Image Processing 2, Exercise 7

1 Image denoising by linear filtering

[basic] A classical usage of filtering is for signal denoising. Here, we investigate two options for the "optimal" design of such filters.

We consider the measurement model $g(\boldsymbol{x}) = f(\boldsymbol{x}) + n(\boldsymbol{x})$ where the unknown signal f is corrupted by some random noise n. Our goal is to design a denoising filter r such that $\tilde{f} = r * g$ is a good approximation of f.

(a) Wiener filtering: Here, we make the assumption that f is a realization of a stochastic process with spectral density $\Phi_f(\boldsymbol{w}) = \|\boldsymbol{\omega}\|^{-4}$ and that n is a white Gaussian noise with $\Phi_n(\boldsymbol{\omega}) = \sigma^2$.

(i) Deduce the frequency response of the corresponding Wiener filter $R_{\text{Wiener}}(\boldsymbol{\omega})$. Is the filter isotropic?

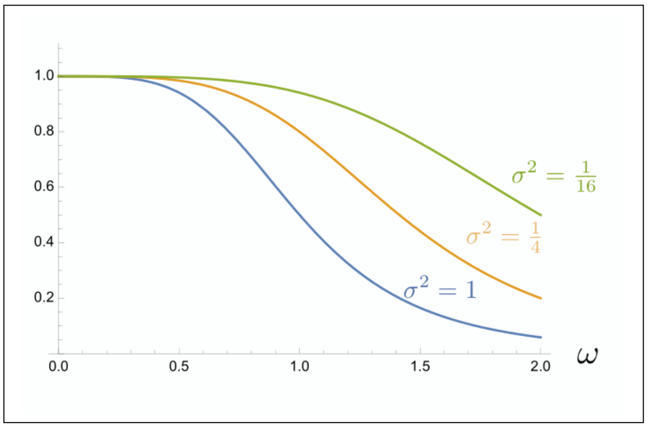
(ii) Plot the frequency response for $\sigma^2 = 1$. What kind of filter is it? What is the limit of the filter as $\sigma^2 \to 0$ (resp., as $\sigma^2 \to \infty$)?

Solution:

(i) This is a special case of the Wiener filter with $h = \delta$. By using the expression of the Wiener filter with $H(\boldsymbol{\omega}) = 1$, we find that

$$R_{\text{Wiener}}(\boldsymbol{\omega}) = \frac{\Phi_f(\boldsymbol{\omega})H^*(\boldsymbol{\omega})}{\Phi_f(\boldsymbol{\omega})|H(\boldsymbol{\omega})|^2 + \Phi_n(\boldsymbol{\omega})} = \frac{\Phi_f(\boldsymbol{\omega})}{\Phi_f(\boldsymbol{\omega}) + \Phi_n(\boldsymbol{\omega})} = \frac{\|\boldsymbol{\omega}\|^{-4}}{\|\boldsymbol{\omega}\|^{-4} + \sigma^2}.$$

The filter is isotropic with $R_{\text{Wiener}}(\boldsymbol{\omega}) = \frac{1}{1+\sigma^2\|\boldsymbol{\omega}\|^4}$, which depends solely on the magnitude $\omega = \|\boldsymbol{\omega}\|$ (radial frequency) and not on the direction of $\boldsymbol{\omega}$.



Frequency response of Wiener filter

(ii) This is a lowpass filter, which is consistent with the property that the signal itself is predominantly lowpass. When $\sigma^2 \to 0$, $R_{\text{Wiener}}(\boldsymbol{\omega}) = 1$ (allpass filter). It means that the filter preserves all frequencies without attenuation. The Wiener filter essentially operates as a noise-free filter, providing an exact estimation of the original signal. By contrast, when $\sigma^2 \to \infty$, $R_{\text{Wiener}}(\boldsymbol{\omega}) = 1$ (allcut filter). The Wiener filter acts as a noise suppressor, heavily suppressing both low and high-frequency components. Indeed, when the noise is overwhelming, there is no chance to recover the underlying signal, so that the safest (e.g. minimum-error) prediction is zero.

(b) Variational denoising : We now adopt an alternative, deterministic formulation where the goal is to minimize the cost functional

$$\tilde{f} = \arg\min_f J(f,g) \quad \text{where} \quad J(f,g) = \int_{\mathbb{R}^d} |g(\boldsymbol{x}) - f(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} + \lambda \int_{\mathbb{R}^d} |\mathrm{L}\{f\}(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x}. \tag{1}$$

Here, $\lambda \in \mathbb{R}^+$ an adjustable regularization parameter and L an LSI operator with frequency response $\widehat{L}(\boldsymbol{\omega})$. The underlying philosophy, which due to the mathematician Tikhonov, is to promote "regular" solutions for which the regularization energy $\|\mathrm{L}\{f\}\|_{L_2}^2$ is reasonably small.

