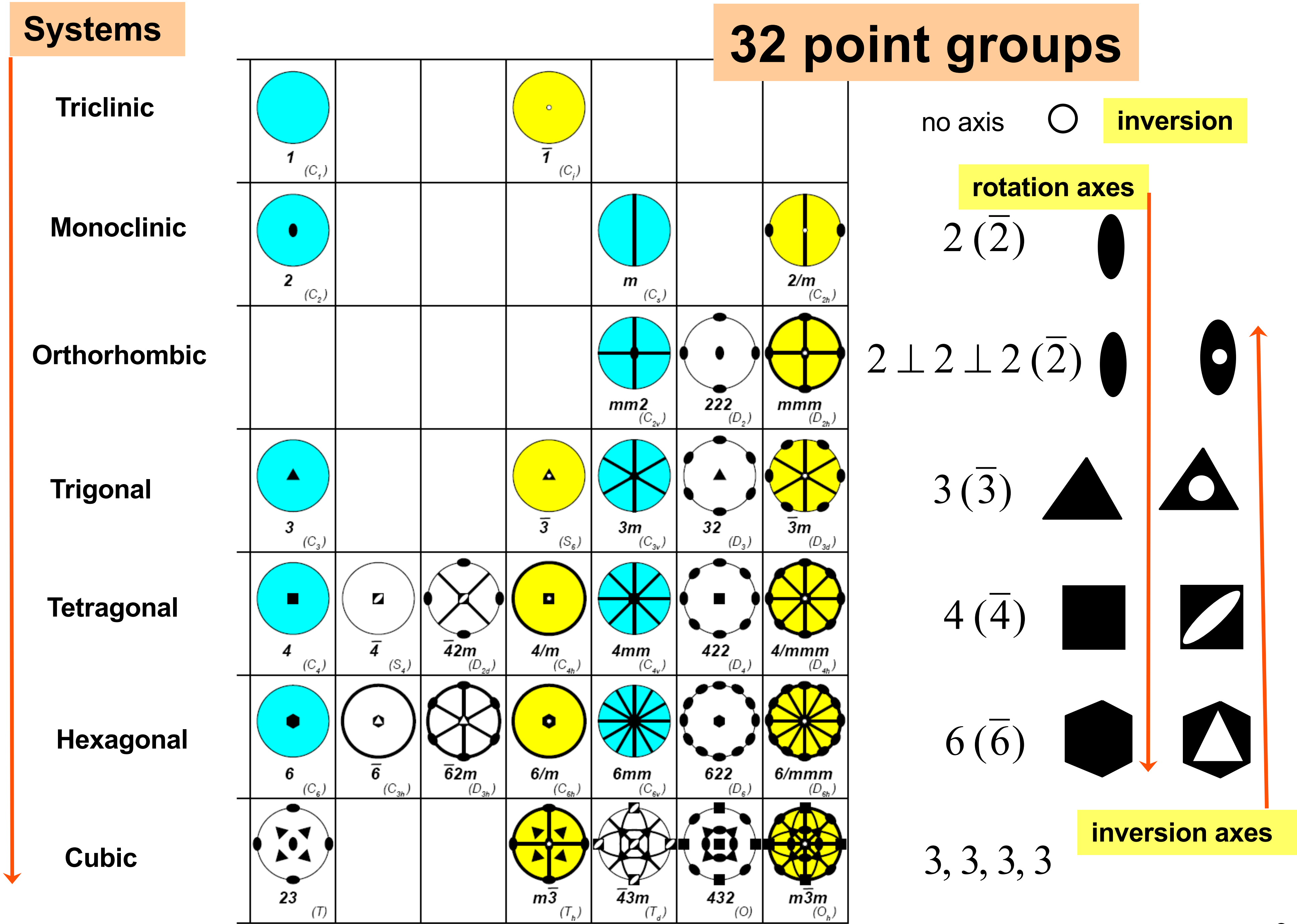


32 point groups



Lecture 2, Exercises

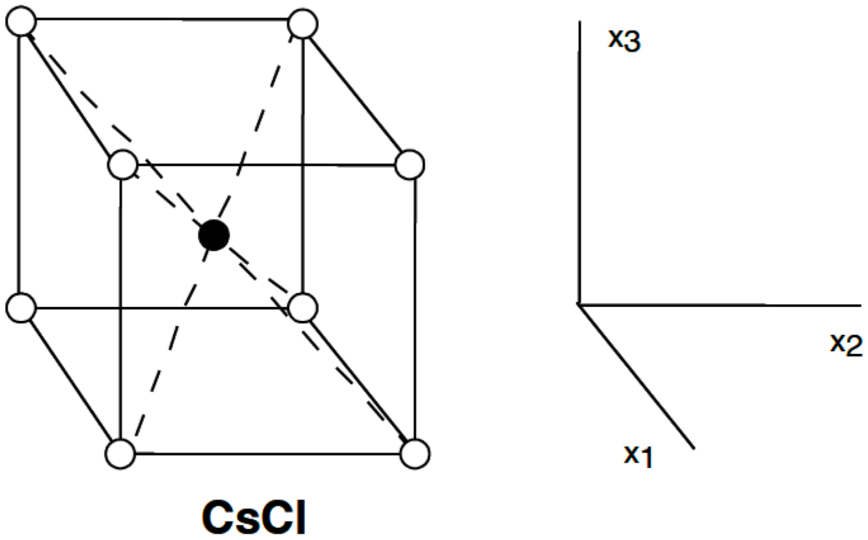


Fig.1.1

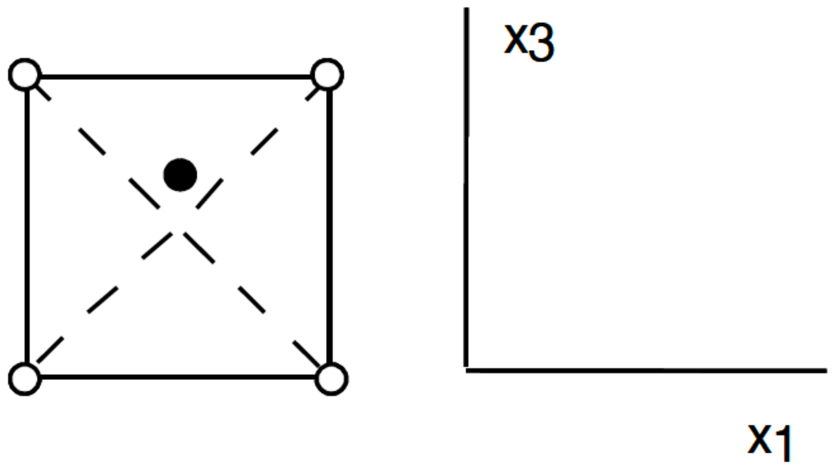
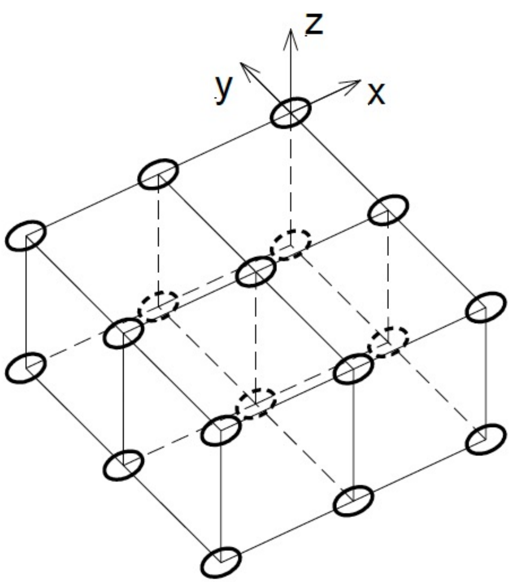


Fig.1.2

symmetry changes to 4mm



symmetry changes to mmm

Tensors

All physical phenomena can be described with tensors

Examples of tensors *(i,j = 1, 2, 3)*

$$P_i = \chi_{ij} E_j$$

Dielectric susceptibility tensor
Connects polarization and el. field

$$\varepsilon_{ij} = s_{ijkl} \sigma_{kl}$$

elastic compliance tensor
Connects strain and stress

Terminology:

Precise:

tensor	**=**	**polar tensor**
pseudo-tensor	**=**	**axial tensor**

Shorthand:

"tensor " is used for "tensor", "polar tensor", "pseudo-tensor", and "axial tensor"

Definition of tensors

1. Tensor of n-th rank is an object having

3^n **components (3D space)**

