EPFL
Spring Semester 2022, **04.07.22**

Probability and statistics for materials science
Prof. Philip Moll

EXAM

Time: The exam starts at 9:15 and finishes at 11:15.

Points: The exam consists of 4 questions. Each question and sub-steps have individual points associated to them, indicated in brackets.

General remarks:

1) Please provide your answers **only** on the supplied quad paper. The question sheets and your draft sheets will not be graded.
2) Explain your reasoning well to justify your answers. For all answers, the computations leading to the results have to be clear to achieve full points.
3) Drafting sheets and additional paper is supplied as well. **Drafting paper will not be graded**. Please write your name and page number on each sheet.
4) Round numerical values to the second digit ($\pi = 3.14$).
5) **Allowed supporting materials for you are one double-sided A4 sheet of notes and a non-programmable calculator.**

Question	Points Total	Points achieved
1	19	
2	22	
3	25	
4	19	
Total	**85**	

1 Winter games

Biathlon is an interesting winter sport, probing both endurance and force but also fine motion and steadiness of the athletes. After a loop of nordic skiing, the athletes have to shoot 5 targets with a rifle, one after another. The score of a game is the number of targets they hit (order is unimportant), so it ranges from 0 (missed all) to 5 (hit all). Being popular with young athletes, the competition to enter the Swiss National team is fierce. Today, there is a competition to select the best shooters for the team. The shooting skills of each athlete boil down to their probability p at which they hit a target. *Hint: Read the entire task and identify first which coefficients you need to compute.*

a) [1 pt] Susy is a good shooter, who usually hits at $p_1 = 0.8$. Say X is the number of targets she hits. What is the probability that she hits at least 3 out of the 5 targets, $P(X \geq 3)$?

b) [2 pt] Given her skills at $p_1 = 0.8$, compute the most likely number X of targets she will hit in a game? Explain your result.

c) [7 pt] She competes with Ben, who is a slightly worse shooter at $p_2 = 0.7$. Despite Susy is the better shooter, there is of course a statistical possibility that Ben will score higher than Susy and win a game. Say X denotes the number of targets Susy hits, and Y those that Ben hits. Compute the probability for Ben to win a competition, $P(Y > X)$. *Hint: As both are fairly good, you may ignore the probabilities that either score only 0 or 1 point, i.e.* $P(X = 0) \approx P(X = 1) \approx P(Y = 0) \approx P(Y = 1) \approx 0$. *This approximation introduces a small error of about 0.5%.*

d) [2 pt] Fantastic news for Susy, she made it into the National team! Congrats! After some time the trainer has to have a word with her. He has the impression she is nervous, and underperforms now that she is on the team. Susy rejects that harshly, walks to the shooting range, and shoots 10 full games with 5 targets each. Here is her data, can you show based on this if she still hits with probability $p_1 = 0.8$?

Points hit (X)	0	1	2	3	4	5
Number of games	0	0	3	2	3	2

Table 1: This table shows in how many games she scored points between 0 and 5. Quite a few hits, impressive!

State the statistical test you would use to verify the trainers suggestion, formulate the null hypothesis H_0 and the alternative hypothesis H_1 for this test.

e) [7 pt] Test your hypothesis identified in d) at a level of significance $\alpha = 0.05$. Is the trainer right?

2 From the Food Lab

All life depends on a constant supply of healthy food. With the growing human population, the side effects of industrial agriculture become apparent - from the CO_2 footprint to the overuse of pesticides and the loss of biodiversity in large monocultures. As part of your PhD thesis in the food lab you develop a new process to chemically modify cellulose to an edible protein - food from wood! Surprised by the elegance and simplicity of your solution, your supervisor admits your protein to an expensive lab testing study. It showed that your product is of high quality, however it found worrying levels of potassium from your chemical process. In order to qualify as food for human use, your protein must not exceed 6% of potassium.