(i) As first step, rewrite $J(f,g)$ in terms of L_2-norms, first in space, and, then in the frequency domain with the Fourier transforms of f and g being denoted by $\hat{f}$ and $\hat{g}$, respectively.

(ii) Since the formal minimization of (1) is an infinite-dimensional problem that would require the use of calculus of variations (which you probably have not yet studied), we shall use an indirect approach where $J(f,g)$ is manipulated such as to make the solution obvious. Specifically, we ask you to show that the latter can be rewritten as

$$J(f,g) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| \hat{f}(\boldsymbol{\omega}) - \frac{\hat{g}(\boldsymbol{\omega})}{1 + \lambda |\widehat{L}(\boldsymbol{\omega})|^2} \right|^2 (1 + \lambda |\widehat{L}(\boldsymbol{\omega})|^2) + \frac{\lambda |\widehat{L}(\boldsymbol{\omega})|^2 |\hat{g}(\boldsymbol{\omega})|^2}{1 + \lambda |\widehat{L}(\boldsymbol{\omega})|^2} \mathrm{d}\boldsymbol{\omega} \tag{2}$$

and to then deduce the solution, which should be of the form $\tilde{f} = r_{\mathrm{Tik}} * g$ where r_{Tik} is a suitable filter.

(ii') Simplified version (for those who are in a hurry): Using some elementary arguments, show that, for any given frequency $\boldsymbol{\omega} \in \mathbb{R}^d$, the minimizer of (2) is $\hat{f}(\boldsymbol{\omega}) = R_{\mathrm{Tik}}(\boldsymbol{\omega}) \hat{g}(\boldsymbol{\omega})$ where

$$R_{\mathrm{Tik}}(\boldsymbol{\omega}) = \frac{1}{1 + \lambda |\widehat{L}(\boldsymbol{\omega})|^2}. \tag{3}$$

(iii) Give the expression of $J_{\min} = \min_f J(f,g)$ and show that $J_{\min} \le \|g\|_{L_2}^2$.

Solution:
(i) Clearly, $J(f,g) = \|f - g\|_{L_2}^2 + \lambda \|\mathrm{L}\{f\}\|_{L_2}^2$, which with the help of Parseval's formula yields

$$\begin{aligned} J(f,g) &= \frac{1}{(2\pi)^d} \|\hat{f} - \hat{g}\|_{L_2}^2 + \lambda \frac{1}{(2\pi)^d} \|\widehat{\mathrm{L}\{f\}}\|_{L_2}^2 \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| \hat{f}(\boldsymbol{\omega}) - \hat{g}(\boldsymbol{\omega}) \right|^2 + \lambda |\widehat{L}(\boldsymbol{\omega})|^2 |\hat{f}(\boldsymbol{\omega})|^2 \mathrm{d}\boldsymbol{\omega} \end{aligned}$$

(ii) To establish (2), we expand the above Fourier-domain formula of $J(f,g)$ as

$$J(f,g) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left(1 + \lambda |\widehat{L}(\boldsymbol{\omega})|^2\right) |\hat{f}(\boldsymbol{\omega})|^2 + \left|\hat{g}(\boldsymbol{\omega})\right|^2 - 2\mathrm{Re}\big(\hat{f}(\boldsymbol{\omega}) \hat{g}(\boldsymbol{\omega})^*\big) \mathrm{d}\boldsymbol{\omega}$$

Likewise, we have that

$$\begin{aligned} &\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| \hat{f}(\boldsymbol{\omega}) - \frac{\hat{g}(\boldsymbol{\omega})}{1 + \lambda |\widehat{L}(\boldsymbol{\omega})|^2} \right|^2 (1 + \lambda |\widehat{L}(\boldsymbol{\omega})|^2) \mathrm{d}\boldsymbol{\omega} \\ &\quad = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 + \lambda |\widehat{L}(\boldsymbol{\omega})|^2) |\hat{f}(\boldsymbol{\omega})|^2 + \frac{|\hat{g}(\boldsymbol{\omega})|^2}{1 + \lambda |\widehat{L}(\boldsymbol{\omega})|^2} - 2\mathrm{Re}\big(\hat{f}(\boldsymbol{\omega}) \hat{g}(\boldsymbol{\omega})^*\big) \mathrm{d}\boldsymbol{\omega}, \end{aligned}$$

which, by putting the pieces together, gives (2). To find the minimizer of (2), we first observe that the factor $(1 + \lambda |\widehat{L}(\boldsymbol{\omega})|^2)$ is real-valued and strictly greater than 0 for all $\boldsymbol{\omega} \in \mathbb{R}^d$, and that the last term, which involves $\hat{g}$ and $\widehat{L}$, is fixed. Accordingly, the obvious minimizer of (2) at any given frequency $\boldsymbol{\omega}$ is

$$\hat{f}(\boldsymbol{\omega}) = \frac{\hat{g}(\boldsymbol{\omega})}{1 + \lambda |\widehat{L}(\boldsymbol{\omega})|^2}.$$

