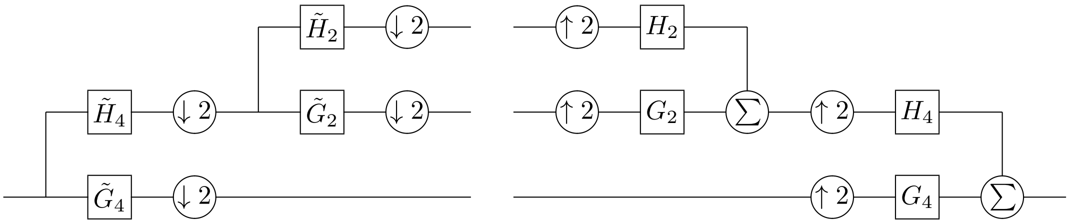
Image Processing 2, Exercise 6

1 The Haar transform

[intermediate] We explicitly write the filters for a small wavelet transform. Although this is a bit difficult, it gives an idea of how the entire transform can be written as a matrix-vector product.

Let the filters of Haar's wavelet transform in one dimension be given by $\tilde{H}(z) = \frac{\sqrt{2}}{2}\,(1+z)$, $\tilde{G}(z) = \frac{\sqrt{2}}{2}\,(1-z)$, $H(z) = \frac{\sqrt{2}}{2}\,\left(1+z^{-1}\right)$, and $G(z) = \frac{\sqrt{2}}{2}\,\left(1-z^{-1}\right)$. Below, we sketch Mallat's tree-structured filterbank and label the occurrences of these four filters for the support $W = 4$, along with the downsampling operations. Note that we only recursively decompose the low-frequency branch of the tree.



Let $\mathbf{x} \in \mathbb{R}^W$ be the vector representation of the one-dimensional sequence x of finite support $W = 4$. Moreover, let $\tilde{\mathbf{W}}_4$ be such that $\mathbf{y} = \tilde{\mathbf{W}}_4\,\mathbf{x}$ is the one-dimensional Haar-wavelet transform of $\mathbf{x}$. By convention, the coefficients in $\mathbf{y}$ are ordered first from lowest to highest frequency, then from past to future. Following the branches on the left part of the Mallat's tree, we can write $\mathbf{y} = \mathbf{P}_{\tilde{\mathbf{H}}\tilde{\mathbf{H}}}\,y_1 + \mathbf{P}_{\tilde{\mathbf{G}}\tilde{\mathbf{H}}}\,y_2 + \mathbf{P}_{\tilde{\mathbf{G}}}\,(y_3, y_4)$ with $\mathbf{P}_{\tilde{\mathbf{H}}\tilde{\mathbf{H}}} = (1,0,0,0)$, $\mathbf{P}_{\tilde{\mathbf{G}}\tilde{\mathbf{H}}} = (0,1,0,0)$, and $\mathbf{P}_{\tilde{\mathbf{G}}} = \left[\ (0,0,1,0)^T,\quad (0,0,0,1)^T\ \right]$. Thus, we can write

$$\tilde{\mathbf{W}}_4 = \left(\mathbf{P}_{\tilde{\mathbf{H}}\tilde{\mathbf{H}}} \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{\mathbf{D}_{2\to1}} \tilde{\mathbf{H}}_2 + \mathbf{P}_{\tilde{\mathbf{G}}\tilde{\mathbf{H}}} \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{\mathbf{D}_{2\to1}} \tilde{\mathbf{G}}_2 \right) \mathbf{D}_{4\to2} \underbrace{\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ (0) & (\frac{\sqrt{2}}{2}) & (\frac{\sqrt{2}}{2}) & (0) \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ (\frac{\sqrt{2}}{2}) & (0) & (0) & (\frac{\sqrt{2}}{2}) \end{pmatrix}}_{\tilde{\mathbf{H}}_4} + \mathbf{P}_{\tilde{\mathbf{G}}}\,\mathbf{D}_{4\to2}\,\tilde{\mathbf{G}}_4.$$

(a) Provide the lowpass $\tilde{\mathbf{H}}_2$, high-pass $\tilde{\mathbf{G}}_2$, downsampling $\mathbf{D}_{4\to2}$, high-pass $\tilde{\mathbf{G}}_4$, and overall transformation $\tilde{\mathbf{W}}_4$.

