Image Processing 2, Exercise 5

1 The DFT matrices

Specific examples of DFT matrices.

(a) A periodic sequence is given by its length-2 period $x = \left[\boxed{x[0]}\ \ x[1]\right]$, with $x[0], x[1] \in \mathbb{C}$. Knowing that the discrete Fourier transform is an orthonormal transform that maps x to y, give the periodic sequence (with length-2 period) as $y = \left[\boxed{y[0]}\ \ y[1]\right]$. Express $y[0]$ and $y[1]$ in terms of $x[0]$ and $x[1]$.

Solution: Use the definition of DFT we have

$$\begin{aligned} y[n] &= \sum_{k=0}^{N-1} \left(\frac{1}{\sqrt{N}} \mathrm{e}^{-\mathrm{j}n\frac{2\pi}{N}k} \right) x[k] \\ &\overset{\mathrm{N=2}}{=} \frac{1}{\sqrt{2}} \sum_{k=0}^{2-1} x[k]\mathrm{e}^{-\mathrm{j}n\pi k} \\ &= \frac{1}{\sqrt{2}} x[0] + \frac{1}{\sqrt{2}} x[1]\mathrm{e}^{-\mathrm{j}n\pi}. \end{aligned}$$

Hence,

$$\underbrace{\begin{bmatrix} y\,[0] \\ y\,[1] \end{bmatrix}}_{y_{\mathrm{col}}} = \underbrace{\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}}_{\mathbf{A}_{\mathrm{col}}} \underbrace{\begin{bmatrix} x\,[0] \\ x\,[1] \end{bmatrix}}_{x_{\mathrm{col}}}$$

(b) A periodic sequence is given by its length-4 period $x = \left[\boxed{x[0]}\ \ x[1]\ \ x[2]\ \ x[3]\right]$. Its discrete Fourier transform is $y = \left[y[-1]\ \ \boxed{y[0]}\ \ y[1]\ \ y[2]\right]$. Express $y[-1]$, $y[0]$, $y[1]$, and $y[2]$ in explicit terms of the samples of x.

Solution: Similar to 1(a), we have

$$\begin{aligned} y[n] &= \frac{1}{2} \sum_{k=0}^{4-1} x[k]\,\mathrm{e}^{-\mathrm{j}\,n\,\frac{2\pi}{4}\,k} \\ &= \frac{1}{2}\,x[0] + \frac{1}{2}\,x[1]\,\mathrm{e}^{-\mathrm{j}\,n\,\frac{\pi}{2}} + \frac{1}{2}\,x[2]\,\mathrm{e}^{-\mathrm{j}\,n\,\pi} + \frac{1}{2}\,x[3]\,\mathrm{e}^{-\mathrm{j}\,n\,\frac{3\pi}{2}}. \end{aligned}$$

Hence,

$$\underbrace{\begin{bmatrix} y\,[-1] \\ y\,[0] \\ y\,[1] \\ y\,[2] \end{bmatrix}}_{y_{\mathrm{row}}} = \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{\mathrm{j}}{2} & -\frac{1}{2} & -\frac{\mathrm{j}}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\mathrm{j}}{2} & -\frac{1}{2} & \frac{\mathrm{j}}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}}_{\mathbf{A}_{\mathrm{row}}} \underbrace{\begin{bmatrix} x\,[0] \\ x\,[1] \\ x\,[2] \\ x\,[3] \end{bmatrix}}_{x_{\mathrm{row}}}$$

(c) Let a periodic image be characterized by the (4×2) rectangular tile

$$x = \begin{bmatrix} \boxed{x[0,0]} & x[1,0] & x[2,0] & x[3,0] \\ x[0,1] & x[1,1] & x[2,1] & x[3,1] \end{bmatrix}.$$

Give the vector representation $\mathbf{x}$ of the tile, according to the image-processing conventions detailed in the Table 1 which is a complete and commented version of the table on pp. 8-13 of your course notes.

Solution:

$$\mathbf{x} = \begin{bmatrix} \boxed{x[0,0]} & x[1,0] & x[2,0] & x[3,0] & x[0,1] & x[1,1] & x[2,1] & x[3,1] \end{bmatrix}^{\mathrm{T}}.$$

(d) Now, we want to compute the discrete Fourier transform of the image. Give the matrix $\mathbf{A}$ such that $(\mathbf{A}\,\mathbf{x})$ corresponds to the vector representation $\mathbf{y}$ of the tile

$$y = \begin{bmatrix} y[-1,0] & \boxed{y[0,0]} & y[1,0] & y[2,0] \\ y[-1,1] & y[0,1] & y[1,1] & y[2,1] \end{bmatrix}.$$

Hint: Take advantage of the separability of the Fourier transform and proceed in two independent steps using the results from 1(a) and 1(b).