You submit 40 samples to the laboratory and have them tested for their potassium concentration X_i (%):

0.1	1.3	2.3	2.4	2.5	2.6	2.6	2.7	2.8	2.9	3.0	3.3	3.6	3.7	3.7	3.7	3.8	3.8	4.1
4.3	4.4	4.6	4.8	5.1	5.2	5.3	5.3	5.6	5.6	5.8	5.8	6.0	6.2	6.4	6.4	6.7	7.7	7.7
7.9	9.2																	

Table 2: Potassium test results in %. *Hint: this question can be solved without typing all these numbers into your calculator.*

a) [3 pt] Compute the mean of the data in table 2. Naturally you use R for this, but unfortunately it only gives strange results. Instead of the mean, it computes for you $\sum_{i=1}^{40}(x_i - 10)^2 = 1345.83$, $\sum_{i=1}^{40} x_i^4 = 37154.21$ and $\sum_{i=1}^{40} x_i^2 = 963.83$ from this dataset.

b) [2 pt] State the empirical quantiles $q_{18\%}, q_{25\%}, q_{50\%}, q_{75\%}$.

c) [6 pt] Analyze your dataset by drawing a boxplot. Describe briefly the relevant parameters of the boxplot, and give their numerical values. What do you notice about the data?

d) [4 pt] A further analysis shows that the datapoint at 9.2% was due to a measurement error, and you remove that datapoint. Next you want to build a model for the remaining $n = 39$ datapoints based on a normal distribution. Estimate the standard deviation σ and mean μ from your remaining 39 datapoints.

e) [5 pt] Assuming your potassium concentration follows a normal distribution $N(\mu, \sigma^2)$, state the probability that a random protein snack exceeds the safe limit of 8% potassium (using the parameters of the previous task d). To be allowed for distribution, the legal requirement is that at most 1% of your products exceeds this limit. Can you sell your product?

f) [2 pt] Compare your results from e) to your empirical probability from your data. Briefly state if this result is reasonable.

3 Clean power

Global warming as well as energy security considerations will require us to find new methods to produce clean energy locally, a big challenge to clean energy in Europe is a "Not in my backyard" mindset. Everybody wants more solar panels and wind turbines, just not close to where they live because they destroy the beautiful natural landscape. You address this issue in your PhD project by changing the color of solar panels to a dark green, so that they blend in nicely into nature.

As a highlight of your thesis, Romande Energie offers to build a small demonstrator next to an existing, normal solar power plant. They ask 6 random people which is more beautiful. They are tasked to rate the "esthetics" of both installations on a scale between 1 (worst) to 10 (best). Here are their grades:

Your green panels (1)	9	7	6	10	8	7
Regular solar panels (2)	4	9	6	8	3	4

Table 3: Beauty rating of solar power plants. Grades between 1 (worst) and 10 (best).

a) [2 pt] Compute the group mean $\bar{X}_1$ and the standard deviation s_1 for your green panels (group 1). For the regular ones (group 2), the same calculation gives $\bar{X}_2 = 5.67$ and $s_2 = 2.42$.

b) [2 pt] Let us figure out if your green panels are more appealing than the existing ones. Formulate the null hypothesis H_0 and alternative hypothesis H_1 for a T-test. Justify your choice of one-sided vs. two-sided T-test.

c) [8 pt] Perform the T-test according to b). Is there a statistically relevant differences at a level of significance $\alpha = 0.05$?

Romande Energie encourages you to approach politicians, and they too see the need to make clean power more socially acceptable! However, they caution that other clean power techniques exist, such as wind farms (2) and hydropower plants (3). To support your product, it is important to see if people approve of it more than these alternatives. Another 6 reviewers are now tasked to grade these other technologies. As you now have 3 groups of power plants to compare, you chose to do an ANOVA analysis. Here is your data:

Your green panels (1)	9	7	6	10	8	7
Wind farms (2)	6	6	5	4	6	5
Hydropower (3)	5	6	7	4	4	6

Table 4: Beauty rating of clean power technologies. Grades between 1 (worst) and 10 (best). *The results of group 1 are the same as in the previous part - no need to recompute mean and standard deviation.*

You use R to compute the statistical parameters of these new groups 2 and 3, and find $\bar{X}_2 = \bar{X}_3 = 5.34$, $s_2 = 0.82$ and $s_3 = 1.21$.

d) [7 pt] Compute the one-factor ANOVA table for this model.

e) [4 pt] Based on your ANOVA analysis, perform an F-test. Can you state a statistically significant difference between these products at a level of significance of $\alpha = 0.05$?

f) [2 pt] Briefly summarize the results of both tests. Where does your product stand?

4 Climate change modeling

Our world is ever changing, and us humans take part in that change. A key challenge of our time is to understand and model this change, to give policymakers the right guidance to ensure we have a livable planet for generations to come. As part of your research project, you study the CO_2 emission increase due to the growing use of servers and computer infrastructure. As a basis for your research, you need to quantify the global computing power and describe quantitatively its change with time.