Solution:

Note: Parentheses indicate matrix elements that have no effect on the transform due to upsampling or downsampling. You would not have to write these parentheses on an exam.

$$
\begin{aligned}
\tilde{\mathbf{W}}_4 \;=\; & \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{P}_{\tilde{\mathbf{H}}\tilde{\mathbf{H}}}}
\underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{\mathbf{D}_{2\to 1}}
\underbrace{\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ (\frac{\sqrt{2}}{2}) & (\frac{\sqrt{2}}{2}) \end{pmatrix}}_{\tilde{\mathbf{H}}_2}
\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{\mathbf{D}_{4\to 2}}
\underbrace{\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ (0) & (\frac{\sqrt{2}}{2}) & (\frac{\sqrt{2}}{2}) & (0) \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ (\frac{\sqrt{2}}{2}) & (0) & (0) & (\frac{\sqrt{2}}{2}) \end{pmatrix}}_{\tilde{\mathbf{H}}_4} \\
& + \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\mathbf{P}_{\tilde{\mathbf{G}}\tilde{\mathbf{H}}}}
\underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{\mathbf{D}_{2\to 1}}
\underbrace{\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ (-\frac{\sqrt{2}}{2}) & (\frac{\sqrt{2}}{2}) \end{pmatrix}}_{\tilde{\mathbf{G}}_2}
\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{\mathbf{D}_{4\to 2}}
\underbrace{\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 \\ (0) & (\frac{\sqrt{2}}{2}) & (\frac{\sqrt{2}}{2}) & (0) \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ (\frac{\sqrt{2}}{2}) & (0) & (0) & (\frac{\sqrt{2}}{2}) \end{pmatrix}}_{\tilde{\mathbf{H}}_4} \\
& + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{P}_{\tilde{\mathbf{G}}}}
\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{\mathbf{D}_{4\to 2}}
\underbrace{\begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \\ (0) & (\frac{\sqrt{2}}{2}) & (-\frac{\sqrt{2}}{2}) & (0) \\ 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ (-\frac{\sqrt{2}}{2}) & (0) & (0) & (\frac{\sqrt{2}}{2}) \end{pmatrix}}_{\tilde{\mathbf{G}}_4} \\
\;=\; & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}
\end{aligned}
$$

Thus we have

$$
\underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}}_{\mathbf{y}} = \underbrace{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}}_{\tilde{\mathbf{W}}_4} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{\mathbf{x}}
$$

This expression is consistent with

$$
y_1 = \underbrace{\frac{\sqrt{2}}{2}\overbrace{\left(\frac{\sqrt{2}}{2}x_1 + \frac{\sqrt{2}}{2}x_2\right)}^{\left(\tilde{h}*x\right)[0]} + \frac{\sqrt{2}}{2}\overbrace{\left(\frac{\sqrt{2}}{2}x_3 + \frac{\sqrt{2}}{2}x_4\right)}^{\left(\tilde{h}*x\right)[2]}}_{\left(\tilde{h}*\left(\tilde{h}*x\right)_{\downarrow 2}\right)[0]}
$$

$$
y_2 = \underbrace{\frac{\sqrt{2}}{2}\overbrace{\left(\frac{\sqrt{2}}{2}x_1 + \frac{\sqrt{2}}{2}x_2\right)}^{\left(\tilde{h}*x\right)[0]} - \frac{\sqrt{2}}{2}\overbrace{\left(\frac{\sqrt{2}}{2}x_3 + \frac{\sqrt{2}}{2}x_4\right)}^{\left(\tilde{h}*x\right)[2]}}_{\left(\tilde{g}*\left(\tilde{h}*x\right)_{\downarrow 2}\right)[0]}
$$

$$
y_3 = \underbrace{\frac{\sqrt{2}}{2}x_1 - \frac{\sqrt{2}}{2}x_2}_{(\tilde{g}*x)[0]}
$$

$$
y_4 = \underbrace{\frac{\sqrt{2}}{2}x_3 - \frac{\sqrt{2}}{2}x_4}_{(\tilde{g}*x)[2]}
$$