Indeed, this solution sets the first term in (2) to zero for all $\boldsymbol{\omega} \in \mathbb{R}^d$, while being always well defined (since $(1 + \lambda|\widehat{L}(\boldsymbol{\omega})|^2)$ is non-vanishing). Consequently, the optimal variational denoising is achieved by convolving the noisy signal with $r_{\mathrm{Tik}} = \mathcal{F}^{-1}\{R_{\mathrm{Tik}}\}$ where $R_{\mathrm{Tik}}(\boldsymbol{\omega})$ is specified by (3).

(iii) Based on (2) and the fact that the minimizer annihilates the first part of the integral, we readily deduce that

$$\begin{aligned} J_{\min} &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\lambda|\widehat{L}(\boldsymbol{\omega})|^2}{1 + \lambda|\widehat{L}(\boldsymbol{\omega})|^2} |\hat{g}(\boldsymbol{\omega})|^2 \mathrm{d}\boldsymbol{\omega} \\ &\leq \sup_{\boldsymbol{\omega}\in\mathbb{R}} \left(\frac{\lambda|\widehat{L}(\boldsymbol{\omega})|^2}{1 + \lambda|\widehat{L}(\boldsymbol{\omega})|^2} \right) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\hat{g}(\boldsymbol{\omega})|^2 \mathrm{d}\boldsymbol{\omega} \leq \|g\|_{L_2}^2 \end{aligned}$$

where the right-hand-side inequality follows from the observation that $\frac{\lambda|\widehat{L}(\boldsymbol{\omega})|^2}{1+\lambda|\widehat{L}(\boldsymbol{\omega})|^2} \leq 1$ and the use of Parseval's relation.

(c) Reconciliation.

(i) By comparing the expression of the two filters, find the operator L that makes the two methods equivalent.

(ii) What is the interpretation of such a filter in the world of stochastic processes?

(iii) By transposing the reasoning of Item b-(ii') to the last integral formula in the derivation of the Wiener filter within the course notes (Slide 9-10), provide the expression of the minimum MSE as a function of Φ_f.

Solution:

(i) The general expression of the Wiener for the Gaussian denoising of a signal with power spectrum $\Phi_f(\boldsymbol{\omega})$ is

$$R_{\mathrm{Wiener}}(\boldsymbol{\omega}) = \frac{\Phi_f(\boldsymbol{\omega})}{\Phi_f(\boldsymbol{\omega}) + \sigma^2} = \frac{1}{1 + \sigma^2/\Phi_f(\boldsymbol{\omega})}, \tag{4}$$

which coincides with (3) if we set $\widehat{L}(\boldsymbol{\omega}) = \frac{\pm 1}{\sqrt{\Phi_f(\boldsymbol{\omega})}}$ and $\lambda = \sigma^2$. In the present example, this yields $\widehat{L}(\boldsymbol{\omega}) = \pm\|\boldsymbol{\omega}\|^2$ so that we can set $\mathrm{L} = \Delta$ (Laplacian).

(ii) The stochastic interpretation is that L will whiten the signal, meaning that $w = \mathrm{L}\{f\}$ has a constant power spectrum:

$$\Phi_w(\boldsymbol{\omega}) = |\widehat{L}(\boldsymbol{\omega})|^2 \Phi_f(\boldsymbol{\omega}) = 1. \tag{5}$$

