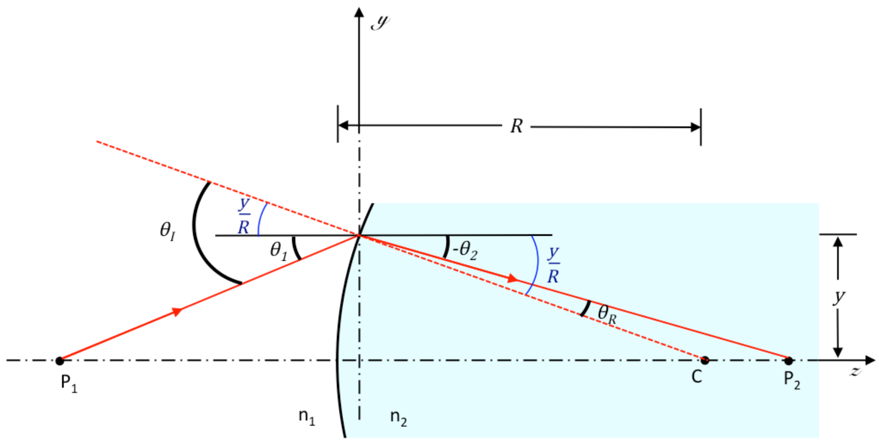
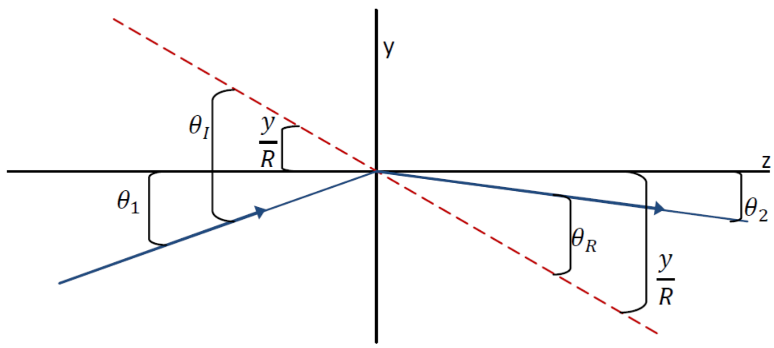
Problem 1





The red dashed line represents the radius of curvature, which is also normal to the surface. Snell's law relates the angle of incidence with the angle of refraction. We wish to derive a relationship between the angles of inc/ref with the angles formed by the z-axis (horizontal).

We can write only under paraxial approximation

$$n_1\theta_I \cong n_2\theta_R$$

$$\theta_I = \theta_1 + \frac{y}{R}$$

$$\theta_R = \theta_2 + \frac{y}{R}$$

Note that all signs are kept positive for the derivation.

$$\frac{n_1}{n_2}\left(\theta_1 + \frac{y}{R}\right) \cong \theta_2 + \frac{y}{R}$$

$$\left(\frac{n_1}{n_2} - 1\right)\frac{y}{R} + \frac{n_1}{n_2}\theta_1 \cong \theta_2$$

$$\boldsymbol{\frac{n_1}{n_2}\theta_1 - \frac{n_2 - n_1}{n_2}\frac{y}{R} \cong \theta_2}$$

Attention: In this question, θ_1 and θ_2 are not the incidence/refraction angles represented in Snell's law.

Problem 2

Begin by using the equation derived in problem #1. Apply it to the two boundaries:

air/lens boundary entering the lens and the lens/air boundary exiting the lens. We will define an intermediate angle.

$$\theta_o \cong \frac{n_a}{n_l}\theta_1 - \frac{n_l - n_a}{n_l}\frac{y}{R_1}$$

$$\theta_2 \cong \frac{n_l}{n_a}\theta_o - \frac{n_a - n_l}{n_a}\frac{y}{R_2}$$

Substitute

$$\theta_2 \cong \frac{n_l}{n_a}\left(\frac{n_a}{n_l}\theta_1 - \frac{n_l - n_a}{n_l}\frac{y}{R_1}\right) - \frac{n_a - n_l}{n_a}\frac{y}{R_2}$$

$$\theta_2 \cong \theta_1 - \frac{n_l - n_a}{n_a}\frac{y}{R_1} - \frac{n_a - n_l}{n_a}\frac{y}{R_2}$$