2. Usually written in the form

$$q - 0^{\text{th}} \text{ rank tensor}$$

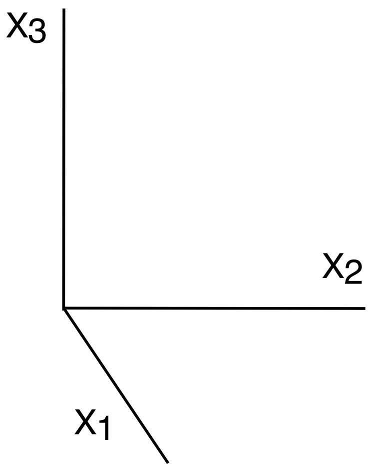
$$q_i - 1^{\text{st}} \text{ rank tensor } i = 1-3$$

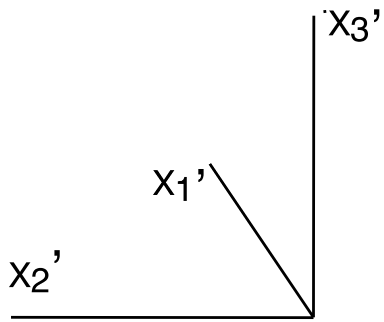
$$q_{ij} - 2^{\text{nd}} \text{ rank tensor } \; i, j = 1-3$$

and so on…………….

Definition of tensors

3. When one changes the reference frame where the phenomenon is described, tensors transform according to strictly specified laws





$$q_{ij} \Rightarrow q'_{ij}$$

Specified transformation law

Description of transformation of the reference frame

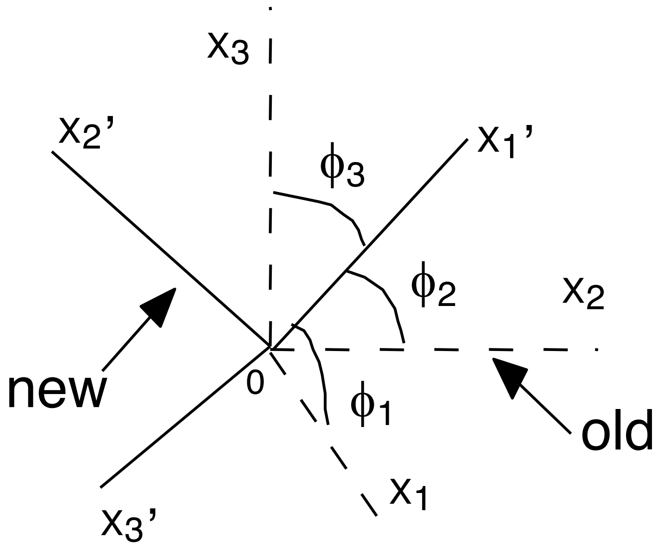


Table of direction cosines

	X_1	X_2	X_3
X_1'	a_{11}	a_{12}	a_{13}
X_2'	a_{21}	a_{22}	a_{23}
X_3'	a_{31}	a_{32}	a_{33}

$$a_{11} = \cos\phi_1 = \cos(X_1' \wedge X_1)$$

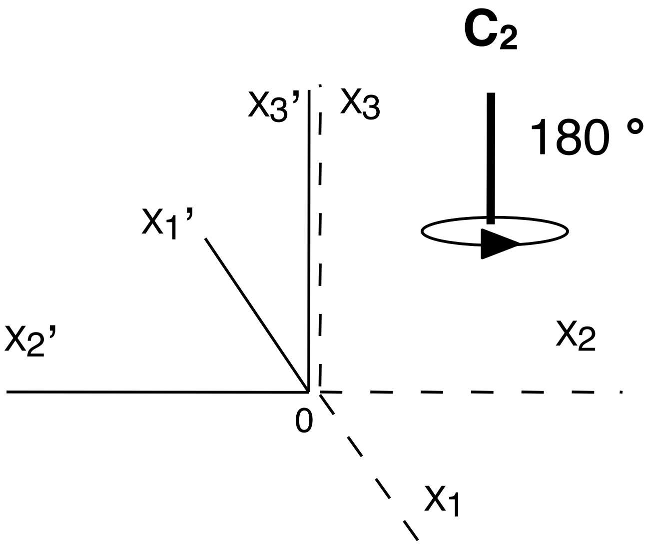
$$a_{12} = \cos\phi_2 = \cos(X_1' \wedge X_2)$$

