

Example 1 (rotational system):

Consider a DC motor equipped with an incremental encoder for position control. This motor is steered in torque mode. This means that we use a servo amplifier configured in current mode. This motor is coupled with a gear and a symmetric load (no gravity).

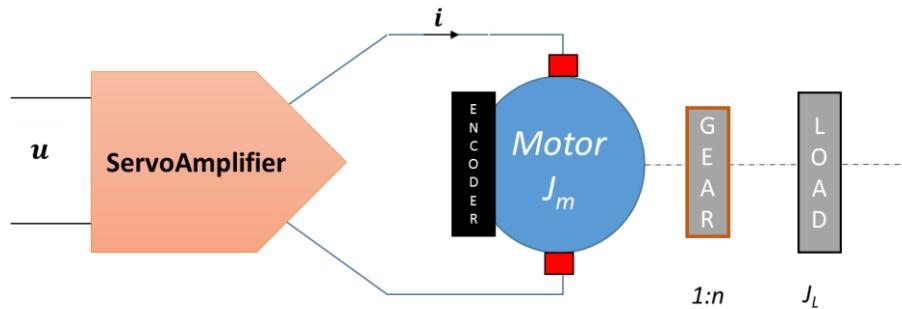


Figure 1- DC Motor, gear and load- The motor is equipped with an incremental encoder and controlled by a servo amplifier

The reason for which this mode is called “Torque mode” is that the motor torque Γ_m provided by the motor is proportional to the current control “ i ”.

Parameters of the system:

n	Gear ratio
k_c	Torque constant of the motor (Constante de couple in French)
J_m	Motor Inertia
k_{vis}	Viscous coefficient reported to the motor side
J_L	Load Inertia
J_{RL}	Total Inertia reported to the load
J_{Rm}	Total Inertia reported to the motor

In this problem we are interested in the position control of this mechanical system (Motor + Gear + Load). Different cases will be considered and discussed.

Case 1: No friction, no gravity, Proportional gain on the position error.

Dynamic Model: 2 cases may be considered- The dynamic model may be written either at the load side or the motor side. Only eq.2 or eq.3 must be considered.

$$\Gamma_m = k_c i \quad (1)$$

$$\sum \Gamma = J_{Rm} * \ddot{\theta}_m \quad (2) \text{ On the side of motor shaft}$$

$$\sum \Gamma = J_{RL} * \ddot{\theta}_L \quad (3) \text{ On the side of the load shaft}$$

$$J_{Rm} = J_m + \frac{J_L}{n^2} \quad \text{is the inertia reported to the motor}$$

$$J_{RL} = J_L + n^2 J_m \quad \text{is the inertia reported to the load}$$

If we consider (eq.1) and (eq.2), we obtain :

$$\sum \Gamma = J_{Rm} \ddot{\theta}_m = \Gamma_m = k_c i \quad (4)$$

The relation between θ and i defines the **open loop** transfer function. It corresponds to a double integrator (Exercise – DC motor step current response).

On the other hand, the closed loop scheme is represented by the following figure:

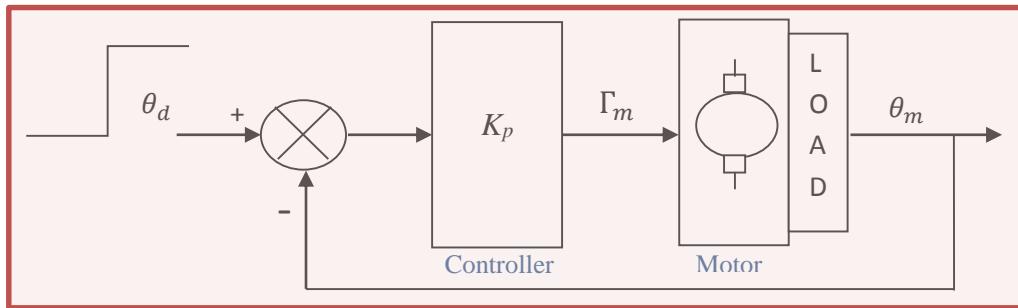


Figure 2- Position control of a DC Motor

By using a proportional controller $\Gamma_m = k_p(\theta_d - \theta_m)$, and replacing the motor torque expression in eq.4, we obtain the following **closed loop expression**:

$$J_{Rm} \ddot{\theta}_m + k_p \theta_m = k_p \theta_d \quad (5)$$

Using the s-transform ($\dot{\theta} = s\theta$, $\ddot{\theta} = s^2\theta$), this gives:

$$\Rightarrow (s^2 J_{Rm} + k_p) \theta_m = k_p \theta_d \quad (6)$$

$$\Rightarrow \frac{\theta_m}{\theta_d} = \frac{k_p}{s^2 J_{Rm} + k_p} \quad (7)$$