$$\theta_2 \cong \theta_1 - y\left(\frac{n_l - n_a}{n_a}\frac{1}{R_1} + \frac{n_a - n_l}{n_a}\frac{1}{R_2}\right)$$

$$\theta_2 \cong \theta_1 - y\left(\frac{n_l - n_a}{n_a}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right)$$

$$\boldsymbol{\theta_2 \cong \theta_1 - \frac{y}{f}}$$

Using the above derivation and the paraxial approximation we can write

$$\theta_1 = \frac{y}{z_1} \qquad and \qquad \theta_2 = -\frac{y}{z_2}$$

$$\frac{y}{f} = \theta_1 - \theta_2 = \frac{y}{z_1} + \frac{y}{z_2}$$

$$\boldsymbol{\frac{1}{f} = \frac{1}{z_1} + \frac{1}{z_2}}$$

Attention: In the equation derived in problem #1, the final form does not depend on the sign conventions as they have been considered already during the derivation. Therefore, do not replace R_1 with '-R_2' when you apply the equation to the second curvature.

Problem 3

Using the Lens-Maker's formula under the thin lens approximation the following can be written:

$$f = \left[(n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)\right]^{-1}$$

where R_1 is infinite (planar) and R_2 is 10cm (positive since concave).

Solve

$$f = \left[-\frac{1}{2}\left(\frac{1}{10cm}\right)\right]^{-1}$$

$$\boldsymbol{f = -20\ cm}$$

Problem 4

Begin using the Lens-Makers formula:

$$\frac{1}{f} = \frac{n_{lens} - n_o}{n_o}\left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

where n_{lens} is 1.5 and n_o is the index of the surroundings. Remember we are trying to prove that $f_{water}=4f_{air}$. To begin, take the ratio of the two focal lengths and make appropriate substitutions.

$$\frac{f_w}{f_a} = \frac{\frac{n_{lens} - n_a}{n_a}\left[\frac{1}{R_1} - \frac{1}{R_2}\right]}{\frac{n_{lens} - n_w}{n_w}\left[\frac{1}{R_1} - \frac{1}{R_2}\right]} = \frac{\frac{n_{lens} - n_a}{n_a}}{\frac{n_{lens} - n_w}{n_w}} = \frac{\frac{1}{2}}{\frac{1}{8}} = \frac{8}{2} = \mathbf{4}$$

Problem 5

Part A

Make substitution

$$s_o = f + x_o \quad and \quad s_i = f + x_i$$

into above the derived equation in problem #5

$$\frac{1}{f} = \frac{1}{f + x_o} + \frac{1}{f + x_i}$$

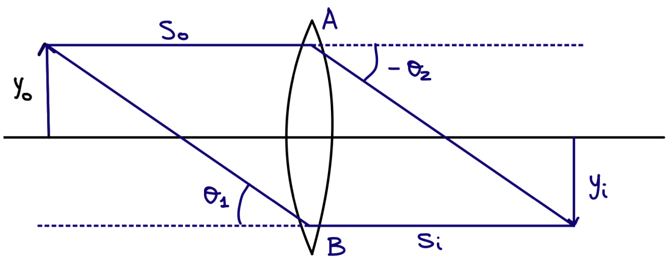
multiply by f, expand, simplify

$$f^2 + f x_o + f x_i + f^2 = f^2 + f x_o + f x_i + x_o x_i$$

$$\boldsymbol{f^2 = x_o x_i}$$

Part B

Apply the equation derived in problem 2 to twice principal rays



Case I

$$\theta_2 = \theta_1 - \frac{y_o}{f} = -\frac{y_o}{f} \quad \text{since } \theta_1 = 0$$

Case II

$$\theta_2 = \theta_1 - \frac{y_i}{f} = 0 \rightarrow \theta_1 = \frac{y_i}{f} \quad \text{since } \theta_2 = 0$$

$$M = \frac{y_i}{y_o} = \frac{y_i}{f} \cdot \frac{f}{y_o} = \frac{\theta_1}{-\theta_2} \qquad \text{paraxial} \qquad \frac{\tan\theta_1}{\tan\theta_2} = \frac{S_i}{\overline{AB}} \cdot \frac{\overline{AB}}{S_o}$$

$$\boxed{M = \frac{y_i}{y_o} = -\frac{\theta_1}{\theta_2} = -\frac{S_i}{S_o}}$$