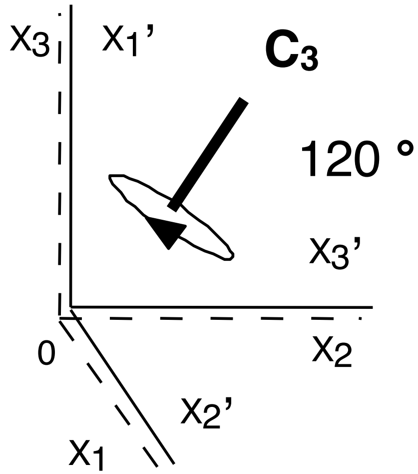
$$a_{13} = \cos\phi_3 = \cos(X_1' \wedge X_3)$$

$$a_{ij} = \cos(X_i' \wedge X_j)$$

new old

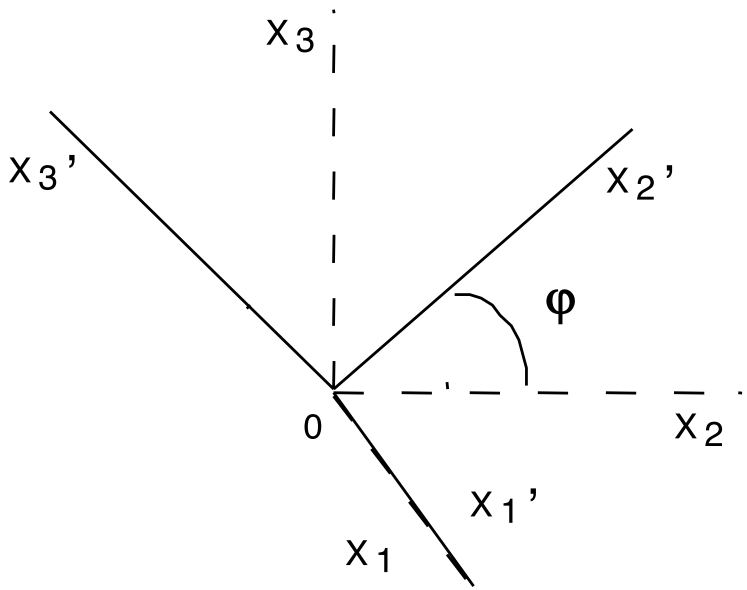
Examples of transformations of the reference frame





$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Leftarrow a_{ij} \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Examples of transformations of the reference frame



$$a_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix}$$

Transformation laws for tensors (polar tensors)

$$\rho - 0^{\text{th}} \text{ rank tensor } = \text{scalar (e.g., density)}$$

No change

Transformation laws for tensors (polar tensors)

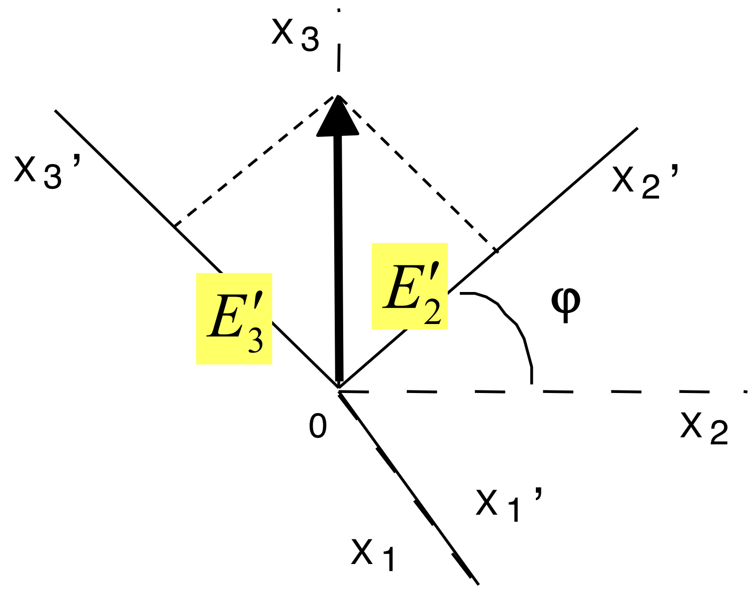
$$E_i - 1^{\text{st}} \text{ rank tensor} = \text{vector (e.g., electric field)}$$

Example: vector ***E*** is parallel to the X_3 direction

$$E_1' = \cos(X_1' \wedge X_3)E_3$$

$$E_2' = \cos(X_2' \wedge X_3)E_3$$

$$E_3' = \cos(X_3' \wedge X_3)E_3$$



$$E_1' = a_{13}E_3$$

$$E_2' = a_{23}E_3$$

$$E_3' = a_{33}E_3$$

Transformation laws for tensors (polar tensors)

E_i – 1st rank tensor = vector (e.g., electric field)

new

$$E'_1 = a_{11}E_1 + a_{12}E_2 + a_{13}E_3$$
$$E'_2 = a_{21}E_1 + a_{22}E_2 + a_{23}E_3$$
$$E'_3 = a_{31}E_1 + a_{32}E_2 + a_{33}E_3$$

old

The transformation law ensures that the vector is independent of the choice of the reference frame

Dummy suffix notation

$$E'_1 = a_{11}E_1 + a_{12}E_2 + a_{13}E_3$$

$$E'_2 = a_{21}E_1 + a_{22}E_2 + a_{23}E_3$$

$$E'_3 = a_{31}E_1 + a_{32}E_2 + a_{33}E_3$$

$$E'_i = \sum_{j=1-3} a_{ij}E_j$$

Einstein summation convention:

$$E'_i = a_{ij}E_j$$

1. When a letter suffix occurs twice in the same term, summation with respect to that suffix is to be automatically understood.

2. Such letter suffix is called *dummy suffix.*

Transformation laws for tensors
(polar tensors)

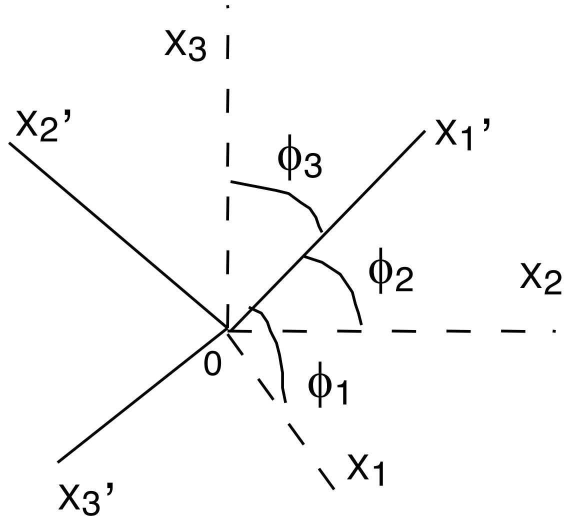
$E_i - 1^{\text{st}}$ rank tensor = vector $\qquad E'_i = a_{ij} E_j$