Eq.7 corresponds to a pure oscillatory system that will never reach the desired position.

Very important remark:

Never control a DC Motor in torque mode with only a proportional –type position control

Case 2: P-Controller in presence of viscous friction

In presence of viscous friction, the model (4) becomes:

$$\sum \Gamma = J_{Rm} \ddot{\theta}_m = \Gamma_m - k_{vis} * \theta_m \quad (8)$$

By closing the loop with a proportional controller $\Gamma_m = k_p(\theta_d - \theta_m)$, we obtain:

$$\Rightarrow s^2 J_{Rm} \theta_m + s k_{vis} \theta_m + k_p \theta_m = k_p \theta_d$$

Which leads to following closed loop transfer function:

$$\frac{\theta_m}{\theta_d} = \frac{k_p}{s^2 J_{Rm} + s k_{vis} + k_p} \quad (9)$$

This transfer function corresponds to a stable system that has a **damping** factor to ensure the convergence of the motor position to the desired position. Nevertheless, this damping may not be sufficient to damp the system and we absolutely **need to add a derivative control component**.

Case 3: PD-controller in presence of viscous friction

$$\text{The PD control law is given by: } \Gamma_m = k_p(\theta_d - \theta_m) - k_d \dot{\theta}_m \quad (10)$$

k_d is the derivative parameter of the PD controller. The open loop dynamic model is given by eq.8

$$\sum \Gamma = J_{Rm} \ddot{\theta}_m = \Gamma_m - k_{vis} \dot{\theta}_m$$

By closing the loop, the dynamic expression becomes:

$$\begin{aligned} \sum \Gamma &= s^2 J_{Rm} \theta_m \\ &= k_p(\theta_d - \theta_m) - k_d s \theta_m - k_{vis} s \theta_m \end{aligned} \quad (11)$$

Which leads to following closed loop transfer function:

$$\frac{\theta_m}{\theta_d} = \frac{k_p}{s^2 J_{Rm} + (k_d + k_{vis})s + k_p} \quad (12)$$

This transfer function between the input (desired motor position) and the output (measured motor position) corresponds to a stable behavior. In this case, it is an asymptotic stability assuring a total convergence of the measured position to the target. The following writing of this transfer function as follows:

$$\frac{\theta_m}{\theta_d} = \frac{1}{\frac{1}{\omega_n^2} s^2 + \left(\frac{2, z}{\omega_n}\right) s + 1} = \frac{1}{s^2 \frac{J_{Rm}}{k_p} + \left(\frac{k_d + k_{vis}}{k_p}\right) s + 1} \quad (13)$$

$$\text{Leads to a proper frequency } \omega_n = \sqrt{\frac{k_p}{J_{Rm}}} \quad (14)$$

$$\text{and a damping ratio } z = \frac{1}{2} \sqrt{\frac{k_p}{J_{Rm}} \left(\frac{k_d + k_{vis}}{k_p}\right)} \quad (15)$$

Case 4: PD-controller in presence of viscous and dry friction

In this case, the dynamic model is expressed by the following equation:

$$\sum \Gamma = J_{Rm} \ddot{\theta}_m = \Gamma_m - \Gamma_{dry} \quad (16)$$

The model of the dry friction is given by the following characteristic

The dry friction torque has two characteristic values :

- The static dry friction corresponding to the value of the dry torque before the starting of the movement ($|\omega| < \varepsilon$).
- The dynamic dry friction corresponding to the value of the dry torque when motion is occurring and ($|\omega| \geq \varepsilon$)