Solution: When applying DFT on a 2D image, thanks to the separability of FT, we can apply DFT first on the columns of this 2D image and then on the rows (or first on rows then on columns) (see lecture slides 8-31).
This means that matrix $\mathbf{A}$ can be decomposed as a multiplication of two matrices $\mathbf{A}_{\text{horizontal}}$ and $\mathbf{A}_{\text{vertical}}$. $\mathbf{A}_{\text{horizontal}}$ ($\mathbf{A}_{\text{vertical}}$) applies DFT on the row (column) vectors of the 2D image.

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \mathbf{A}_{\text{row}}\mathbf{A}_{\text{col}}\mathbf{x}.$$

From 1(b) we have

$$y_{\text{row}} = \mathbf{A}_{\text{row}} x_{\text{row}}. \tag{1}$$

From 1(a) we have

$$y_{\text{col}} = \mathbf{A}_{\text{col}} x_{\text{col}}. \tag{2}$$

The first row of the image y can be obtained by applying the transformation in (1) to the first row of x. The same applies for the second row of y and the second row of x.

Hence, $\mathbf{A}_{\text{horizontal}} = \begin{bmatrix} \mathbf{A}_{\text{row}} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}_{\text{row}} \end{bmatrix}$, where $\mathbf{O}$ is a 4×4 zero matrix.

Similarly, the columns of y can be obtained by applying the transformation in (2) to the corresponding columns of x.

Hence, $A_{\text{vertical}} = \begin{bmatrix} \mathbf{A}_{\text{col}}[1,1]\mathbf{I} & \mathbf{A}_{\text{col}}[1,2]\mathbf{I} \\ \mathbf{A}_{\text{col}}[2,1]\mathbf{I} & \mathbf{A}_{\text{col}}[2,2]\mathbf{I} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}\mathbf{I} & \frac{\sqrt{2}}{2}\mathbf{I} \\ \frac{\sqrt{2}}{2}\mathbf{I} & -\frac{\sqrt{2}}{2}\mathbf{I} \end{bmatrix}$, where $\mathbf{I}$ is a 4×4 identity matrix.

Precisely,

$$\mathbf{A}_{\text{horizontal}} = \begin{pmatrix} \frac{1}{2} & \frac{\mathrm{j}}{2} & -\frac{1}{2} & -\frac{\mathrm{j}}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{\mathrm{j}}{2} & -\frac{1}{2} & \frac{\mathrm{j}}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{\mathrm{j}}{2} & -\frac{1}{2} & -\frac{\mathrm{j}}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{\mathrm{j}}{2} & -\frac{1}{2} & \frac{\mathrm{j}}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix},$$

$$\mathbf{A}_{\text{vertical}} = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} \end{pmatrix}.$$

$$\mathbf{A} = \mathbf{A}_{\text{horizontal}}\mathbf{A}_{\text{vertical}} = \frac{\sqrt{2}}{4} \begin{pmatrix} 1 & \mathrm{j} & -1 & -\mathrm{j} & 1 & \mathrm{j} & -1 & -\mathrm{j} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -\mathrm{j} & -1 & \mathrm{j} & 1 & -\mathrm{j} & -1 & \mathrm{j} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \mathrm{j} & -1 & -\mathrm{j} & -1 & -\mathrm{j} & 1 & \mathrm{j} \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -\mathrm{j} & -1 & \mathrm{j} & -1 & \mathrm{j} & 1 & -\mathrm{j} \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{pmatrix}.$$

(e) Give the basis $\mathbf{U}$ associated to matrix $\mathbf{A}$.