$T_{ij} - 2^{\text{nd}}$ rank tensor $\qquad T'_{ip} = a_{ij} a_{pq} T_{jq}$

$M_{ijk} - 3^{\text{rd}}$ rank tensor $\qquad M'_{ips} = a_{ij} a_{pq} a_{st} M_{jqt}$

n^{th} rank tensor $\quad M'_{\underbrace{ij.....pt}_{n}} = \underbrace{a_{ii'} a_{jj'} ......a_{pp'} a_{tt'}}_{n} M_{\underbrace{i'j'.....p't'}_{n}}$

Reverse transformation



$$a_{11} = \cos\phi_1$$

$$a_{12} = \cos\phi_2$$

$$a_{13} = \cos\phi_3$$

Direct

$$(X_1, X_2, X_3) \Rightarrow (X_1', X_2', X_3')$$

	X_1	X_2	X_3
X_1'	a_{11}	a_{12}	a_{13}
X_2'	a_{21}	a_{22}	a_{23}
X_3'	a_{31}	a_{32}	a_{33}

Reverse

$$(X_1', X_2', X_3') \Rightarrow (X_1, X_2, X_3)$$

	X_1'	X_2'	X_3'
X_1	a_{11}	a_{21}	a_{31}
X_2	a_{12}	a_{22}	a_{32}
X_3	a_{13}	a_{23}	a_{33}

Important property of direction cosines

For rotations and reflections (“orthogonal transformations”)

$$a_{is}a_{js} = \delta_{ij}$$

$$a_{si}a_{sj} = \delta_{ij}$$

Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases}$$

Why $a_{is}a_{js} = \delta_{ij}$?

Components of a vector should not change after changing the reference frame from one system to another and back

Axes $(X_1, X_2, X_3) \Rightarrow (X'_1, X'_2, X'_3)$ $(X'_1, X'_2, X'_3) \Rightarrow (X_1, X_2, X_3)$

Dir. cos a_{ij} $a^T_{ij} = a_{ji}$

Vectors $E'_i = a_{ij}E_j$ $E''_s = a^T_{si}E'_i$

$$E''_s = a^T_{si}a_{ij}E_j = a_{is}a_{ij}E_j$$

$E''_s = E_s$ → $a_{is}a_{ij} = \delta_{sj}$

Characteristic property of orthogonal transformations

Reflection and rotations
should note change the norm of the vector

$$\|E\|^2 = E_j E_j \qquad E'_i = a_{ij} E_j \qquad \|E'\|^2 = E'_j E'_j$$

$$\|E'\|^2 = a_{ij} E_j a_{is} E_s = a_{ij} a_{is} E_j E_s = \delta_{js} E_j E_s = E_j E_j = \|E\|^2$$

Why tensors?

Any physical phenomenon or property should be independent of the reference frame used for its description

↓

All physical phenomena and properties are described by tensors

Why tensors?

Ohm's law as an illustration

Electric field is a vector

E_i

Current density is a vector

J_i

Ohm's law describes the linear response of the current to the application of an electric field

Why tensors?

Ohm's law describes the linear response of the current to the application of an electric field

$$J_1 = \tau_{11}E_1 + \tau_{12}E_2 + \tau_{13}E_3$$
$$J_2 = \tau_{21}E_1 + \tau_{22}E_2 + \tau_{23}E_3$$
$$J_3 = \tau_{31}E_1 + \tau_{32}E_2 + \tau_{33}E_3$$

⟺

$$J_i = \tau_{ij}E_j$$

↑ **Conductivity**

Is conductivity a tensor?

Why tensors?

$$a_{ij}$$

$$(X_1, X_2, X_3) \longrightarrow (X'_1, X'_2, X'_3)$$

$$J_i = \tau_{ij} E_j \qquad J'_i = \tau'_{ij} E'_j$$

$$E'_i = a_{ij} E_j$$

$$J'_i = a_{ij} J_j$$

$$\tau'_{ij} = ?$$

We can demonstrate that if ***E*** and ***J*** both transform like 1st rank tensors, then τ_{ij} will transform like second rank tensor

How to live peacefully with tensors?
(Simple rules)

1. Dummy and free suffixes

2. Order of terms

3. Order of suffixes

4. Use of letters for suffixes

Dummy and free suffixes

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}$$

free

dummy

has no sense

$$\sigma_{ij} = c_{ijkl}\varepsilon_{il}$$

Order of terms

Order of terms is of no importance

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}$$

$$\sigma_{ij} = \varepsilon_{kl}c_{ijkl}$$

are equivalent

Order of suffixes

Order of suffixes is, in general, of importance

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}$$

$$\sigma_{ij} = c_{ikjl}\varepsilon_{kl}$$

$$\sigma_{ij} = c_{ljki}\varepsilon_{kl}$$

are NOT, in general, equivalent

Use of letters for suffixes

Change of letters used for suffixes does not change the meaning of a tensor expression

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}$$

$$\sigma_{ij} = c_{ijst}\varepsilon_{st}$$

$$\sigma_{mn} = c_{mnst}\varepsilon_{st}$$

are equivalent

How to read tensor expressions?

1. Identify dummy and free suffixes

2. Take a set of the free suffixes

3. Make summation over the dummy suffixes

4. Take another set of free suffices

5. Continue until all sets of free suffixes are taken

How to read tensor expressions?

$$G_{ijk} = b_i M_{jks} B_s$$

1. Identify dummy and free suffixes

s- dummy, i,j,k- free

2. Take a set of free suffixes

i=1, j=1, k=2

$$G_{112} = b_1 M_{12s} B_s$$

3. Make summation over the dummy suffixes

$$G_{112} = b_1 M_{121} B_1 + b_1 M_{122} B_2 + b_1 M_{123} B_3$$

4. Take another set of free suffices

i=1, j=1, k=1

$$G_{111} = b_1 M_{11s} B_s$$

5. Continue until all sets of free suffixes are taken

$$G_{111} = b_1 M_{111} B_1 + b_1 M_{112} B_2 + b_1 M_{113} B_3$$

How to read tensor expression?

$$G_{ijk} = b_i M_{jks} B_s$$

gives 3^3=27 equations

$$G_{111} = b_1 M_{111} B_1 + b_1 M_{112} B_2 + b_1 M_{113} B_3$$
$$G_{112} = b_1 M_{121} B_1 + b_1 M_{122} B_2 + b_1 M_{123} B_3$$
$$G_{113} = b_1 M_{131} B_1 + b_1 M_{132} B_2 + b_1 M_{133} B_3$$
$$G_{211} = b_2 M_{111} B_1 + b_2 M_{112} B_2 + b_2 M_{113} B_{33}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Further reading : book of Nye (English), suppl. notes on moodle (French)

Neumann principle

Any tensor describing a physical property of a material should not be affected when one changes the reference frame according to the symmetry elements of the material

Should be invariant with respect to the symmetry elements of the material!