Here are your experimental results:

Time since the year 2000 (T)	0	1	2	3
Normalized computing power (C)	-0.1	0.9	2.3	9.6

Table 5: 4 datapoints of your computing power study

a) [6 pt] Clearly, the computing power C is growing every year. Your first guess is a linear relation, $C = A \cdot T$, where A is a parameter of your model. Apply the principle of least squares to this simple model $\hat{C}^{(1)} = A \cdot T$ and compute the coefficient A that best describes your data.

b) [5 pt] At a research conference, you see that most of your colleagues use different models. Technology is evolving rapidly, so that the computing power really grows faster than a linear model. Instead, they model the time dependence in quadrature. Hence, you try a second model and describe your data by $\hat{C}^{(2)} = B \cdot T^2$. Apply the principle of least squares to model 2 and compute the coefficient B that best describes your data.

c) [1 pt] Are $\hat{C}^{(1)}$ and $\hat{C}^{(2)}$ nested models? Briefly explain your answer.

d) [7 pt] Compute the Error Sum of Squares (SSE) for both models. Which model explains more of the total error?

Tableau de la fonction de répartition normale. La fonction $\Phi(z)$ est égale à la proportion des réalisations inférieures ou égales à z d'une varable aléatoire normale centrée et réduite.

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0,00	0,500	0,72	0,764	1,44	0,9251	2,16	0,9846	2,88	0,99801	3,80	0,9999277
0,02	0,508	0,74	0,770	1,46	0,9279	2,18	0,9854	2,90	0,99813	3,84	0,9999385
0,04	0,516	0,76	0,776	1,48	0,9306	2,20	0,9861	2,92	0,99825	3,88	0,9999478
0,06	0,524	0,78	0,782	1,50	0,9332	2,22	0,9868	2,94	0,99836	3,92	0,9999557
0,08	0,532	0,80	0,788	1,52	0,9357	2,24	0,9875	2,96	0,99846	3,96	0,9999625
0,10	0,540	0,82	0,794	1,54	0,9382	2,26	0,9881	2,98	0,99856	4,00	0,9999683
0,12	0,548	0,84	0,800	1,56	0,9406	2,28	0,9887	3,00	0,99865	4,04	0,9999733
0,14	0,556	0,86	0,805	1,58	0,9429	2,30	0,9893	3,02	0,99874	4,08	0,9999775
0,16	0,564	0,88	0,811	1,60	0,9452	2,32	0,9898	3,04	0,99882	4,12	0,9999811
0,18	0,571	0,90	0,816	1,62	0,9474	2,34	0,9904	3,06	0,99889	4,16	0,9999841
0,20	0,579	0,92	0,821	1,64	0,9495	2,36	0,9909	3,08	0,99996	4,20	0,9999867
0,22	0,587	0,94	0,826	1,66	0,9515	2,38	0,9913	3,10	0,99903	4,24	0,9999888
0,24	0,595	0,96	0,831	1,68	0,9535	2,40	0,9918	3,12	0,99910	4,28	0,9999907
0,26	0,603	0,98	0,836	1,70	0,9554	2,42	0,9922	3,14	0,99916	4,32	0,9999922
0,28	0,610	1,00	0,841	1,72	0,9573	2,44	0,9927	3,16	0,99921	4,36	0,9999935
0,30	0,618	1,02	0,846	1,74	0,9591	2,46	0,9931	3,18	0,99926	4,40	0,9999946
0,32	0,626	1,04	0,851	1,76	0,9608	2,48	0,9934	3,20	0,99931	4,44	0,9999955
0,34	0,633	1,06	0,855	1,78	0,9625	2,50	0,9938	3,22	0,99936	4,48	0,9999963
0,36	0,641	1,08	0,860	1,80	0,9641	2,52	0,9941	3,24	0,99940	4,52	0,9999969
0,38	0,648	1,10	0,864	1,82	0,9656	2,54	0,9945	3,26	0,99944	4,56	0,9999974
0,40	0,655	1,12	0,869	1,84	0,9671	2,56	0,9948	3,28	0,99948	4,60	0,9999979
0,42	0,663	1,14	0,873	1,86	0,9686	2,58	0,9951	3,30	0,99952	4,64	0,9999983
0,44	0,670	1,16	0,877	1,88	0,9799	2,60	0,9953	3,32	0,99955	4,68	0,9999986
0,46	0,677	1,18	0,881	1,90	0,9713	2,62	0,9956	3,34	0,99958	4,72	0,9999988
0,48	0,684	1,20	0,885	1,92	0,9726	2,64	0,9959	3,36	0,99961	4,76	0,9999990
0,50	0,691	1,22	0,889	1,94	0,9738	2,66	0,9961	3,38	0,99964	4,80	0,9999992
0,52	0,698	1,24	0,893	1,96	0,9750	2,68	0,9963	3,40	0,99966	4,84	0,9999994
0,54	0,705	1,26	0,896	1,98	0,9761	2,70	0,9965	3,42	0,99969	4,88	0,9999995
0,56	0,712	1,28	0,900	2,00	0,9772	2,72	0,9967	3,44	0,99971	4,92	0,9999996
0,58	0,719	1,30	0,903	2,02	0,9783	2,74	0,9969	3,46	0,99973	4,96	0,9999996
0,60	0,726	1,32	0,907	2,04	0,9793	2,76	0,9971	3,48	0,99975	5,00	0,9999997
0,62	0,732	1,34	0,910	2,06	0,9803	2,78	0,9973	3,50	0,99977	5,04	0,9999998
0,64	0,739	1,36	0,913	2,08	0,9812	2,80	0,9974	3,52	0,99978	5,08	0,9999998
0,66	0,745	1,38	0,916	2,10	0,9821	2,82	0,9976	3,54	0,99980	5,12	0,9999998
0,68	0,752	1,40	0,919	2,12	0,9830	2,84	0,9977	3,56	0,99981	5,16	0,9999999
0,70	0,758	1,42	0,922	2,14	0,9838	2,86	0,9979	3,58	0,99983	5,20	0,9999999