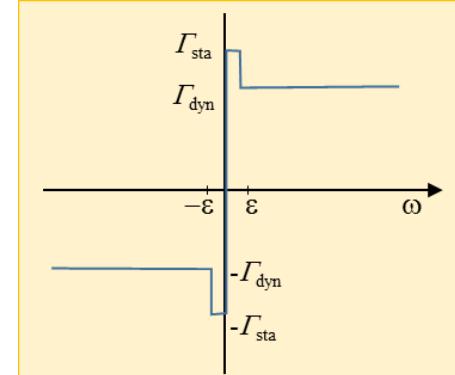


Figure 2- Dry friction characteristic

However, in the dynamic model the dry friction will be considered as constant (Γ_{dry}). By closing the loop of the system with a PD controller, the dynamic expression of our mechanical system is then written as follows:

$$\sum \Gamma = J_{Rm} \ddot{\theta}_m = k_p(\theta_d - \theta_m) - k_D \dot{\theta}_m - k_{vis} \dot{\theta}_m - \Gamma_{dry} \quad (17)$$

To understand what happen in the static phase, we only need to cancel the first and second derivatives of the desired and measured positions.

$$\dot{\theta} \rightarrow 0, \quad \dot{\theta}_d \rightarrow 0 \quad \ddot{\theta} \rightarrow 0 \quad \text{and} \quad \ddot{\theta}_d \rightarrow 0$$

$$\sum \Gamma = J_{Rm} \ddot{\theta}_m = k_p(\theta_d - \theta_m) - k_D \dot{\theta}_m - k_{vis} \dot{\theta}_m - \Gamma_{dry} \quad (18)$$

$$\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & \varepsilon_{sta} & 0 & 0 \end{array}$$

$$\Rightarrow k_p \underbrace{(\theta_d - \theta_m)}_{\varepsilon_{sta}} = C_{sec}$$

The static error ε_{sta} may then be expressed as follows:

$$\varepsilon_{sta} = \frac{\Gamma_{dry}}{k_p} \quad (19)$$

First, eq. 16 implies that in presence of dry friction, the PD controller is unable to cancel the static error (ε_{sta} **is not null**). The good news is that the static error may be reduced by increasing the proportional gain k_p . Increasing k_p increases the stiffness of the controller. Nevertheless, high values of the gain k_p may make the system instable because of the stability criterion (Nyquist plot) and also because of the saturation of the control. The PD controller is then not sufficient to totally remove the static error and the PID controller may be considered.

Case 5: PID-controller in presence of dry friction

As in the previous case (eq.16, case 4), the physical model is given by the following equation:

$$\sum \Gamma = J_{Rm} \ddot{\theta}_m = \Gamma_m - \Gamma_{dry}$$

The PID control law is given by:

$$\Gamma_m = \mathbf{k}_p(\theta_d - \theta_m) - \mathbf{k}_d \dot{\theta}_m + \mathbf{k}_I \int_0^t (\theta_d - \theta_m) dt \quad (20)$$

Closing the loop of the system (eq.16) by using the PID control law (eq.20) and using the s-transform leads to the following behavior:

$$J_{Rm} s^2 \theta_m = k_p(\theta_d - \theta_m) - s k_D \theta_m + \frac{k_I}{s} \theta_d - \frac{k_I}{s} \theta_m - \Gamma_{dry} \quad (21)$$

Below, eq.21 is rewritten in a very interesting manner ☺.

$$\left(J_{Rm} s^2 + k_D s + k_p + \frac{k_I}{S} \right) \theta_m = \left(k_p + \frac{k_I}{S} \right) \theta_d - \Gamma_{dry}$$

By multiplying by “s” in two sides, we obtain:

$$(J_{Rm} s^2 + k_D s^2 + k_p s + k_I) \theta_m = \theta_d - s \Gamma_{dry} \quad (22)$$

$$\Rightarrow \theta_m = \frac{k_p s + k_I}{J_{Rm} s^3 + k_D s^2 + k_p s + k_I} \theta_d - \frac{s}{J_{Rm} s^3 + k_D s^2 + k_p s + k_I} \Gamma_{dry} \quad (23)$$

What eq.23 is meaning?