Neumann principle

$$K_{i'j'} = a_{i'i}(t_1)a_{j'j}(t_1)K_{ij}$$

$$K_{i'j'} = a_{i'i}(t_2)a_{j'j}(t_2)K_{ij}$$

$$K_{i'j'} = a_{i'i}(t_3)a_{j'j}(t_3)K_{ij}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$K_{i'j'} = a_{i'i}(t_n)a_{j'j}(t_n)K_{ij}$$

$t_1, t_2, t_3 \ldots\ldots t_n$ **- symmetry elements of the material**

Neumann principle – another formulation

Symmetry elements of any physical property of a material must include the macroscopic symmetry elements of the material

Very often the symmetry of the property is higher than that of the material

How to make simple transformation of tensors?

"Simple transformations"
- rotations by 60^0, 90^0, 120^0, 180^0, and reflections

Transformations of vectors are always simple

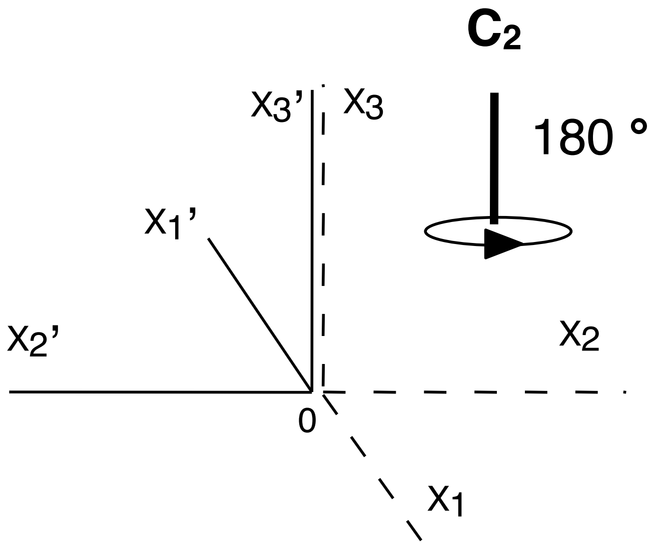
$$E'_i = a_{ij} E_j$$

Transformations of higher order tensors can also be simple

$$T'_{ip} = a_{ij} a_{pq} T_{jq}$$

$$M'_{ips} = a_{ij} a_{pq} a_{st} M_{jqt}$$

Transformation of vectors



$$p'_i = a_{ij} p_j$$

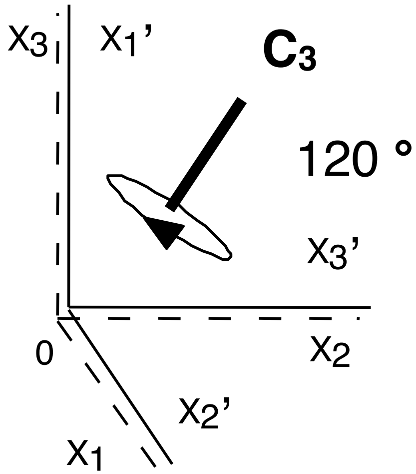
$$a_{ij} \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$p'_3 = p_3$$

$$p'_2 = -p_2$$

$$p'_1 = -p_1$$

Transformation of vectors



$$p'_i = a_{ij} p_j$$

$$p'_3 = p_2$$

$$p'_2 = p_1$$

$$p'_1 = p_3$$

$$a_{ij} \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Transformation of a tensor as products of components of a vector

$$p_i' = a_{ij} p_j$$

Two vectors

$$q_\alpha' = a_{\alpha\beta} q_\beta$$

$$p_i' q_\alpha' = a_{ij} p_j a_{\alpha\beta} q_\beta = a_{ij} a_{\alpha\beta} \; p_j q_\beta$$

Product of components of two vectors transforms same way as 2nd rank tensor

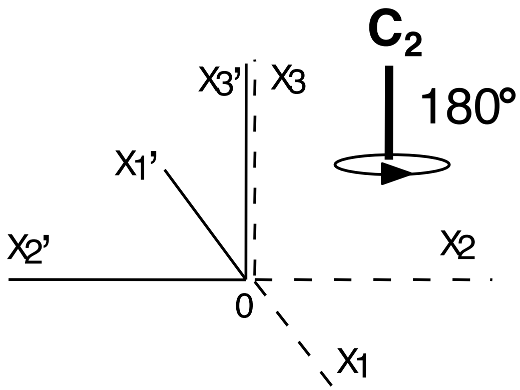
$$T_{i\alpha}' = a_{ij} \; a_{\alpha\beta} T_{j\beta}$$

2nd rank tensor

This can be extended to a tensor of any rank

Transformation of a tensor as products of components of a vector - example

Here p and q are 2 randomly chosen vectors



$$p'_3 = p_3 \qquad q'_3 = q_3$$

$$p'_2 = -p_2 \qquad q'_2 = -q_2$$

$$p'_1 = -p_1 \qquad q'_1 = -q_1$$

$$p'_1 q'_1 = p_1 q_1 \Rightarrow T'_{11} = T_{11} \qquad p'_1 q'_3 = -p_1 q_3 \Rightarrow T'_{13} = -T_{13}$$

$$p'_2 q'_2 = p_2 q_2 \Rightarrow T'_{22} = T_{22} \qquad p'_2 q'_3 = -p_2 q_3 \Rightarrow T'_{23} = -T_{23}$$

$$p'_3 q'_3 = p_3 q_3 \Rightarrow T'_{33} = T_{33}$$

and so on....

$$p'_1 q'_2 = p_1 q_2 \Rightarrow T'_{12} = T_{12}$$

Matrix and tensor notations

Simplification of notations

$$E_1' = a_{11}E_1 + a_{12}E_2 + a_{13}E_3$$
$$E_2' = a_{21}E_1 + a_{22}E_2 + a_{23}E_3$$
$$E_3' = a_{31}E_1 + a_{32}E_2 + a_{33}E_3$$

$$E_i' = \sum_{j=1-3} a_{ij}E_j \qquad E_i' = a_{ij}E_j$$

When the number of suffixes is 2 or less one can further simplify the notation

One can drop suffixes !