Quantiles de la loi t_ν de Student

ν	$qt_\nu(95\%)$	$qt_\nu(97{,}5\%)$	$qt_\nu(99\%)$	ν	$qt_\nu(95\%)$	$qt_\nu(97{,}5\%)$	$qt_\nu(99\%)$
1	6,314	12,71	31,82	21	1,721	2,080	2,518
2	2,920	4,303	6,965	22	1,717	2,074	2,508
3	2,353	3,182	4,541	23	1,714	2,069	2,500
4	2,132	2,776	3,747	24	1,711	2,064	2,492
5	2,015	2,571	3,365	25	1,708	2,060	2,485
6	1,943	2,447	3,143	26	1,706	2,056	2,479
7	1,895	2,365	2,998	27	1,703	2,052	2,473
8	1,860	2,306	2,896	28	1,701	2,048	2,467
9	1,833	2,262	2,821	29	1,699	2,045	2,462
10	1,812	2,228	2,764	30	1,697	2,042	2,457
11	1,796	2,201	2,718	32	1,694	2,037	2,449
12	1,782	2,179	2,681	34	1,691	2,032	2,441
13	1,771	2,160	2,650	36	1,688	2,028	2,434
14	1,761	2,145	2,624	38	1,686	2,024	2,429
15	1,753	2,131	2,602	40	1,684	2,021	2,423
16	1,746	2,120	2,583	50	1,676	2,009	2,403
17	1,740	2,110	2,567	60	1,671	2,000	2,390
18	1,734	2,101	2,552	120	1,658	1,980	2,358
19	1,729	2,093	2,539	∞	1,645	1,960	2,326
20	1,725	2,086	2,528				

Table 1 – Quantiles des distributions t_ν. Pour ν suffisamment grand, on peut substituer aux quantiles de la loi t_ν les quantiles de la loi normale.