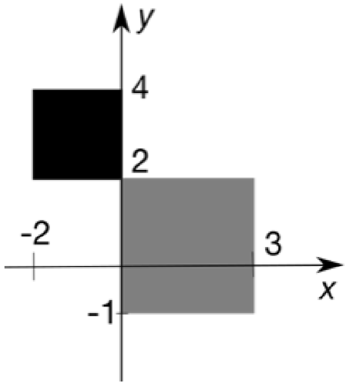
(iii) By identifying the non-vanishing term in the relevant integral formula of the MSE in the course notes (cf. Slide "Derivation of the Wiener filter") and setting $H(\boldsymbol{\omega}) = 1$, we get

$$\mathrm{MSE}_{\min} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{-\Phi_f^2(\boldsymbol{\omega})}{\Phi_f(\boldsymbol{\omega}) + \sigma^2} + \Phi_f(\boldsymbol{\omega}) \mathrm{d}\boldsymbol{\omega} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\sigma^2 \Phi_f(\boldsymbol{\omega})}{\Phi_f(\boldsymbol{\omega}) + \sigma^2} \mathrm{d}\boldsymbol{\omega} \leq \mathbb{E}\{|f(\boldsymbol{x})|^2\}.$$

2 Center of mass and Radon transform

Consider the following image $f(x, y)$:

(a) Give an expression of $f(x, y)$ in terms of a sum of shifted and scaled separable 2D B-splines. Assume the indicated grayscale with white being 0, gray 50 and black 100.



Solution: Since the image is piecewise constant we rewrite it in terms of rectangular pulses, i.e. B-splines β^0 of degree zero. Each of the squares is the product of two β^0s, one in x and the other in y. They have respective intensities (spline weights) 100 and 50; centers (offsets) $(-1, 3)$ and $(3/2, 1/2)$; and lengths (scalings) 2 and 3:

$$f(x, y) = 100 \cdot \beta^0 \left(\frac{x+1}{2}\right) \cdot \beta^0 \left(\frac{y-3}{2}\right) + 50 \cdot \beta^0 \left(\frac{x-3/2}{3}\right) \cdot \beta^0 \left(\frac{y-1/2}{3}\right)$$

We will use both $\cdot$ and the absence of operator to denote scalar multiplication.

(b) Calculate the coordinates $\bar{\mathbf{x}} = (\bar{x}, \bar{y})$ of the center of mass of $f(x, y)$.

Solution: The coordinates can be computed using the integral definition of the center of mass, which can be computed separately for both axes:

$$\bar{x} = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x\, f(x, y)\, \mathrm{d}x \mathrm{d}y}{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)\, \mathrm{d}x \mathrm{d}y} = \frac{-1(2 \cdot 2)100 + 1.5(3 \cdot 3)50}{(2 \cdot 2)100 + (3 \cdot 3)50} = 0.3235$$

$$\bar{y} = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} y\, f(x, y)\, \mathrm{d}x \mathrm{d}y}{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)\, \mathrm{d}x \mathrm{d}y} = \frac{3(2 \cdot 2)100 + 0.5(3 \cdot 3)50}{(2 \cdot 2)100 + (3 \cdot 3)50} = 1.6765$$

Notice that the integrals amount to weighted (area and intensity) averages of the $x-$ or $y-$centers of the two squares. For example, -1 is the $x-$center of the 100-intensity square with area $2 \cdot 2$, hence the term $(-1)(2 \cdot 2)100$.

(c) Calculate the projection $t_{\bar{\mathbf{x}},\theta}$ of the center of mass $\bar{\mathbf{x}}$ onto the projection line for a projection direction $\theta = 0°, 45°$.

Solution: We calculate the projection $t_{\bar{\mathbf{x}},\theta}$ of the center of mass $\bar{\mathbf{x}}$ onto a projection line with direction $\boldsymbol{\theta} = (\cos\theta, \sin\theta)^\top$ using the Euclidean inner product

$$t_{\bar{\mathbf{x}},\theta} = \langle \bar{\mathbf{x}}, \boldsymbol{\theta} \rangle.$$

For $\theta = 0°$:

$$t_{\bar{\mathbf{x}},0°} = \langle (\bar{x}, \bar{y})^\top, (1, 0)^\top \rangle = \bar{x} \cdot 1 + \bar{y} \cdot 0 = 0.3235.$$

And for $\theta = 45°$:

$$t_{\bar{\mathbf{x}},45°} = \left\langle (\bar{x}, \bar{y})^\top, \frac{1}{\sqrt{2}}(1, 1)^\top \right\rangle = \frac{0.3235 + 1.6765}{\sqrt{2}} = \sqrt{2}$$

(d) Calculate and sketch the Radon transform projections $p_\theta(t)$ of f for $\theta = 0°, 45°$.

Solution: We will calculate the Radon transform projections using the formula:

$$p_\theta(t) = \mathcal{R}\{f\}(t,\theta) := \int_{\mathbb{R}^2} f(x,y)\ \delta(x\cos\theta + y\sin\theta - t)\ \mathrm{d}x\mathrm{d}y$$

where $\mathcal{R}$ is the Radon transform operator and we call $p_\theta(t)$ the result from applying the transform to $f(x,y)$, which is the projection living in the sinogram domain. Notice the integral is taken over the **implicit** line fulfilling the following equation: $x\cos\theta + y\sin\theta - t = 0$. This is a line **perpendicular** to the direction $(\cos\theta, \sin\theta)$ and with an offset t from the origin; so be careful with the angles!