Eq.23 means that the motor position θ_m is an output of two dynamic systems.

$$\theta_m = \underbrace{\frac{k_p s + k_I}{J_{Rm} s^3 + k_D s^2 + k_p s + k_I} \theta_d}_{\substack{\text{Regulation transfer function} \\ H_R(s)}} - \underbrace{\frac{s}{J_{Rm} s^3 + k_D s^2 + k_p s + k_I} \Gamma_{dry}}_{\substack{\text{Disturbance rejection transfer} \\ \text{function } H_D(s)}}$$

We first notice that the dry friction is considered as a disturbance because it is assumed to be unknown. The transfer function $H_R(s)$ represents the dynamic behavior of the output (motor position) with respect to the target (desired position) and the transfer function $H_D(s)$ corresponds to the dynamic effect of the disturbance Γ_{dry} on the output. $H_R(s)$ represents the closed loop regulation performances and $H_D(s)$ gives the disturbance rejection performances.

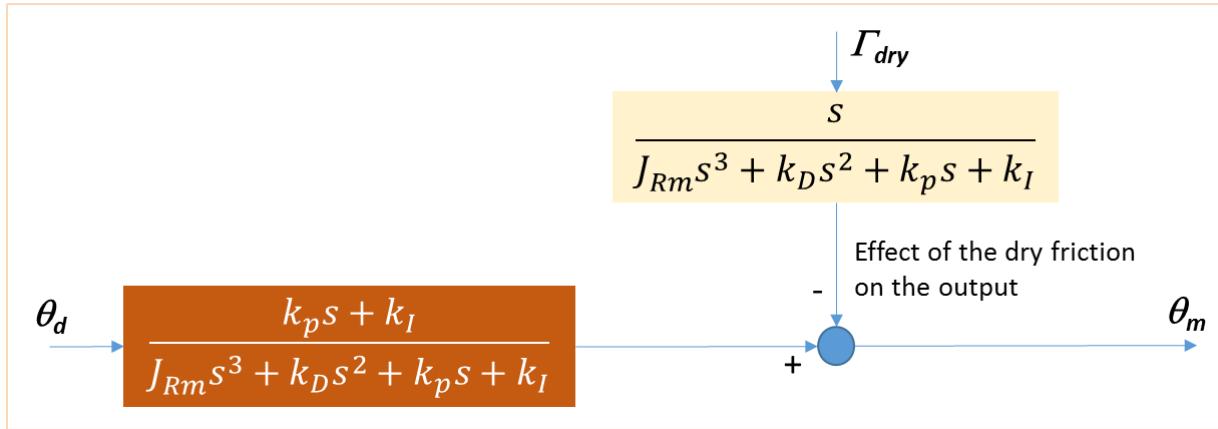


Figure 3- PID control and dry friction effect

In static phase by assuming that Γ_{dry} is constant, the static gain of the output transfer function is 1 and the static gain of the friction transfer function is 0. This leads to the total cancellation of the static error.

Notice that, biggest is K_I , More the disturbance rejection is fast!

What happen in the static phase?

Regarding previous analysis, we demonstrated that the static error reaches the zero-value. The closed loop behavior may be represented by the (eq. 23) or by the following temporal representation:

$$J_{Rm} \ddot{\theta}_m = k_p \varepsilon - k_D \dot{\theta}_m + k_I \int_0^t \varepsilon(\tau) d\tau - \Gamma_{dry} \quad (24)$$

ε is the regulation error.

In the static phase, $\dot{\theta}_m = \ddot{\theta}_m = \dot{\theta}_d = \ddot{\theta}_d = 0$. This leads to the following **very important** equality:

$$k_I \int_0^t \varepsilon(\tau) d\tau = \Gamma_{dry} \quad (25)$$

Observations:

- This relation shows that at the end of the regulation phase (in the static phase and when ε reaches 0) **the integrator identifies the dry friction**. It also shows that the “Integral contribution” of the controller works as a dry friction compensation.
- If the dry friction changes with respect to the motor position, take care that this identification is only valid at the position on which the motor is stopped (ie, at the position θ_d).
- Previous remarks are not valid only for the case of the dry friction but for any type of constant unknown torques (or slowly variable disturbances). We can give the example of the gravity, external constant force or others.