(b) Let $\mathbf{P_{HH}} = \mathbf{P}^{\mathsf{T}}_{\tilde{\mathbf{H}}\tilde{\mathbf{H}}}$, $\mathbf{P_{HG}} = \mathbf{P}^{\mathsf{T}}_{\tilde{\mathbf{G}}\tilde{\mathbf{H}}}$, and $\mathbf{P}^{\mathsf{T}}_{\tilde{\mathbf{G}}}$ represent the matrices used for honoring the order of the coefficients. Then, give the matrix $\mathbf{W}_4$ that will recover $\mathbf{x} = \mathbf{W}_4\,\mathbf{y}$ from its one-dimensional Haar-wavelet transform $\mathbf{y}$.

Solution: Because $\tilde{\mathbf{W}}_4$ is orthonormal,

$$
\begin{aligned}
\mathbf{W}_4 &= \tilde{\mathbf{W}}_4^{\mathsf{H}} \\
&= \underbrace{\begin{pmatrix} \frac{\sqrt{2}}{2} & (0) & 0 & (\frac{\sqrt{2}}{2}) \\ \frac{\sqrt{2}}{2} & (\frac{\sqrt{2}}{2}) & 0 & (0) \\ 0 & (\frac{\sqrt{2}}{2}) & \frac{\sqrt{2}}{2} & (0) \\ 0 & (0) & \frac{\sqrt{2}}{2} & (\frac{\sqrt{2}}{2}) \end{pmatrix}}_{\mathbf{H}_4} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\mathbf{U}_{2\to 4}} \underbrace{\begin{pmatrix} \frac{\sqrt{2}}{2} & (\frac{\sqrt{2}}{2}) \\ \frac{\sqrt{2}}{2} & (\frac{\sqrt{2}}{2}) \end{pmatrix}}_{\mathbf{H}_2} \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\mathbf{U}_{1\to 2}} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}}_{\mathbf{P_{HH}}} \\
&\quad + \underbrace{\begin{pmatrix} \frac{\sqrt{2}}{2} & (0) & 0 & (\frac{\sqrt{2}}{2}) \\ \frac{\sqrt{2}}{2} & (\frac{\sqrt{2}}{2}) & 0 & (0) \\ 0 & (\frac{\sqrt{2}}{2}) & \frac{\sqrt{2}}{2} & (0) \\ 0 & (0) & \frac{\sqrt{2}}{2} & (\frac{\sqrt{2}}{2}) \end{pmatrix}}_{\mathbf{H}_4} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\mathbf{U}_{2\to 4}} \underbrace{\begin{pmatrix} \frac{\sqrt{2}}{2} & (-\frac{\sqrt{2}}{2}) \\ -\frac{\sqrt{2}}{2} & (\frac{\sqrt{2}}{2}) \end{pmatrix}}_{\mathbf{G}_2} \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\mathbf{U}_{1\to 2}} \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}}_{\mathbf{P_{GH}}} \\
&\quad + \underbrace{\begin{pmatrix} \frac{\sqrt{2}}{2} & (0) & 0 & (-\frac{\sqrt{2}}{2}) \\ -\frac{\sqrt{2}}{2} & (\frac{\sqrt{2}}{2}) & 0 & (0) \\ 0 & (-\frac{\sqrt{2}}{2}) & \frac{\sqrt{2}}{2} & (0) \\ 0 & (0) & -\frac{\sqrt{2}}{2} & (\frac{\sqrt{2}}{2}) \end{pmatrix}}_{\mathbf{G}_4} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\mathbf{U}_{2\to 4}} \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\mathbf{P_{G}}} \\
&= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{\sqrt{2}}{2} \end{pmatrix}
\end{aligned}
$$