Quantiles de la loi khi–deux

ν	$q\chi^2_\nu(1\%)$	$q\chi^2_\nu(2{,}5\%)$	$q\chi^2_\nu(5\%)$	$q\chi^2_\nu(95\%)$	$q\chi^2_\nu(97{,}5\%)$	$q\chi^2_\nu(99\%)$
1	$0,0^3 1571$	$0,0^3 9821$	0,003932	3,841	5,024	6,635
2	0,02010	0,05064	0,1026	5,991	7,378	9,210
3	0,1148	0,2158	0,3518	7,815	9,348	11,34
4	0,2971	0,4844	0,7107	9,488	11,14	13,28
5	0,5543	0,8312	1,145	11,07	12,83	15,09
6	0,8721	1,237	1,635	12,59	14,45	16,81
7	1,239	1,690	2,167	14,07	16,01	18,48
8	1,646	2,180	2,733	15,51	17,53	20,09
9	2,088	2,700	3,325	16,92	19,02	21,67
10	2,558	3,247	3,940	18,31	20,48	23,21
11	3,053	3,816	4,575	19,68	21,92	24,72
12	3,571	4,404	5,226	21,03	23,34	26,22
13	4,107	5,009	5,892	22,36	24,74	27,69
14	4,660	5,629	6,571	23,68	26,12	29,14
15	5,229	6,262	7,261	25,00	27,49	30,58
16	5,812	6,908	7,962	26,30	28,85	32,00
17	6,408	7,564	8,672	27,59	30,19	33,41
18	7,015	8,231	9,390	28,87	31,53	34,81
19	7,633	8,907	10,12	30,14	32,85	36,19
20	8,260	9,591	10,85	31,41	34,17	37,57

TABLE 5 – Quantiles des distributions χ^2_ν. Le chiffres pour $\nu = 21, \ldots, 100$ se trouvent à la page suivante.

ν	$q\chi^2_\nu(1\%)$	$q\chi^2_\nu(2{,}5\%)$	$q\chi^2_\nu(5\%)$	$q\chi^2_\nu(95\%)$	$q\chi^2_\nu(97{,}5\%)$	$q\chi^2_\nu(99\%)$
21	8,897	10,28	11,59	32,67	35,48	38,93
22	9,542	10,98	12,34	33,92	36,78	40,29
23	10,20	11,69	13,09	35,17	38,08	41,64
24	10,86	12,40	13,85	36,42	39,36	42,98
25	11,52	13,12	14,61	37,65	40,65	44,31
26	12,20	13,84	15,38	38,89	41,92	45,64
27	12,88	14,57	16,15	40,11	43,19	46,96
28	13,56	15,31	16,93	41,34	44,46	48,28
29	14,26	16,05	17,71	42,56	45,72	49,59
30	14,95	16,79	18,49	43,77	46,98	50,89
32	16,36	18,29	20,07	46,19	49,48	53,49
34	17,79	19,81	21,66	48,60	51,97	56,06
36	19,23	21,34	23,27	51,00	54,44	58,62
38	20,69	22,88	24,88	53,38	56,90	61,16
40	22,16	24,43	26,51	55,76	59,34	63,69
50	29,71	32,36	34,76	67,50	71,42	76,15
60	37,48	40,48	43,19	79,08	83,30	88,38
70	45,44	48,76	51,74	90,53	95,02	100,4
80	53,54	57,15	60,39	101,9	106,6	112,3
90	61,75	65,65	69,13	113,1	118,1	124,1
100	70,06	74,22	77,93	124,3	129,6	135,8

TABLE 6 – Quantiles des distributions χ^2_ν, dont la figure A.8 montre quelques densités.
L'approximation normale est $q\chi^2_\nu(p) \approx \nu + \sqrt{2\nu}\, q_{\text{normal}}(p)$, valable pour ν suffisamment grand. Pour $\nu = 20$, par exemple, cette formule donne $20 + 6,32 \times 1,64 = 30,37$ comme valeur approximative du 95%-quantile d'une loi χ^2_{20}.
Une meilleure approximation est celle de Wilson-Hilferty, définie par
$q\chi^2_\nu(p) \approx \nu \left(1 - 2/(9\nu) + \sqrt{2/(9\nu)}\, q_{\text{normal}}(p)\right)^3$. Dans l'exemple précédent cette formule donne la valeur approximative de $20(0,99 + 0,11 \times 1,64)^3 = 31,36$ pour $q\chi^2_{20}(95\%)$.

