# Image Processing 2, Exercise 7

## 1 Image denoising by linear filtering

*[basic] A classical usage of filtering is for signal denoising. Here, we investigate two options for the "optimal" design of such filters.*

We consider the measurement model $g(\boldsymbol{x}) = f(\boldsymbol{x}) + n(\boldsymbol{x})$ where the unknown signal $f$ is corrupted by some random noise $n$. Our goal is to design a denoising filter $r$ such that $\tilde{f} = r * g$ is a good approximation of $f$.

(a) Wiener filtering: Here, we make the assumption that $f$ is a realization of a stochastic process with spectral density $\Phi_f(\boldsymbol{w}) = \|\boldsymbol{\omega}\|^{-4}$ and that $n$ is a white Gaussian noise with $\Phi_n(\boldsymbol{\omega}) = \sigma^2$.

(i) Deduce the frequency response of the corresponding Wiener filter $R_{\text{Wiener}}(\boldsymbol{\omega})$. Is the filter isotropic?

(ii) Plot the frequency response for $\sigma^2 = 1$. What kind of filter is it? What is the limit of the filter as $\sigma^2 \to 0$ (resp., as $\sigma^2 \to \infty$)?

(b) Variational denoising : We now adopt an alternative, deterministic formulation where the goal is to minimize the cost functional

$$\tilde{f} = \arg\min_f J(f,g) \quad \text{where} \quad J(f,g) = \int_{\mathbb{R}^d} |g(\boldsymbol{x}) - f(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} + \lambda \int_{\mathbb{R}^d} |\mathrm{L}\{f\}(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x}. \tag{1}$$

Here, $\lambda \in \mathbb{R}^+$ an adjustable regularization parameter and L an LSI operator with frequency response $\widehat{L}(\boldsymbol{\omega})$. The underlying philosophy, which due to the mathematician Tikhonov, is to promote "regular" solutions for which the regularization energy $\|\mathrm{L}\{f\}\|_{L_2}^2$ is reasonably small.

(i) As first step, rewrite $J(f,g)$ in terms of $L_2$-norms, first in space, and, then in the frequency domain with the Fourier transforms of $f$ and $g$ being denoted by $\hat{f}$ and $\hat{g}$, respectively.

(ii) Since the formal minimization of (1) is an infinite-dimensional problem that would require the use of calculus of variations (which you probably have not yet studied), we shall use an indirect approach where $J(f,g)$ is manipulated such as to make the solution obvious. Specifically, we ask you to show that the latter can be rewritten as

$$J(f,g) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| \hat{f}(\boldsymbol{\omega}) - \frac{\hat{g}(\boldsymbol{\omega})}{1 + \lambda |\widehat{L}(\boldsymbol{\omega})|^2} \right|^2 (1 + \lambda |\widehat{L}(\boldsymbol{\omega})|^2) + \frac{\lambda |\widehat{L}(\boldsymbol{\omega})|^2 |\hat{g}(\boldsymbol{\omega})|^2}{1 + \lambda |\widehat{L}(\boldsymbol{\omega})|^2} \mathrm{d}\boldsymbol{\omega} \tag{2}$$

and to then deduce the solution, which should be of the form $\tilde{f} = r_{\text{Tik}} * g$ where $r_{\text{Tik}}$ is a suitable filter.

(ii') Simplified version (for those who are in a hurry): Using some elementary arguments, show that, for any given frequency $\boldsymbol{\omega} \in \mathbb{R}^d$, the minimizer of (2) is $\hat{f}(\boldsymbol{\omega}) = R_{\text{Tik}}(\boldsymbol{\omega}) \hat{g}(\boldsymbol{\omega})$ where

$$R_{\text{Tik}}(\boldsymbol{\omega}) = \frac{1}{1 + \lambda |\widehat{L}(\boldsymbol{\omega})|^2}. \tag{3}$$

(iii) Give the expression of $J_{\min} = \min_f J(f,g)$ and show that $J_{\min} \leq \|g\|_{L_2}^2$.

(c) Reconciliation.

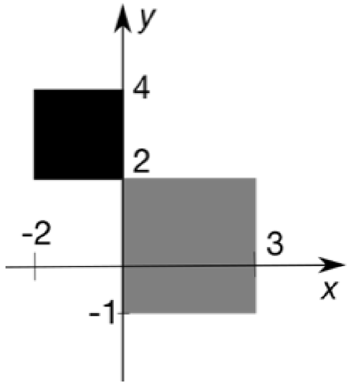
(i) By comparing the expression of the two filters, find the operator L that makes the two methods equivalent.

(ii) What is the interpretation of such a filter in the world of stochastic processes?

(iii) By transposing the reasoning of Item b-(ii') to the last integral formula in the derivation of the Wiener filter within the course notes (Slide 9-10), provide the expression of the minimum MSE as a function of $\Phi_f$.

