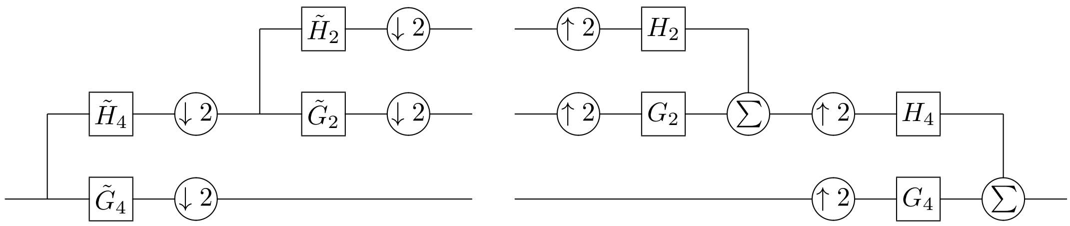
Image Processing 2, Exercise 6

1 The Haar transform

[intermediate] We explicitly write the filters for a small wavelet transform. Although this is a bit difficult, it gives an idea of how the entire transform can be written as a matrix-vector product.

Let the filters of Haar's wavelet transform in one dimension be given by $\tilde{H}(z) = \frac{\sqrt{2}}{2}\,(1+z)$, $\tilde{G}(z) = \frac{\sqrt{2}}{2}\,(1-z)$, $H(z) = \frac{\sqrt{2}}{2}\,\left(1+z^{-1}\right)$, and $G(z) = \frac{\sqrt{2}}{2}\,\left(1-z^{-1}\right)$. Below, we sketch Mallat's tree-structured filterbank and label the occurrences of these four filters for the support $W = 4$, along with the downsampling operations. Note that we only recursively decompose the low-frequency branch of the tree.



Let $\mathbf{x} \in \mathbb{R}^W$ be the vector representation of the one-dimensional sequence x of finite support $W = 4$. Moreover, let $\tilde{\mathbf{W}}_4$ be such that $\mathbf{y} = \tilde{\mathbf{W}}_4\,\mathbf{x}$ is the one-dimensional Haar-wavelet transform of $\mathbf{x}$. By convention, the coefficients in $\mathbf{y}$ are ordered first from lowest to highest frequency, then from past to future. Following the branches on the left part of the Mallat's tree, we can write $\mathbf{y} = \mathbf{P}_{\tilde{\mathbf{H}}\tilde{\mathbf{H}}}\,y_1 + \mathbf{P}_{\tilde{\mathbf{G}}\tilde{\mathbf{H}}}\,y_2 + \mathbf{P}_{\tilde{\mathbf{G}}}\,(y_3, y_4)$ with $\mathbf{P}_{\tilde{\mathbf{H}}\tilde{\mathbf{H}}} = (1,0,0,0)$, $\mathbf{P}_{\tilde{\mathbf{G}}\tilde{\mathbf{H}}} = (0,1,0,0)$, and $\mathbf{P}_{\tilde{\mathbf{G}}} = \left[\ (0,0,1,0)^T,\quad (0,0,0,1)^T\ \right]$. Thus, we can write

$$\tilde{\mathbf{W}}_4 = \left(\mathbf{P}_{\tilde{\mathbf{H}}\tilde{\mathbf{H}}}\,\underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{\mathbf{D}_{2\to1}}\,\tilde{\mathbf{H}}_2 + \mathbf{P}_{\tilde{\mathbf{G}}\tilde{\mathbf{H}}}\,\underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{\mathbf{D}_{2\to1}}\,\tilde{\mathbf{G}}_2\right)\mathbf{D}_{4\to2}\,\underbrace{\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ (0) & (\frac{\sqrt{2}}{2}) & (\frac{\sqrt{2}}{2}) & (0) \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ (\frac{\sqrt{2}}{2}) & (0) & (0) & (\frac{\sqrt{2}}{2}) \end{pmatrix}}_{\tilde{\mathbf{H}}_4} + \mathbf{P}_{\tilde{\mathbf{G}}}\,\mathbf{D}_{4\to2}\,\tilde{\mathbf{G}}_4.$$

(a) Provide the lowpass $\tilde{\mathbf{H}}_2$, high-pass $\tilde{\mathbf{G}}_2$, downsampling $\mathbf{D}_{4\to2}$, high-pass $\tilde{\mathbf{G}}_4$, and overall transformation $\tilde{\mathbf{W}}_4$.