Quantiles de la loi F_{ν_1,ν_2} de Fisher

	$\nu_1 = 1$	2	3	4	5	6	7	8	10	12	24	∞
$\nu_2 = 1$	161,4	199,5	215,7	224,6	230,2	234,0	236,8	238,9	241,9	243,9	249,1	254,3
2	18,51	19,00	19,16	19,25	19,30	19,33	19,35	19,37	19,40	19,41	19,45	19,50
3	10,13	9,552	9,277	9,117	9,013	8,941	8,887	8,845	8,786	8,745	8,639	8,526
4	7,709	6,944	6,591	6,388	6,256	6,163	6,094	6,041	5,964	5,912	5,774	5,628
5	6,608	5,786	5,409	5,192	5,050	4,950	4,876	4,818	4,735	4,678	4,527	4,365
6	5,987	5,143	4,757	4,534	4,387	4,284	4,207	4,147	4,060	4,000	3,841	3,669
7	5,591	4,737	4,347	4,120	3,972	3,866	3,787	3,726	3,637	3,575	3,410	3,230
8	5,318	4,459	4,066	3,838	3,687	3,581	3,500	3,438	3,347	3,284	3,115	3,928
9	5,117	4,256	3,863	3,633	3,482	3,374	3,293	3,230	3,137	3,073	2,900	2,707
10	4,965	4,103	3,708	3,478	3,326	3,217	3,135	3,072	2,978	2,913	2,737	2,538
11	4,844	3,982	3,587	3,357	3,204	3,095	3,012	2,948	2,854	2,788	2,609	2,404
12	4,747	3,885	3,490	3,259	3,106	2,996	2,913	2,849	2,753	2,687	2,505	2,296
13	4,667	3,806	3,411	3,179	3,025	2,915	2,832	2,767	2,671	2,604	2,420	2,206
14	4,600	3,739	3,344	3,112	2,958	2,848	2,764	2,699	2,602	2,534	2,349	2,131
15	4,543	3,682	3,287	3,056	2,901	2,790	2,707	2,641	2,544	2,475	2,288	2,066
16	4,494	3,634	3,239	3,007	2,852	2,741	2,657	2,591	2,494	2,425	2,235	2,010
17	4,451	3,592	3,197	2,965	2,810	2,699	2,614	2,548	2,450	2,381	2,190	1,960
18	4,414	3,555	3,160	2,928	2,773	2,661	2,577	2,510	2,412	2,342	2,150	1,917
19	4,381	3,522	3,127	2,895	2,740	2,628	2,544	2,477	2,378	2,308	2,114	1,878
20	4,351	3,493	3,098	2,866	2,711	2,599	2,514	2,447	2,348	2,278	2,082	1,843
21	4,325	3,467	3,072	2,840	2,685	2,573	2,488	2,420	2,321	2,250	2,054	1,812
22	4,301	3,443	3,049	2,817	2,661	2,549	2,464	2,397	2,297	2,226	2,028	1,783
23	4,279	3,422	3,028	2,796	2,640	2,528	2,442	2,375	2,275	2,204	2,005	1,757
24	4,260	3,403	3,009	2,776	2,621	2,508	2,423	2,355	2,255	2,183	1,984	1,733
25	4,242	3,385	2,991	2,759	2,603	2,490	2,405	2,337	2,236	2,165	1,964	1,711
26	4,225	3,369	2,975	2,743	2,587	2,474	2,388	2,321	2,220	2,148	1,946	1,691
27	4,210	3,354	2,960	2,728	2,572	2,459	2,373	2,305	2,204	2,132	1,930	1,672
28	4,196	3,340	2,947	2,714	2,558	2,445	2,359	2,291	2,190	2,118	1,915	1,654
29	4,183	3,328	2,934	2,701	2,545	2,432	2,346	2,278	2,177	2,104	1,901	1,638
30	4,171	3,316	2,922	2,690	2,534	2,421	2,334	2,266	2,165	2,092	1,887	1,622
32	4,149	3,295	2,901	2,668	2,512	2,399	2,313	2,244	2,142	2,070	1,864	1,594
34	4,130	3,276	2,883	2,650	2,494	2,380	2,294	2,225	2,123	2,050	1,843	1,569
36	4,113	3,259	2,866	2,634	2,477	2,364	2,277	2,209	2,106	2,033	1,824	1,547
38	4,098	3,245	2,852	2,619	2,463	2,349	2,262	2,194	2,091	2,017	1,808	1,527
40	4,085	3,232	2,839	2,606	2,449	2,336	2,249	2,180	2,077	2,003	1,793	1,509
60	4,001	3,150	2,758	2,525	2,368	2,254	2,167	2,097	1,993	1,917	1,700	1,389
120	3,920	3,072	2,680	2,447	2,290	2,175	2,087	2,016	1,910	1,834	1,608	1,254
∞	3,841	2,996	2,605	2,372	2,214	2,099	2,010	1,938	1,831	1,752	1,517	1,000

TABLE 5 – Les 95%-quantiles, $qF_{\nu_1,\nu_2}(95\%)$, des distributions F_{ν_1,ν_2}.