Image Processing 2, Exercise 8

1 Classical image reconstruction

[basic] This exercise will teach you how to design a "classical" image reconstruction algorithm. Given some physical model, it will take you through all the design steps from the discretization to the specification of an iterative reconstruction scheme.

The term "classical" refers to the reconstruction algorithms that can be derived from the minimization of a quadratic functional. The idea is that such functionals have a unique minimum, which can usually be determined in closed form. This then yields a linear reconstruction which often has a direct matrix-free implementation. Otherwise, the reconstruction can be performed iteratively by steepest descent or some faster variant (e.g., conjugate gradient).

The discretization step is a prerequisite for computational imaging. In practice, one always start by designing a classical algorithm, which is easier to debug and which encompasses all the knowledge of the physics. In a later stage, one can refine the reconstruction by incorporating additional constraints such as positivity, some improved form of regularization, and/or even the appropriate deployment of neural networks.

The continuous-domain modelling of our imaging system (1D for simplicity) in the noise-free scenario is $g_0(x) = (h * s)(x)$ where $h(x) = \mathrm{rect}(x/3)$. To reconstruct the signal, we propose to represent the signal in a "pixelated" basis as $s(x) = \sum_{k=1}^{K} s_k \beta^0(x-k)$ where $\beta^0(x) = \mathrm{rect}(x)$.

Given the noisy measurements $\mathbf{y} = (y_m) \in \mathbb{R}^M$ with $y_m = g_0(m) + n_m$, the task is to reconstruct the unknown signal coefficients $\mathbf{s} = (s_k) \in \mathbb{R}^K$ such as to minimise a suitable cost functional.

(a) Discretization: For simplicity, we take $K = M$. Specify the system matrix $\mathbf{H} \in \mathbb{R}^{M\times M}$ such that the forward imaging model can be written as $\mathbf{y} = \mathbf{Hs} + \mathbf{n}$ where $\mathbf{n} = (n_m)$ represents the measurement noise.

(b) We now formulate the reconstruction problem as the minimization of the cost functional

$$J(\mathbf{y}, \mathbf{s}) = \|\mathbf{\Lambda}(\mathbf{y} - \mathbf{Hs})\|^2 + \lambda\|\mathbf{s} - \mathbf{s}_0\|^2, \tag{1}$$

where $\mathbf{s}_0$ is a known reference signal, $\lambda \geq 0$ a tuning parameter and $\mathbf{\Lambda} = \mathrm{diag}(a_1, \dots, a_M)$ is a diagonal matrix of weights $a_m > 0$.

Motivation: The use of this particular cost functional is our way of injecting prior information. Specifically, we want the reconstruction to be "reasonably" close to $\mathbf{s}_0$, while we are also weighting the contribution of a measurement in inverse proportion to the standard deviation of the noise which may be detector-dependent; e.g., $a_m \propto 1/\sigma_m$.

Your first task is to calculate the gradient of $J(\mathbf{y}, \mathbf{s})$ with respect to $\mathbf{s}$ (the unknown).

(c) Give the closed form of the signal reconstruction: $\arg\min_{\mathbf{s}\in\mathbb{R}^M} J(\mathbf{y}, \mathbf{s})$.

(d) (i) Give a short description of an iterative gradient-based reconstruction algorithm with initial condition $\mathbf{s}^{(0)} = \mathbf{s}_0$.
(ii) Modify your algorithm in order to impose the positivity of your reconstruction.

2 Convex functionals

Most image reconstruction algorithms are derived from the minimization of a convex functional. You should therefore be able to recognize and manipulate such functionals.

Motivation: The convexity property ensures that the standard iterative procedures (based on steepest descent and proximal updates) converge to the solution.

We recall that a functional $f : \mathbb{R}^d \to \mathbb{R}$ is convex if, for all $\lambda \in [0, 1]$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$,

$$f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda)f(\mathbf{y}). \tag{3}$$

Under the assumption that the functions $\phi, \psi : \mathbb{R} \to \mathbb{R}$ are convex, prove the following properties.

(a) $a_0\phi(x)$ is convex for any fixed $a_0 \in \mathbb{R}^+$.

(b) $g(x) = \phi(x) + \psi(x)$ is convex.

(c) The function $x \mapsto \phi(x) + a_2x^2 + a_1x + a_0$ with $a_2, a_1 \in \mathbb{R}^+$ and $a_0 \in \mathbb{R}$ is convex.

(d) The function $g : \mathbb{R}^d \to \mathbb{R}$ where $g(\mathbf{x}) = \phi(\mathbf{w}^\mathsf{T}\mathbf{x} + b_0)$ with parameters $\mathbf{w} \in \mathbb{R}^d$ and $b_0 \in \mathbb{R}$ is convex.

(e) Variation of a theme. Let us consider a convex function $\Phi : \mathbb{R}^d \to \mathbb{R}$ and an affine map $\mathbb{R}^{d_0} \to \mathbb{R}^d : \mathbf{x} \mapsto \mathbf{A}\mathbf{x} + \mathbf{b}$ where $\mathbf{A} \in \mathbb{R}^{d \times d_0}$ and $\mathbf{b} \in \mathbb{R}^d$ are some fixed parameters. Prove that the composed function $g : \mathbb{R}^{d_0} \to \mathbb{R}$ with $g(\mathbf{x}) = \Phi(\mathbf{A}\mathbf{x} + \mathbf{b})$ is convex .

3 Proximal Operators

[basic] Proximal operators are extremely useful components for the design of efficient iterative optimization/reconstruction algorithms.

We recall that the proximal operator of a convex and l.s.c. function $f : \mathbb{R}^d \to \mathbb{R}$ is given by

$$\operatorname{prox}_f(\mathbf{x}) = \arg\min_{\mathbf{z}\in\mathbb{R}^d} \tfrac{1}{2}\|\mathbf{z} - \mathbf{x}\|_2^2 + f(\mathbf{z}). \tag{4}$$

(a) Scalar scenario. Let $g(x) = \lambda|x|^3$ where $\lambda > 0$ is a fixed constant. First, show that g is convex, and, then determine its proximal operator $\operatorname{prox}_g(x)$.

(b) Quadratic regularization functional. Your task is to derive the explicit form of the proximal operator of the functional $f_2 : \mathbb{R}^d \to \mathbb{R}$ with $f_2(\mathbf{x}) = \frac{\lambda}{2}\|\mathbf{L}\mathbf{x}\|_2^2$ where $\mathbf{L} \in \mathbb{R}^{d\times d}$ is a square matrix and $\lambda > 0$ is an adjustable (regularization) parameter.