For $\theta = 0°$:

$$\begin{aligned} p_{\theta=0^\circ}(t) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\ \delta(x-t)\ \mathrm{d}x\mathrm{d}y \\ &= \int_{-\infty}^{+\infty} f(t,y)\ \mathrm{d}y \\ &= 2\cdot 100\cdot \beta^0\left(\frac{t+1}{2}\right) + 3\cdot 50\cdot\beta^0\left(\frac{t-3/2}{3}\right) \end{aligned}$$

The projection of the image onto the $x-$axis (recall the angle convention, this means we integrate over y for $\theta = 0°$) preserves the two squares because they do not overlap in x. Each square stays homogeneous because the original squares are homogeneous and the integral length over the square is the same for every t. In fact, the result is just the multiplication of the square by the length of the y side.

For $\theta = 45°$:

Let us first get some insight. Since $\theta = 45°$ aligns precisely with the diagonal of the squares we expect a single maximum. The maximum will be offset from the origin half the distance between $(-2,0)$ and $(0,2)$, i.e. $t = \sqrt{2}$. The length of the projection will decrease linearly as we move away from the diagonal. It will do so symmetrically and thus we expect β^1. In addition, the projections are linear so the result will be the projection of the two squares separately. That is a sum of two β^1s with different scaling and weight but the same center. We also expect each weight to be the product between the intensity value (e.g., 50) and the length of the diagonal (e.g., $3\sqrt{2}$). Can you figure out the scaling? Think of where the linear function has to get to zero.

Let's check mathematically whether our intuition is correct:

$$\begin{aligned} p_{\theta=45^\circ}(t) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\ \delta\left(\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} - t\right)\ \mathrm{d}x\mathrm{d}y \\ &= \sqrt{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(\tilde{x}\sqrt{2}, y)\ \delta\left(\tilde{x} + \frac{y}{\sqrt{2}} - t\right)\ \mathrm{d}\tilde{x}\mathrm{d}y \\ &= \sqrt{2}\int_{-\infty}^{+\infty} f(t\sqrt{2} - y, y)\ \mathrm{d}y \\ &= 100\sqrt{2}\underbrace{\int_{-\infty}^{+\infty}\beta^0\left(\frac{t\sqrt{2} - y + 1}{2}\right)\beta^0\left(\frac{y-3}{2}\right)\ \mathrm{d}y}_{I_1} \\ &\quad + 50\sqrt{2}\underbrace{\int_{-\infty}^{+\infty}\beta^0\left(\frac{t\sqrt{2} - y - 3/2}{3}\right)\beta^0\left(\frac{y-1/2}{3}\right)\ \mathrm{d}y}_{I_2} \end{aligned}$$

where we first changed variables, then integrated the delta function and finally used the linearity of the integral to split the image. To simplify the expression for I_1, let $a = \frac{y-3}{2}$

and $\tau_1 = \frac{t\sqrt{2}+1-3}{2} = \frac{t\sqrt{2}-2}{2}$. We now have:

$$\begin{aligned} I_1 &= 2\int_{-\infty}^{+\infty} \beta^0(\tau_1 - a)\beta^0(a)\ \mathrm{d}a \\ &= 2\beta^1(\tau_1) \end{aligned}$$

because the convolution of the two splines is a spline one degree higher. To simplify the expression for I_2, let $b = \frac{y-1/2}{3}$ and $\tau_2 = \frac{t\sqrt{2}-3/2-1/2}{2} = \frac{t\sqrt{2}-2}{2}$. We now have:

$$\begin{aligned} I_1 &= 3\int_{-\infty}^{+\infty} \beta^0(\tau_2 - b)\beta^0(b)\ \mathrm{d}b \\ &= 3\beta^1(\tau_2). \end{aligned}$$

Therefore,

$$p_{\theta=45^\circ}(t) = 100 \cdot 2\sqrt{2} \cdot \beta^1\left(\frac{t\sqrt{2}-2}{2}\right) + 50 \cdot 3\sqrt{2} \cdot \beta^1\left(\frac{t\sqrt{2}-2}{3}\right)$$

(e) Calculate the center of mass $\bar{t}_\theta$ of the projections $p_\theta(t)$ for $\theta = 0^\circ, 45^\circ$.