Solution: Because DFT is a linear transform,

$$\mathbf{U} = \mathbf{A}^{-1} \overset{\text{orthonormality}}{=} \mathbf{A}^{\mathsf{H}} = \frac{\sqrt{2}}{4} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -\mathrm{j} & 1 & \mathrm{j} & -1 & -\mathrm{j} & 1 & \mathrm{j} & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ \mathrm{j} & 1 & -\mathrm{j} & -1 & \mathrm{j} & 1 & -\mathrm{j} & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -\mathrm{j} & 1 & \mathrm{j} & -1 & \mathrm{j} & -1 & -\mathrm{j} & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ \mathrm{j} & 1 & -\mathrm{j} & -1 & -\mathrm{j} & -1 & \mathrm{j} & 1 \end{pmatrix}.$$

(f) Is $\mathbf{U}$ symmetric? Hermitian? Toeplitz? circulant? Motivate your answers.

Solution:

$$\left([\mathbf{U}]_{1,2} = 1 \neq -\mathrm{j} = \left[\mathbf{U}^{\mathsf{T}}\right]_{1,2}\right) \Rightarrow (\neg\text{symmetric})$$

$$\left([\mathbf{U}]_{1,2} = 1 \neq \mathrm{j} = \left[\mathbf{U}^{\mathsf{H}}\right]_{1,2}\right) \Rightarrow (\neg\text{Hermitian})$$

$$\left([\mathbf{U}]_{1,2} = 1 \neq \mathrm{j} = [\mathbf{U}]_{2,3}\right) \Rightarrow (\neg\text{Toeplitz})$$

$$(\neg\text{Toeplitz}) \Rightarrow (\neg\text{circulant})$$

2 DFT properties

[basic] Proving simple properties of the DFT.

(a) Show that $\sum_{k=0}^{N-1} \mathrm{e}^{-\mathrm{j}\, n\, \frac{2\pi}{N}\, k} = N \sum_{m\in\mathbb{Z}} \delta[n - m\, N]$ for $n \in \mathbb{Z}$, $N \in \mathbb{N}\backslash\{0\}$. Suggestion: Pay attention to distinguish Dirac from unit sample.

Solution:

	2-D Arrays ($K_1 \times K_2$)	Rasterization	Vector Components	Vectors
Array Elements	$x[k_1, k_2]$	$x[i] = x[k_2\, K_1 + k_1]$	$[\mathbf{x}]_{i+1}$	$\mathbf{x}$
Basis Functions	$\varphi_{l,k}[k_1, k_2]$	$\varphi_j[i] = \varphi_{(l-1)\, K_1 + k}[i]$	$[\mathbf{U}]_{i+1,j}$	$[\mathbf{U}]_{\cdot,j} = \mathbf{u}_j$
Reconstruction	$x[k_1, k_2] = \sum_{l=1}^{K_2} \sum_{k=1}^{K_1} \varphi_{l,k}[k_1, k_2]\, y_{l,k}$	$x[i] = \sum_{j=1}^{K_1\, K_2} \varphi_j[i]\, y_j$	$[\mathbf{x}]_{i+1} = \sum_{j=1}^{K_1\, K_2} [\mathbf{U}]_{i+1,j}\, [\mathbf{y}]_j$	$\mathbf{x} = \mathbf{U}\, \mathbf{y} = \mathbf{A}^{-1}\, \mathbf{y}$
Inner Products	$x[k_1, k_2] = \langle \varphi_\cdot^*[k_1, k_2], y_\cdot \rangle_{\ell_2^2}$	$x[i] = \langle \varphi_\cdot^*[i], y_\cdot \rangle_{\ell_2}$	$[\mathbf{x}]_{i+1} = \left\langle \left([\mathbf{U}]_{i+1,\cdot}\right)^{\mathsf{H}}, \mathbf{y} \right\rangle$	$\mathbf{x} = \mathbf{U}\, \mathbf{y} = \mathbf{A}^{-1}\, \mathbf{y}$
Coefficients	$y_{l,k}$	$y_j = y[j-1]$	$[\mathbf{y}]_j$	$\mathbf{y}$
Analysis Functions	$\tilde{\varphi}_{l,k}[k_1, k_2]$	$\tilde{\varphi}_j[i]$	$\left[\widetilde{\mathbf{U}}^{\mathsf{T}}\right]_{j,i+1}$	$\left[\widetilde{\mathbf{U}}\right]_{\cdot,j} = \tilde{\mathbf{u}}_j$
Analysis	$y_{l,k} = \sum_{k_2=0}^{K_2-1} \sum_{k_1=0}^{K_1-1} \tilde{\varphi}_{l,k}^*[k_1, k_2]\, x[k_1, k_2]$	$y_j = \sum_{i=0}^{K_1\, K_2-1} \tilde{\varphi}_j^*[i]\, x[i]$	$[\mathbf{y}]_j = \sum_{i=0}^{K_1\, K_2-1} \left[\widetilde{\mathbf{U}}^{\mathsf{H}}\right]_{j,i+1} [\mathbf{x}]_{i+1}$	$\mathbf{y} = \widetilde{\mathbf{U}}^{\mathsf{H}}\, \mathbf{x} = \mathbf{A}\, \mathbf{x}$
Transform Inner Products	$y_{l,k} = \langle \tilde{\varphi}_{l,k}[\cdot], x[\cdot] \rangle_{\ell_2^2}$	$y_j = \langle \tilde{\varphi}_j[\cdot], x[\cdot] \rangle_{\ell_2}$	$[\mathbf{y}]_j = \left\langle [\tilde{\mathbf{u}}]_j, \mathbf{x} \right\rangle$	$\mathbf{y} = \widetilde{\mathbf{U}}^{\mathsf{H}}\, \mathbf{x} = \mathbf{A}\, \mathbf{x}$
Invertibility	$\langle \tilde{\varphi}_{l',k'}[\cdot], \varphi_{l'',k''}[\cdot] \rangle_{\ell_2^2} = \delta[k'-k'', l'-l'']$	$\langle \tilde{\varphi}_{j'}[\cdot], \varphi_{j''}[\cdot] \rangle_{\ell_2} = \delta[j'-j'']$		$\widetilde{\mathbf{U}}^{\mathsf{H}}\, \mathbf{U} = \mathbf{I}$
Orthonormality	$\tilde{\varphi}_{l,k} = \varphi_{l,k}$	$\tilde{\varphi}_j = \varphi_j$		$\widetilde{\mathbf{U}} = \mathbf{U}$