which is consistent with

$$
x_1 = \underbrace{\frac{\sqrt{2}}{2}\left(\overbrace{\frac{\sqrt{2}}{2}\,y_1}^{\left(h*(P_{hh}y)_{\uparrow 2}\right)[0]} + \overbrace{\frac{\sqrt{2}}{2}\,y_2}^{\left(g*(P_{hg}y)_{\uparrow 2}\right)[0]} \right)}_{\left(h*\left(\left(h*(P_{hh}y)_{\uparrow 2}+g*(P_{hg}y)_{\uparrow 2}\right)\right)_{\uparrow 2}\right)[0]} + \underbrace{\frac{\sqrt{2}}{2}\,y_3}_{\left(g*(P_g y)_{\uparrow 2}\right)[0]}
$$

$$
x_2 = \underbrace{\frac{\sqrt{2}}{2}\left(\overbrace{\frac{\sqrt{2}}{2}\,y_1}^{\left(h*(P_{hh}y)_{\uparrow 2}\right)[1]} + \overbrace{\frac{\sqrt{2}}{2}\,y_2}^{\left(g*(P_{hg}y)_{\uparrow 2}\right)[1]} \right)}_{\left(h*\left(\left(h*(P_{hh}y)_{\uparrow 2}+g*(P_{hg}y)_{\uparrow 2}\right)\right)_{\uparrow 2}\right)[1]} - \underbrace{\frac{\sqrt{2}}{2}\,y_3}_{\left(g*(P_g y)_{\uparrow 2}\right)[1]}
$$

$$
x_3 = \underbrace{\frac{\sqrt{2}}{2}\left(\overbrace{\frac{\sqrt{2}}{2}\,y_1}^{\left(h*(P_{hh}y)_{\uparrow 2}\right)[2]} - \overbrace{\frac{\sqrt{2}}{2}\,y_2}^{\left(g*(P_{hg}y)_{\uparrow 2}\right)[2]} \right)}_{\left(h*\left(\left(h*(P_{hh}y)_{\uparrow 2}+g*(P_{hg}y)_{\uparrow 2}\right)\right)_{\uparrow 2}\right)[2]} + \underbrace{\frac{\sqrt{2}}{2}\,y_4}_{\left(g*(P_g y)_{\uparrow 2}\right)[2]}
$$

$$x_4 = \underbrace{\frac{\sqrt{2}}{2}\left(\overbrace{\frac{\sqrt{2}}{2}\, y_1}^{\left(h*(P_{hh}y)_{\uparrow 2}\right)[3]} \quad - \quad \overbrace{\frac{\sqrt{2}}{2}\, y_2}^{\left(g*(P_{hg}y)_{\uparrow 2}\right)[3]} \right)}_{\left(h*\left(\left(h*(P_{hh}y)_{\uparrow 2}+g*(P_{hg}y)_{\uparrow 2}\right)\right)_{\uparrow 2}\right)[3]} - \underbrace{\frac{\sqrt{2}}{2}\, y_4}_{\left(g*(P_g y)_{\uparrow 2}\right)[3]}$$

(c) Give the vector **y** of wavelet coefficients for the sequence $x = \begin{bmatrix} \boxed{5} & 7 & 6 & -2 \end{bmatrix}$.

Solution:

$$\underbrace{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}}_{\tilde{\mathbf{W}}_4} \underbrace{\begin{pmatrix} 5 \\ 7 \\ 6 \\ -2 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 8 \\ 4 \\ -\sqrt{2} \\ \sqrt{32} \end{pmatrix}}_{\mathbf{y}}$$

(d) Apply a soft-thresholding operation to obtain $\bar{\mathbf{y}} = T_\lambda(\mathbf{y})$ with $\lambda = 2$. Proceed component-wise, except for the lowest-frequency component—which remains intact, so that $\bar{y}_1 = y_1$.