Solution: We calculate the center of mass $\bar{t}_\theta$ of the projections $p_\theta(t)$ for $\theta = 0^\circ, 45^\circ$ as:

$$\bar{t}_{\theta=0^\circ}(t) = \frac{-1 \cdot 2 \cdot (2 \cdot 100) + 1.5 \cdot 3 \cdot (3 \cdot 5)}{850} = 0.3235$$

and

$$\bar{t}_{\theta=45^\circ}(t) = \sqrt{2} = 1.4142$$

(f) Compare the results you obtained in c) and e). Comment on how this result could be useful for practical applications.

Solution: The results obtained in c) and e) are identical. Calculating the first moment (center of mass) of the projections can be used to determine a common center of rotation in case it is unknown.

3 Radon transform with the "magical" help of Fourier

[basic] The Radon transform of a function provides a complete characterization of its line integrals. Computing it explicitly can be laborious. Fortunately, we can take advantage of the Fourier-slice theorem *which states that* $\mathrm{R}\{f\}(t,\theta)$ *for a fixed* θ *is the inverse Fourier transform of the 1D function* $\omega \mapsto \hat{f}(\omega\boldsymbol{\theta})$ *where* $\boldsymbol{\theta} = (\cos\theta, \sin\theta)$ *and* $\hat{f} : \mathbb{R}^2 \to \mathbb{C}$ *is the 2D Fourier transform of* f.

The Fourier-slice theorem is fundamental in that it remains valid for distributions whose Fourier transform is a well-defined function. It allows us to calculate the Radon transform of objects such as $\|\boldsymbol{x}\|^s$*, which are not integrable in the conventional Lebesgue sense.*

(a) Basic calculations. Compute the Radon transform of the following functions.

1. $\delta(\cdot - \boldsymbol{x}_0)$ with $\boldsymbol{x}_0 = (x_0, y_0) = r(\cos\phi, \sin\phi) \in \mathbb{R}^2$
2. $g(\boldsymbol{x}) = \frac{1}{2\pi}\mathrm{e}^{-\frac{1}{2}\|\boldsymbol{x}\|^2}$ (standardized Gaussian)

Solution:

1. We first evaluate the 2D Fourier transform $\mathcal{F}\{\delta(\cdot - \boldsymbol{x}_0)\}(\boldsymbol{\omega}) = \mathrm{e}^{-\mathrm{j}\langle\boldsymbol{\omega},\boldsymbol{x}_0\rangle} = \mathrm{e}^{-\mathrm{j}\omega\langle\boldsymbol{\theta},\boldsymbol{x}_0\rangle}$. Correspondingly, we recall that the inverse 1D Fourier transform of $\mathrm{e}^{-\mathrm{j}\omega t_0}$ is $\mathcal{F}^{-1}\{\mathrm{e}^{-\mathrm{j}\omega t_0}\}(t) = \delta(t - t_0)$. By setting, $t_0 = \langle\boldsymbol{\theta}, \boldsymbol{x}_0\rangle = x_0\cos\theta + y_0 \sin\theta$, this yields

$$\mathrm{R}\{\delta(\cdot - \boldsymbol{x}_0)\}(t,\theta) = \delta(t - x_0\cos\theta + y_0\sin\theta) = \delta(t - r\cos(\theta - \phi)) \qquad \text{(sinogram)}$$

2. $\mathcal{F}\{g\}(\boldsymbol{\omega}) = \mathrm{e}^{-\frac{1}{2}\|\boldsymbol{\omega}\|^2} = \mathrm{e}^{-\frac{1}{2}\omega^2\|\boldsymbol{\theta}\|^2} = \mathrm{e}^{-\frac{1}{2}\omega^2}$, which is independent of θ since g is isotropic. Consequently,

$$\mathcal{R}\{g\}(t,\theta) = \mathcal{F}^{-1}_{1\mathrm{D}}\{\mathrm{e}^{-\frac{\omega^2}{2}}\}(t) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{t^2}{2}}.$$

(b) Prove the following properties of the Radon transform where $p_\theta(t) = \mathcal{R}\{f\}(t,\theta)$.