Matrix notations

Zero-suffix quantity

$$b \Rightarrow b$$

1-suffix quantity
(vector)

$$E_i \Rightarrow \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \Rightarrow \vec{E}$$

2-suffix quantity
(matrix)

$$T_{ij} \Rightarrow \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \Rightarrow \underline{T}$$

"Column-row" multiplication rule

$$A = q_j p_j$$

$$A = \vec{q}\,\vec{p}$$

$$A = \downarrow \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \downarrow$$

$$\longrightarrow$$

$$p'_i = a_{ij} p_j$$

$$\vec{p} = \underline{a}\vec{p}$$

$$\begin{pmatrix} p'_1 \\ p'_2 \\ p'_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \downarrow$$

Order plays a role!

“Column-row” multiplication rule

$$a_{ij} = b_{ik}c_{kj}$$

$$\underline{a} = \underline{b}\,\underline{c}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

The matrix-vector multiplication rule is applied to each column of the second matrix

$$\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} c_{11} \\ c_{21} \\ c_{31} \end{pmatrix}$$

Matrix and "tensor" notations

Tensor notation	**Matrix notation**	
$A = q_j p_j$	$A = \vec{q}\,\vec{p}$	**1 equation**
$p'_i = a_{ij} p_j$	$\vec{p} = \underline{a}\vec{p}$	**3 equations**
$a_{ij} = b_{ik} c_{kj}$	$\underline{a} = \underline{b}\,\underline{c}$	**9 equations**

One can easily pass from tensor to matrix notation if the dummy suffixes (in all pairs of them) are in the neighboring positions

When in matrix form, in contrast to the tensor form the <u>order of factors</u> is important

Matrix and “tensor” notations

How to pass from tensor to matrix notation for the case of 2nd rank tensor if the dummy suffixes are NOT in the neighboring positions?

Tensor notation

$$T'_{ip} = a_{ij} a_{pq} T_{jq}$$

Matrix notation

$$\underline{T'} = \underline{a}\,\underline{T}\,\underline{a^T}$$

Second rank tensors

Second- rank tensor can be represented as a matrix

(however 2nd rank tensor ≠ a matrix!)

Some matrix terminology applies for second rank tensors:

1. Transpose of a matrix $B_{ij}^{T} = B_{ji}$

2. Symmetric matrix $B_{ij}^{T} = B_{ij}$

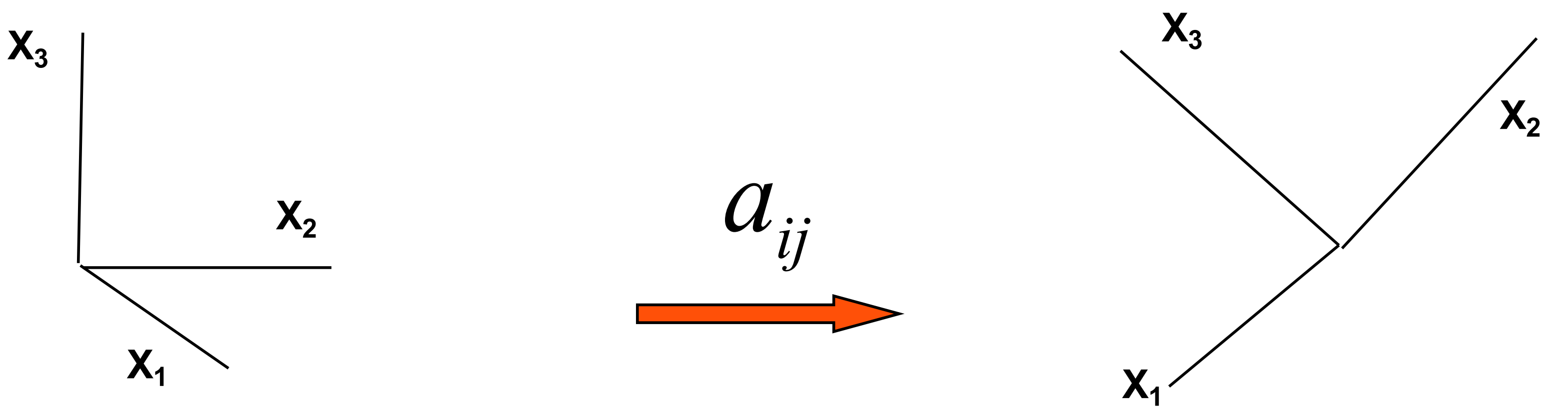
3. Anti-symmetric matrix $B_{ij}^{T} = -B_{ij}$

4. Diagonal matrix

$$\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$$

Symmetric second rank tensors

Any symmetric 2nd rank tensor can be diagonalized by a rotation of the reference frame