Solution: Following the definition of $T_\lambda(y)$ from lecture slide 8-63

$$T_\lambda(y) = \begin{cases} y - \lambda, & y > \lambda \\ 0, & |y| \leq \lambda \\ y + \lambda, & y < -\lambda \end{cases}$$

we can obtain $\bar{\mathbf{y}}$

$$T_2(4) = 2 \qquad T_2(-\sqrt{2}) = 0 \qquad T_2(\sqrt{32}) = \sqrt{32} - 2$$

$$\bar{\mathbf{y}} = (8, 2, 0, -2 + \sqrt{32})$$

(e) From $\bar{\mathbf{y}}$, reconstruct the denoised version $\bar{\mathbf{x}}$ of $\mathbf{x}$.

Solution: Using the final results from 1(b) and vector $\bar{\mathbf{y}}$ of denoised wavelet coefficients from 1(d)

$$\begin{aligned} \bar{\mathbf{x}} &= \mathbf{W}_4\, \bar{\mathbf{y}} \\ &= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 8 \\ 2 \\ 0 \\ -2 + \sqrt{32} \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 5 \\ 7 - \sqrt{2} \\ -1 + \sqrt{2} \end{pmatrix} \end{aligned}$$

2 JPEG 2000

[intermediate] In this exercise, we will explore the JPEG2000 compression standard which uses the wavelet transform.

(a) Let four digital filters be described by their z-transform $H(z) = \frac{1}{4}\left(z^{-1} + 2 + z\right)$, $\tilde{G}(z) = \frac{1}{4}\left(-1 + 2\,z - z^2\right)$, $\tilde{H}(z) = \frac{1}{4}\left(-z^{-2} + 2\,z^{-1} + 6 + 2\,z - z^2\right)$, and $G(z) = \frac{1}{4}\left(-z^{-3} - 2\,z^{-2} + 6\,z^{-1} - 2 - z\right)$. (These filters form the core of the JPEG 2000 standard.) Report in a table the discrete impulse response of each filter for indices $k \in [-4 \ldots 4]$.

Solution:

k	-4	-3	-2	-1	0	1	2	3	4
$h[k]$	0	0	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0	0	0
$g[k]$	0	0	0	$-\frac{1}{4}$	$-\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	$-\frac{1}{4}$	0
$\tilde{h}[k]$	0	0	$-\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{4}$	0	0
$\tilde{g}[k]$	0	0	$-\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	0	0	0	0

(b) Verify explicitly that the four JPEG 2000 filters satisfy the perfect-reconstruction condition PR-1 (distortion-free).

Solution: Following the definition of the condition (1) for perfect-reconstruction in Slide 8-46, we only need to verify $\tilde{H}(z)H(z) + \tilde{G}(z)G(z) = 2$:

$$\begin{aligned}
\tilde{H}(z)\,H(z) + \tilde{G}(z)\,G(z) &= \frac{1}{4}\left(-z^{-2} + 2\,z^{-1} + 6 + 2\,z - z^2\right)\frac{1}{4}\left(z^{-1} + 2 + z\right) \\
&\quad + \frac{1}{4}\left(-1 + 2\,z - z^2\right)\frac{1}{4}\left(-z^{-3} - 2\,z^{-2} + 6\,z^{-1} - 2 - z\right) \\
&= \frac{1}{16}\left(-z^{-3} - 2\,z^{-2} - z^{-1} + 2\,z^{-2} + 4\,z^{-1} + 2 + 6\,z^{-1} + 12 + 6\,z\right. \\
&\quad \left. + 2 + 4\,z + 2\,z^2 - z - 2\,z^2 - z^3\right) \\
&\quad + \frac{1}{16}\left(z^{-3} + 2\,z^{-2} - 6\,z^{-1} + 2 + z - 2\,z^{-2} - 4\,z^{-1} + 12 - 4\,z - 2\,z^2\right. \\
&\quad \left. + z^{-1} + 2 - 6\,z + 2\,z^2 + z^3\right) \\
&= 2
\end{aligned}$$

(c) Verify explicitly that these four filters satisfy the perfect-reconstruction condition PR-2 (aliasing-free).