1. Laplacian:
$$\mathcal{R}\{\Delta f\}(t,\theta) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}p_\theta(t).$$

2. Conservation of the integral.
$$\int_{\mathbb{R}} p_\theta(t)\mathrm{d}t = \hat{f}(\mathbf{0}) = \int_{\mathbb{R}^2} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

3. Second-order moments:
$$\int_{\mathbb{R}} t^2 p_\theta(t)\mathrm{d}t = \boldsymbol{\theta}^\mathsf{T}\mathbf{H}\boldsymbol{\theta} \quad \text{where} \quad \mathbf{H} = \begin{pmatrix} \int_{\mathbb{R}^2} x^2 f(x,y)\mathrm{d}x\mathrm{d}y & \int_{\mathbb{R}^2} xyf(x,y)\mathrm{d}x\mathrm{d}y \\ \int_{\mathbb{R}^2} yxf(x,y)\mathrm{d}x\mathrm{d}y & \int_{\mathbb{R}^2} y^2 f(x,y)\mathrm{d}x\mathrm{d}y \end{pmatrix}$$

Solution:

1. The frequency response of the Laplace operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is $(\mathrm{j}\omega_1)^2 + (\mathrm{j}\omega_2)^2 = -\omega_1^2 + -\omega_2^2 = -\|\boldsymbol{\omega}\|^2$. Consequently, $\mathcal{F}\{\Delta f\}(\boldsymbol{\omega}) = -\|\boldsymbol{\omega}\|^2\hat{f}(\boldsymbol{\omega})$. For the Fourier "slice" $\boldsymbol{\omega} = \omega\boldsymbol{\theta}$, we have $\mathcal{F}\{\Delta f\}(\omega\boldsymbol{\theta}) = -|\omega|^2\hat{f}(\omega\boldsymbol{\theta}) = -\omega^2\hat{p}_\theta(\omega)$, whose inverse 1D Fourier, $\frac{\mathrm{d}^2}{\mathrm{d}t^2}p_\theta(t)$, then yields the Radon transform of Δf.

2. The property directly follows from the standard relation $\hat{f}(\mathbf{0}) = \int_{\mathbb{R}^d} f(\boldsymbol{x})\mathrm{d}x$, which holds in any number of dimensions including for $\int_{\mathbb{R}} p_\theta(t)\mathrm{d}t = \hat{p}_\theta(0)$. It then suffice to note that $\hat{p}_\theta(0) = \hat{f}(\mathbf{0})$, as obvious consequence of the Fourier-slice theorem.

3. The relevant relations there are the moment-generating properties of the 1D and 2D Fourier transform:

$$\int_{\mathbb{R}} t^n p_\theta(t)\mathrm{d}t = \mathrm{j}^n \frac{\mathrm{d}^n\hat{p}_\theta(\omega)}{\mathrm{d}\omega^n}\bigg|_{\omega=0} \tag{6}$$

$$\int_{\mathbb{R}^2} x^{n_1}y^{n_2}f(x,y)\mathrm{d}x\mathrm{d}y = \mathrm{j}^{n_1+n_2}\frac{\partial^{n_1+n_2}\hat{f}(\boldsymbol{\omega})}{\partial\omega_1^m\partial\omega_2^n}\bigg|_{\boldsymbol{\omega}=\mathbf{0}}. \tag{7}$$

The case of interest for us is $n = 2$ and $n_1 + n_2 = 2$ so that the matrix $\mathbf{H}$ can be identified as being $\mathrm{j}^2 = -1$ "times" the Hessian of the $\hat{f}$ evaluated at $\boldsymbol{\omega} = \mathbf{0}$. By virtue of the Fourier-slice theorem, the second derivative of $\hat{p}_\theta$ coincides with the second directional derivative of $\hat{f}$ along $\boldsymbol{\theta}$, which can be calculated as

$$\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\hat{p}_\theta(\omega) = \mathrm{D}^2_{\boldsymbol{\theta}}\hat{f}(\boldsymbol{\omega})\Big|_{\boldsymbol{\omega}=\omega\boldsymbol{\theta}} = \boldsymbol{\theta}^\mathsf{T}\begin{pmatrix} \frac{\partial^2}{\partial\omega_1^2}\hat{f}(\boldsymbol{\omega}) & \frac{\partial^2}{\partial\omega_1\partial\omega_2}\hat{f}(\boldsymbol{\omega}) \\ \frac{\partial^2}{\partial\omega_2\partial\omega_1}\hat{f}(\boldsymbol{\omega}) & \frac{\partial^2}{\partial\omega_2^2}\hat{f}(\boldsymbol{\omega}) \end{pmatrix}\boldsymbol{\theta}, \tag{8}$$

using the directional derivative calculus from Chapter 6 (which is obviously also valid in the Fourier domain). The result then follows from the determination of (8) for $\boldsymbol{\omega} = \mathbf{0}$ as

$$\int_{\mathbb{R}} t^2 p_\theta(t)\mathrm{d}t = -\left.\mathrm{D}^2_{\boldsymbol{\theta}} \hat{f}(\boldsymbol{\omega})\right|_{\boldsymbol{\omega}=0} = \boldsymbol{\theta}^\mathsf{T}\mathbf{H}\boldsymbol{\theta}. \tag{9}$$