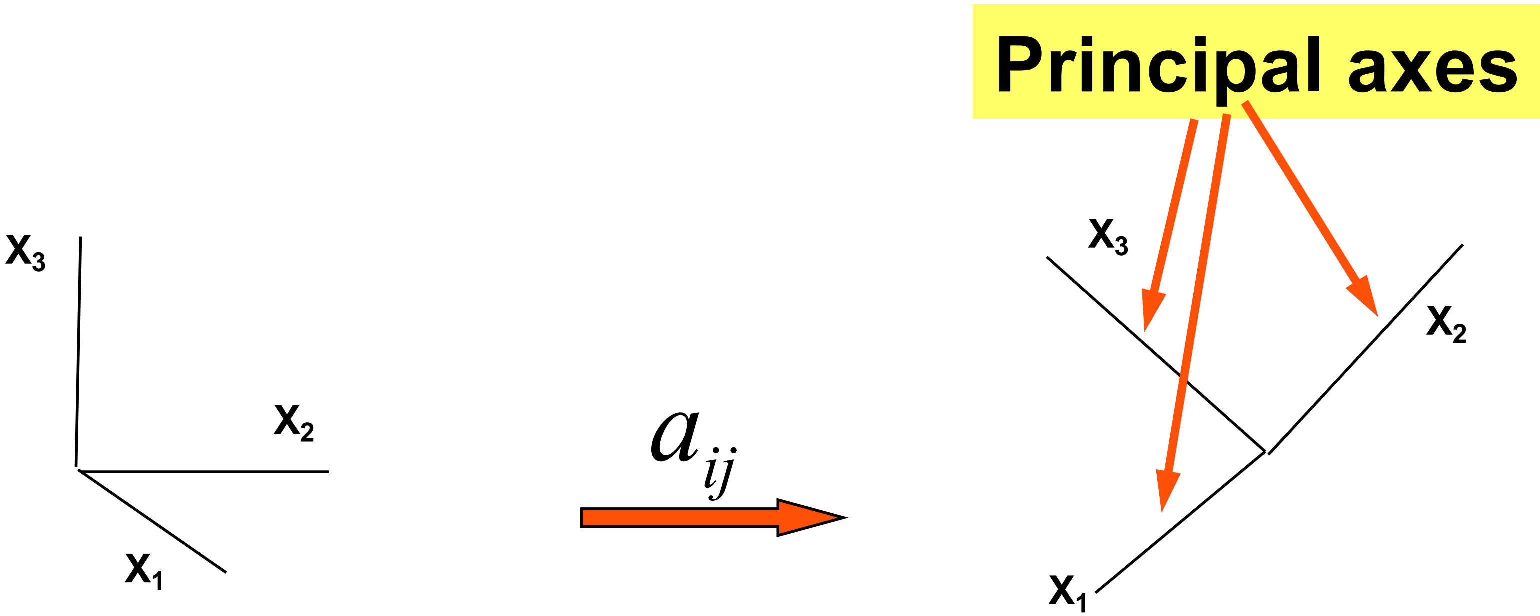


$$T_{ij} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{pmatrix}$$

$$T'_{ip} = a_{ij} a_{pq} T_{jq}$$

$$T'_{ij} = \begin{pmatrix} T^{(1)} & 0 & 0 \\ 0 & T^{(2)} & 0 \\ 0 & 0 & T^{(3)} \end{pmatrix}$$

Symmetric second rank tensors



Principal components

$$T_{ij} = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{pmatrix} \qquad T'_{ip} = a_{ij} a_{pq} T_{jq} \qquad T'_{ij} = \begin{pmatrix} T^{(1)} & 0 & 0 \\ 0 & T^{(2)} & 0 \\ 0 & 0 & T^{(3)} \end{pmatrix}$$

Symmetric second rank tensors

In principal axes, life is simpler

Arbitrary axes

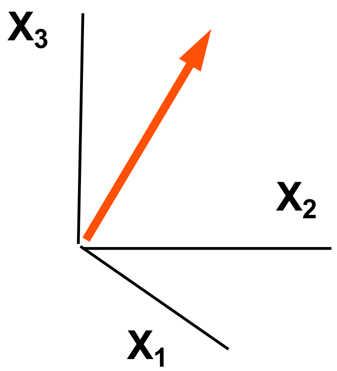
$$J_1 = \sigma_{11}E_1 + \sigma_{12}E_2 + \sigma_{13}E_3$$
$$J_2 = \sigma_{21}E_1 + \sigma_{22}E_2 + \sigma_{23}E_3$$
$$J_3 = \sigma_{31}E_1 + \sigma_{32}E_2 + \sigma_{33}E_3$$

Principal axes

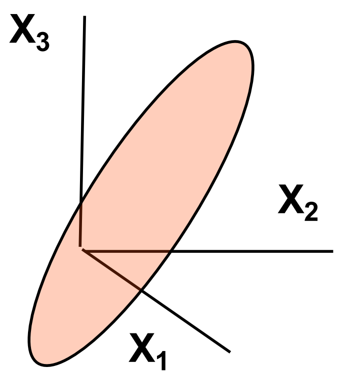
$$J_1' = \sigma^{(1)}E_1'$$
$$J_2' = \sigma^{(2)}E_2'$$
$$J_3' = \sigma^{(3)}E_3'$$

Geometric representation of tensors

Vector



Symmetric second rank tensor with positive principle components



Geometric representation of tensors

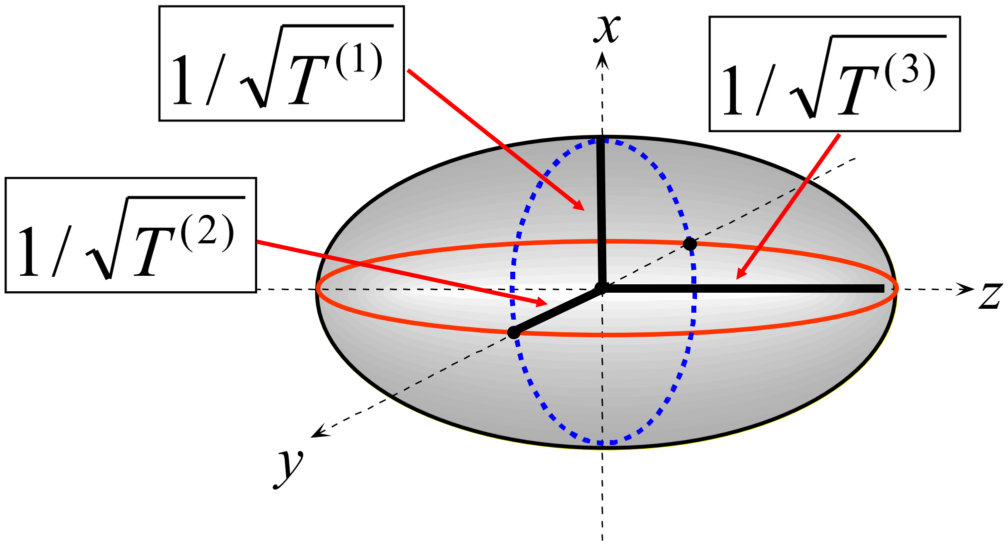
Symmetric second rank tensor

$$T_{ij}X_iX_j = 1$$

"Representation quadric"

- a geometrical representation of a second rank tensor
- The representation surface – visual image of the tensor
- Useful for calculation of materials properties

In principle axes of the tensor

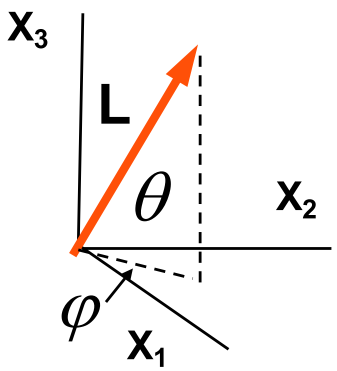


$$T'_{ij} = \begin{pmatrix} T^{(1)} & 0 & 0 \\ 0 & T^{(2)} & 0 \\ 0 & 0 & T^{(3)} \end{pmatrix}$$