Solution: Following the definition of the condition (2) for perfect-reconstruction in Slide 8-46, we only need to verify that $\tilde{H}(-z)H(z) + \tilde{G}(-z)G(z) = 0$:

$$\begin{aligned}
\tilde{H}(-z)\,H(z) + \tilde{G}(-z)\,G(z) &= \frac{1}{4}\left(-z^{-2} - 2\,z^{-1} + 6 - 2\,z - z^2\right)\frac{1}{4}\left(z^{-1} + 2 + z\right) \\
&\quad + \frac{1}{4}\left(-1 - 2\,z - z^2\right)\frac{1}{4}\left(-z^{-3} - 2\,z^{-2} + 6\,z^{-1} - 2 - z\right) \\
&= \frac{1}{16}\left(-z^{-3} - 2\,z^{-2} - z^{-1} - 2\,z^{-2} - 4\,z^{-1} - 2 + 6\,z^{-1} + 12 + 6\,z\right. \\
&\quad \left. - 2 - 4\,z - 2\,z^2 - z - 2\,z^2 - z^3\right) \\
&\quad + \frac{1}{16}\left(z^{-3} + 2\,z^{-2} - 6\,z^{-1} + 2 + z + 2\,z^{-2} + 4\,z^{-1} - 12 + 4\,z + 2\,z^2\right. \\
&\quad \left. + z^{-1} + 2 - 6\,z + 2\,z^2 + z^3\right) \\
&= 0
\end{aligned}$$

(d) Does H satisfy the conjugate-quadrature condition? Justify your answer.

Solution: The conjugate-quadrature condition is $G(z) = -z^{-1}H(-z^{-1})$, here

$$\begin{aligned} -z^{-1}\,H(-z^{-1}) &= -z^{-1}\,\frac{1}{4}\left(-z+2-z^{-1}\right) \\ &= \frac{1}{4}\left(z^{-2}-2\,z^{-1}+1\right) \\ &\neq G(z) \\ &\Rightarrow \text{No} \end{aligned}$$

(e) Given the sequence x below, compute its full wavelet analysis and report intermediate results in a table as indicated.

k	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$x[k]$	0	0	0	0	0	16	-2	-20	14	-32	0	0	0	0	0
$(x * \tilde{h})[k]$	—	—												—	—
$y_1[k] = [x * \tilde{h}]_{\downarrow 2 \uparrow 2}[k]$	—	—												—	—
$(x * \tilde{g})[k]$														—	—
$y_2[k] = [x * \tilde{g}]_{\downarrow 2 \uparrow 2}[k]$														—	—
$(y_1 * h)[k]$	—	—	—										—	—	—
$(y_2 * g)[k]$	—	—	—										—	—	—
$(y_1 * h)[k] + (y_2 * g)[k]$	—	—	—										—	—	—

Solution:

k	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$x[k]$	0	0	0	0	0	16	-2	-20	14	-32	0	0	0	0	0
$(x * \tilde{h})[k]$	—	—	0	-4	$\frac{17}{2}$	28	$-\frac{17}{2}$	-20	$-\frac{9}{2}$	-36	$-\frac{39}{2}$	8	0	—	—
$y_1[k] = [x * \tilde{h}]_{\downarrow 2 \uparrow 2}[k]$	—	—	0	-4	0	28	0	-20	0	-36	0	8	0	—	—
$(x * \tilde{g})[k]$	0	0	0	-4	$\frac{17}{2}$	0	-13	20	$-\frac{39}{2}$	8	0	0	0	—	—
$y_2[k] = [x * \tilde{g}]_{\downarrow 2 \uparrow 2}[k]$	0	0	0	-4	0	0	0	20	0	8	0	0	0	—	—
$(y_1 * h)[k]$	—	—	—	-2	6	14	2	-10	-14	-18	-7	4	—	—	—
$(y_2 * g)[k]$	—	—	—	2	-6	2	-4	-10	28	-14	7	-4	—	—	—
$(y_1 * h)[k] + (y_2 * g)[k]$	—	—	—	0	0	16	-2	-20	14	-32	0	0	—	—	—