(c) Transform of a separable function: Let $\varphi(\boldsymbol{x}) = \varphi_1(x)\varphi_2(y)$. Show that $\mathrm{R}\{\varphi\}(t,\theta) = \varphi_\theta(t)$ where

$$\varphi_\theta(t) = \left(\tfrac{1}{|\cos\theta|}\varphi_1\big(\frac{\cdot}{\cos\theta}\big) * \tfrac{1}{|\sin\theta|}\varphi_2\big(\frac{\cdot}{\sin\theta}\big)\right)(t)$$

with the convention that $\frac{1}{|a|}\varphi\big(\frac{\cdot}{a}\big) \to \hat{\varphi}(0)\delta$ as $a \to 0$.

Solution: Once again, we start by computing $\mathcal{F}\{\varphi\}(\omega_1,\omega_2) = \hat{\varphi}_1(\omega_1)\hat{\varphi}_2(\omega_2)$, which when evaluated along the Fourier "slice" $\boldsymbol{\omega} = (\omega_1,\omega_2) = \omega\boldsymbol{\theta}$ yields $\hat{\varphi}_\theta(\omega) = \hat{\varphi}_1(\omega\cos\theta)\hat{\varphi}_2(\omega\sin\theta)$. We then interpret this product as the Fourier transform of the convolution $(f_1 * f_2)$ of the 1D functions $f_1 = \mathcal{F}^{-1}\{\hat{\varphi}_1(\omega\cos\theta)\}$ and $f_2 = \mathcal{F}^{-1}\{\hat{\varphi}_2(\omega\sin\theta)\}$. By invoking the classical scaling relation

$$\mathcal{F}\Big\{f\Big(\frac{t}{a}\Big)\Big\}(\omega) = |a|\hat{f}(a\omega), \tag{10}$$

we readily deduce that $f_1(t) = \frac{1}{|\cos\theta|}\varphi_1\big(\frac{t}{\cos\theta}\big)$ and $f_2(t) = \frac{1}{|\sin\theta|}\varphi_2\big(\frac{t}{\sin\theta}\big)$, which is the desired result. The "singular" cases where either $\cos\theta$ or $\sin\theta$ are zero are also covered by the formula provided that we interpret $\lim_{a\to 0} f\Big(\frac{t}{a}\Big) = \delta(t)\int_{\mathbb{R}} f(t)\mathrm{d}t$ as a weighted Dirac distribution. For example, the Fourier-slice theorem gives $\hat{\varphi}_{\theta=0}(\omega) = \hat{\varphi}_1(\omega) \times \hat{\varphi}_2(0)$, which directly translates into $\mathcal{R}\{\varphi\}(t,0) = a_0\varphi_1(t)$ where $a_0 = \hat{\varphi}_2(0) = \int_{\mathbb{R}}\varphi_2(t)\mathrm{d}t$.

(d) Radon transform of a B-spline. Compute $\mathrm{R}\{\varphi\}(t,0)$ and $\mathrm{R}\{\varphi\}(t,\frac{\pi}{4})$ for the case where $\varphi(\boldsymbol{x}) = \beta^n(x)\beta^n(y)$ is a separable B-spline of degree n.

Solution: By using the result of Item (c) (or the Fourier-slice theorem) and the property that $\int_{\mathbb{R}}\beta^n(x)\mathrm{d}x = 1$, we readily deduce that

$$\mathcal{R}\{\varphi\}(t,0) = (\beta^n * \delta)(t) = \beta^n(t) \tag{11}$$

Likewise, for $\theta = \frac{\pi}{4}$ with $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, we get that

$$\mathcal{R}\{\varphi\}(t,\tfrac{\pi}{4}) = \left(\tfrac{1}{1/\sqrt{2}}\beta^n\big(\frac{\cdot}{1/\sqrt{2}}\big) * \tfrac{1}{1/\sqrt{2}}\beta^n\big(\frac{\cdot}{1/\sqrt{2}}\big)\right)(t) = \sqrt{2}\beta^{2n+1}\big(\frac{t}{1/\sqrt{2}}\big) \tag{12}$$

where we used the B-spline convolution property: $(\beta^n * \beta^n)(t) = \beta^{2n+1}(t)$. Note that the scaling by $\sqrt{2}$ in (12) is consistent with the integral preservation property $\int_{\mathbb{R}}\varphi_\theta(t)\mathrm{d}t = 1$, which can always be used as a cross-check.