## 2 Center of mass and Radon transform

Consider the following image $f(x, y)$:




(a) Give an expression of $f(x, y)$ in terms of a sum of shifted and scaled separable 2D B-splines. Assume the indicated grayscale with white being 0, gray 50 and black 100.

(b) Calculate the coordinates $\bar{\mathbf{x}} = (\bar{x}, \bar{y})$ of the center of mass of $f(x, y)$.

(c) Calculate the projection $t_{\bar{\mathbf{x}},\theta}$ of the center of mass $\bar{\mathbf{x}}$ onto the projection line for a projection direction $\theta = 0°, 45°$.

(d) Calculate and sketch the Radon transform projections $p_\theta(t)$ of $f$ for $\theta = 0°, 45°$.

(e) Calculate the center of mass $\bar{t}_\theta$ of the projections $p_\theta(t)$ for $\theta = 0°, 45°$.

(f) Compare the results you obtained in c) and e). Comment on how this result could be useful for practical applications.

## 3 Radon transform with the "magical" help of Fourier

*[basic] The Radon transform of a function provides a complete characterization of its line integrals. Computing it explicitly can be laborious. Fortunately, we can take advantage of the* Fourier-slice theorem *which states that* $\mathrm{R}\{f\}(t, \theta)$ *for a fixed* $\theta$ *is the inverse Fourier transform of the 1D function* $\omega \mapsto \hat{f}(\omega\boldsymbol{\theta})$ *where* $\boldsymbol{\theta} = (\cos\theta, \sin\theta)$ *and* $\hat{f} : \mathbb{R}^2 \to \mathbb{C}$ *is the 2D Fourier transform of* $f$.

*The Fourier-slice theorem is fundamental in that it remains valid for distributions whose Fourier transform is a well-defined function. It allows us to calculate the Radon transform of objects such as* $\|\boldsymbol{x}\|^s$*, which are not integrable in the conventional Lebesgue sense.*

(a) Basic calculations. Compute the Radon transform of the following functions.

1. $\delta(\cdot - \boldsymbol{x}_0)$ with $\boldsymbol{x}_0 = (x_0, y_0) = r(\cos\phi, \sin\phi) \in \mathbb{R}^2$
2. $g(\boldsymbol{x}) = \frac{1}{2\pi}\mathrm{e}^{-\frac{1}{2}\|\boldsymbol{x}\|^2}$ (standardized Gaussian)

(b) Prove the following properties of the Radon transform where $p_\theta(t) = \mathcal{R}\{f\}(t, \theta)$.

1. Laplacian:
$$\mathcal{R}\{\Delta f\}(t, \theta) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} p_\theta(t).$$
2. Conservation of the integral.
$$\int_{\mathbb{R}} p_\theta(t)\mathrm{d}t = \hat{f}(\mathbf{0}) = \int_{\mathbb{R}^2} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$
3. Second-order moments:
$$\int_{\mathbb{R}} t^2 p_\theta(t)\mathrm{d}t = \boldsymbol{\theta}^\mathsf{T}\mathbf{H}\boldsymbol{\theta} \quad \text{where} \quad \mathbf{H} = \begin{pmatrix} \int_{\mathbb{R}^2} x^2 f(x,y)\mathrm{d}x\mathrm{d}y & \int_{\mathbb{R}^2} xy f(x,y)\mathrm{d}x\mathrm{d}y \\ \int_{\mathbb{R}^2} yx f(x,y)\mathrm{d}x\mathrm{d}y & \int_{\mathbb{R}^2} y^2 f(x,y)\mathrm{d}x\mathrm{d}y \end{pmatrix}$$

(c) Transform of a separable function: Let $\varphi(\boldsymbol{x}) = \varphi_1(x)\varphi_2(y)$. Show that $\mathrm{R}\{\varphi\}(t,\theta) = \varphi_\theta(t)$ where

$$\varphi_\theta(t) = \left(\tfrac{1}{|\cos\theta|}\varphi_1\big(\frac{\cdot}{\cos\theta}\big) * \tfrac{1}{|\sin\theta|}\varphi_2\big(\frac{\cdot}{\sin\theta}\big)\right)(t)$$

with the convention that $\frac{1}{|a|}\varphi\left(\frac{\cdot}{a}\right) \to \hat{\varphi}(0)\delta$ as $a \to 0$.

(d) Radon transform of a B-spline. Compute $\mathrm{R}\{\varphi\}(t,0)$ and $\mathrm{R}\{\varphi\}(t,\frac{\pi}{4})$ for the case where $\varphi(\boldsymbol{x}) = \beta^n(x)\beta^n(y)$ is a separable B-spline of degree $n$.