(b) Let $\mathbf{P}_{\mathbf{HH}} = \mathbf{P}^{\mathsf{T}}_{\tilde{\mathbf{H}}\tilde{\mathbf{H}}}$, $\mathbf{P}_{\mathbf{HG}} = \mathbf{P}^{\mathsf{T}}_{\tilde{\mathbf{G}}\tilde{\mathbf{H}}}$, and $\mathbf{P}^{\mathsf{T}}_{\tilde{\mathbf{G}}}$ represent the matrices used for honoring the order of the coefficients. Then, give the matrix $\mathbf{W}_4$ that will recover $\mathbf{x} = \mathbf{W}_4\,\mathbf{y}$ from its one-dimensional Haar-wavelet transform $\mathbf{y}$.

(c) Give the vector $\mathbf{y}$ of wavelet coefficients for the sequence $x = \left[\ \boxed{5}\quad 7\quad 6\quad -2\ \right]$.

(d) Apply a soft-thresholding operation to obtain $\bar{\mathbf{y}} = T_\lambda(\mathbf{y})$ with $\lambda = 2$. Proceed component-wise, except for the lowest-frequency component—which remains intact, so that $\bar{y}_1 = y_1$.

(e) From $\bar{\mathbf{y}}$, reconstruct the denoised version $\bar{\mathbf{x}}$ of $\mathbf{x}$.

2 JPEG 2000

[intermediate] In this exercise, we will explore the JPEG2000 compression standard which uses the wavelet transform.

(a) Let four digital filters be described by their z-transform $H(z) = \frac{1}{4}\,\left(z^{-1}+2+z\right)$, $\tilde{G}(z) = \frac{1}{4}\,\left(-1+2\,z-z^2\right)$, $\tilde{H}(z) = \frac{1}{4}\,\left(-z^{-2}+2\,z^{-1}+6+2\,z-z^2\right)$, and $G(z) = \frac{1}{4}\,\left(-z^{-3}-2\,z^{-2}+6\,z^{-1}-2-z\right)$. (These filters form the core of the JPEG 2000 standard.) Report in a table the discrete impulse response of each filter for indices $k \in [-4 \ldots 4]$.

(b) Verify explicitly that the four JPEG 2000 filters satisfy the perfect-reconstruction condition PR-1 (distortion-free).

(c) Verify explicitly that these four filters satisfy the perfect-reconstruction condition PR-2 (aliasing-free).

(d) Does H satisfy the conjugate-quadrature condition? Justify your answer.

(e) Given the sequence x below, compute its full wavelet analysis and report intermediate results in a table as indicated.

k	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$x[k]$	0	0	0	0	0	16	-2	-20	14	-32	0	0	0	0	0
$(x * \tilde{h})[k]$	—	—												—	—
$y_1[k] = [x * \tilde{h}]_{\downarrow 2 \uparrow 2}[k]$	—	—												—	—
$(x * \tilde{g})[k]$														—	—
$y_2[k] = [x * \tilde{g}]_{\downarrow 2 \uparrow 2}[k]$														—	—
$(y_1 * h)[k]$	—	—	—										—	—	—
$(y_2 * g)[k]$	—	—	—										—	—	—
$(y_1 * h)[k] + (y_2 * g)[k]$	—	—	—										—	—	—

3 Wiener Filter

[intermediate] Performing Wiener filtering on a vector signal.

Assume that $\boldsymbol{s}$ is a vector of signals and the goal is to recover them using the MMSE criteria. The signal model is given as $\boldsymbol{y} = A\boldsymbol{s} + \boldsymbol{n}$ where A is a fixed known invertible mixture matrix and $\boldsymbol{n}$ is a Gaussian noise with covariance matrix $C_{\boldsymbol{n}} = \sigma^2 \mathrm{I}$ that is independent from the signal. Our linear estimator is in the form of $\tilde{\boldsymbol{s}} = \alpha A^{-1}\boldsymbol{y}$.

(a) Determine the optimal value of α that minimizes the MMSE loss defined as $\epsilon^2 = E\{\|\tilde{\boldsymbol{s}} - \boldsymbol{s}\|_2^2\}$

(b) Explain your result qualitatively: how should α change as the norm of the signal and noise change? What role does A play?

(c) Show that if A is a unitary matrix, then α coincides with the formula given in Slide 8-62.