3 Wiener Filter

[intermediate] Performing Wiener filtering on a vector signal.

Assume that $\boldsymbol{s}$ is a vector of signals and the goal is to recover them using the MMSE criteria. The signal model is given as $\boldsymbol{y} = A\boldsymbol{s} + \boldsymbol{n}$ where A is a fixed known invertible mixture matrix and $\boldsymbol{n}$ is a Gaussian noise with covariance matrix $C_{\boldsymbol{n}} = \sigma^2 \mathrm{I}$ that is independent from the signal. Our linear estimator is in the form of $\tilde{\boldsymbol{s}} = \alpha A^{-1}\boldsymbol{y}$.

(a) Determine the optimal value of α that minimizes the MMSE loss defined as $\epsilon^2 = E\{\|\tilde{\boldsymbol{s}} - \boldsymbol{s}\|_2^2\}$

Solution:

$$\epsilon^2 = E\{\|\tilde{\boldsymbol{s}} - \boldsymbol{s}\|_2^2\} = E\{\|\alpha A^{-1}\left(A\boldsymbol{s} + \boldsymbol{n}\right) - \boldsymbol{s}\|_2^2\} = E\{\|(\alpha - 1)\boldsymbol{s} + \alpha A^{-1}\boldsymbol{n}\|_2^2\}.$$

The vectors $\boldsymbol{n}$ and $\boldsymbol{s}$ are independent (by assumption). Thus, so are $A^{-1}\boldsymbol{n}$ and $\boldsymbol{s}$. Hence,

$$\epsilon^2 = (\alpha - 1)^2 E\{\|\boldsymbol{s}\|_2^2\} + \alpha^2 E\{\|A^{-1}\boldsymbol{n}\|_2^2\}.$$

By taking the derivative with respect to α and setting it to zero, we obtain the optimal value of α as

$$\alpha = \frac{E\{\|\boldsymbol{s}\|_2^2\}}{E\{\|\boldsymbol{s}\|_2^2\} + E\{\|A^{-1}\boldsymbol{n}\|_2^2\}}$$

(b) Explain your result qualitatively: how should α change as the norm of the signal and noise change? What role does A play?

Solution: When the norm of the signal increases, α should increase towards a maximum of one. When the norm of $A^{-1}\boldsymbol{n}$ increases, α should decrease towards a minimum of zero. The matrix A^{-1} mixes the noise: what matters is $\|A^{-1}\boldsymbol{n}\|_2^2$, not $\|\boldsymbol{n}\|_2^2$.

(c) Show that if A is a unitary matrix, then α coincides with the formula given in Slide 8-62.

Solution: We have

$$\|A^{-1}\boldsymbol{n}\|_2^2 = (A^{-1}\boldsymbol{n})^T A^{-1}\boldsymbol{n} = \boldsymbol{n}^T A^{-T} A^{-1}\boldsymbol{n} = \boldsymbol{n}^T\boldsymbol{n} = \|\boldsymbol{n}\|_2^2. \tag{1}$$

Plugging in $\|A^{-1}\boldsymbol{n}\|_2^2 = \|\boldsymbol{n}\|_2^2$, we have

$$\alpha = \frac{E\{\|\boldsymbol{s}\|_2^2\}}{E\{\|\boldsymbol{s}\|_2^2\} + E\{\|\boldsymbol{n}\|_2^2\}}.$$