$$T^{(1)}, T^{(2)}, T^{(3)} > 0$$

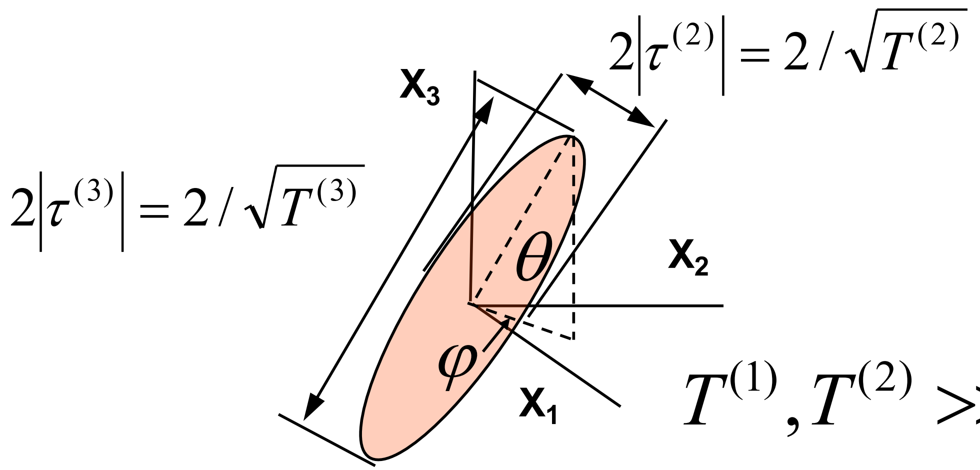
Geometric representation of tensors

Vector



$$\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} \Rightarrow \begin{matrix} L \\ \theta \\ \varphi \end{matrix}$$

Symmetric second rank tensor



$$T^{(1)}, T^{(2)} >> T^{(3)} > 0$$

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ & T_{22} & T_{23} \\ & & T_{33} \end{pmatrix} \Rightarrow \begin{matrix} \tau^{(1)} & \theta \\ \tau^{(2)} & \varphi \\ \tau^{(3)} & \psi \end{matrix}$$

Pseudo-tensors

	tensor	pseudo-tensor
Rotations	$T'_{ip} = a_{ij} a_{pq} T_{jq}$	$P'_{ip} = a_{ij} a_{pq} P_{jq}$
Reflections (Inversions)	$T'_{ip} = a_{ij} a_{pq} T_{jq}$	$P'_{ip} = -a_{ij} a_{pq} P_{jq}$

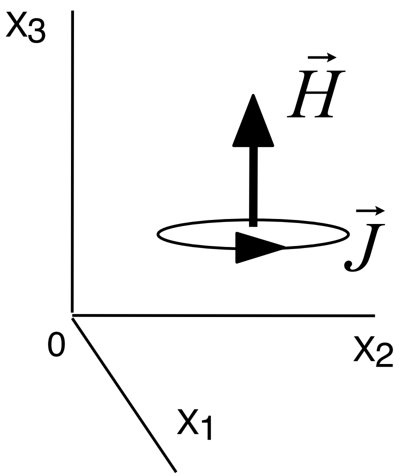
n^{th} rank pseudo - tensor

$$P'_{ij.....pt} = a_{ii'} a_{jj'} ......a_{pp'} a_{tt'} P_{i'j'.....p'} \det|a|$$

rotations $\det|a| = 1$ **reflections** $\det|a| = -1$

Pseudo-tensors

Example - magnetic field
– 1st rank pseudo-tensor (pseudo-vector)



(X_1 X_2) mirror

$$\vec{J} \rightarrow \vec{J}$$
$$\vec{H} \rightarrow \vec{H}$$

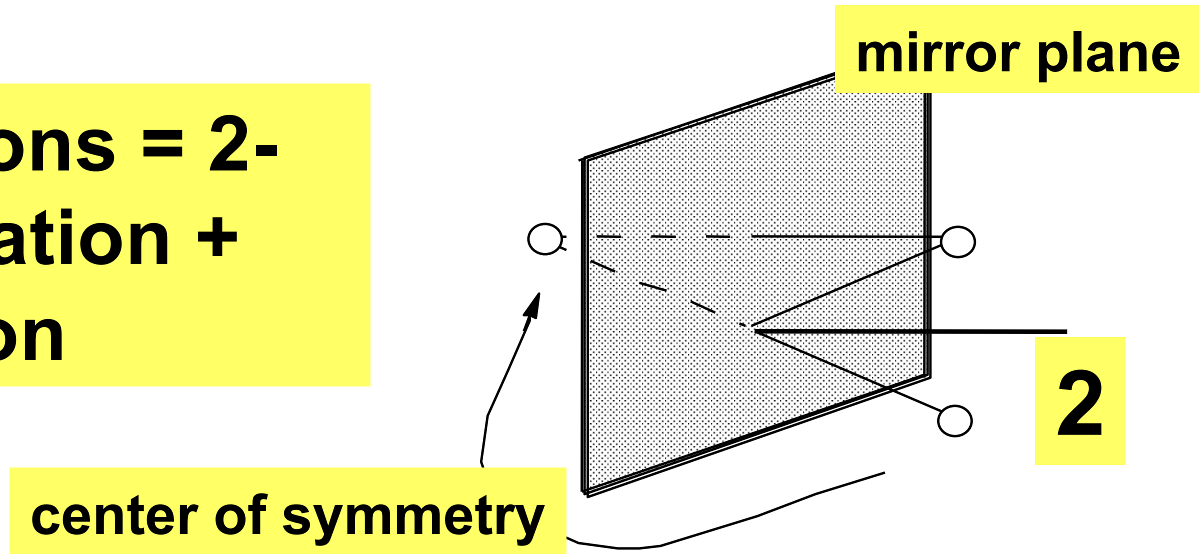
(X_2 X_3) mirror

$$\vec{J} \rightarrow -\vec{J}$$
$$\vec{H} \rightarrow -\vec{H}$$

Pseudo-tensors

All what has been said about manipulations with tensors can be applied to pseudo-tensors, one should only multiply the result of a manipulation by (-1) when dealing with a symmetry operation containing reflection.

Note: Inversions = 2-fold axis rotation + reflection



Invariant tensors

Kronecker delta
is an invariant symmetric second rank tensor

$$\delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases}$$

$$\delta'_{ij} = a_{ik} a_{js} \delta_{ks} = a_{ik} a_{jk} = \delta_{ij}$$

$$a_{is} a_{js} = \delta_{ij}$$

Essential

1. Tensors and pseudo-tensors are characterized by their transformation laws.

$$T'_{ip} = a_{ij} a_{pq} T_{jq}$$

$$M'_{ips} = a_{ij} a_{pq} a_{st} M_{jqt}$$

2. “Tensor” and matrix notations are possible

3. There exists a simple way to transform tensors (technique of vector components)

4. Life is simpler in the principal axes