Table 1: Let $\mathbf{X}$ be some $(L \times K)$ matrix with L rows and K columns. When writing $x_{l,k} = [\mathbf{X}]_{l,k}$, the ordinal index l indicates the lth line of the matrix $\mathbf{X}$ and k its kth column. Because l and k are ordinals and not cardinals, neither can take the value 0. The "first" element of $\mathbf{X}$ is therefore $x_{1,1}$. Now, let x be some $(W \times H)$ array of width W and height H—in other words, the array has H rows and W columns. When writing $x[k_1, k_2]$, the cardinal variable k_1 indicates the first component of the argument $\mathbf{k}$. It is associated with $\mathbf{e}_1$ which, in an image-processing context, is made to correspond to the horizontal orientation. Contrarily to matrices, the origin of the array x is therefore $(0, 0)$. Important: if someone ever tries to store a two-dimensional array x in a matrix $\mathbf{X}$ (which would be a very bad idea), confusion would arise because one would have to write $x[p, q] = [\mathbf{X}]_{q+1,p+1}$. Observe that not only the indices are off-by-one, but, in addition, they do not come in the same order; likewise, the size would have to be indicated by $(X_1 \times X_2)$ in the array context, and the *same* size would be written $(X_2 \times X_1)$ in the matrix context. Instead, a much better formulation is to store the array x in a *vector* $\mathbf{x}$. An added benefit is dimensionality independence. By notational convention, $\mathbf{v} = (v_1, v_2) = [v_1\ \ v_2]^{\mathsf{T}}$ is a column vector, while $\mathbf{v}^{\mathsf{T}} = (v_1, v_2)^{\mathsf{T}} = [v_1\ \ v_2]$ is a line vect or—observe the presence or absence of commas, and the use of parentheses or brackets. Also, note that the absence of indication of the origin distinguishes between the line vector $[v_1\ \ v_2]$ and the sequence $[\boxed{v_1}\ \ v_2]$

$$n = m\,N : \sum_{k=0}^{N-1} \mathrm{e}^{-\mathrm{j}\,n\,\frac{2\,\pi}{N}\,k} = \sum_{k=0}^{N-1} \underbrace{\mathrm{e}^{-\mathrm{j}\,m\,2\,\pi\,k}}_{1} = N$$

$$\begin{aligned}
n \neq m\,N : \sum_{k=0}^{N-1} \mathrm{e}^{-\mathrm{j}\,n\,\frac{2\,\pi}{N}\,k} &= \sum_{k=0}^{N-1} \left(\mathrm{e}^{-\mathrm{j}\,n\,\frac{2\,\pi}{N}}\right)^{k} \\
&= \frac{1-\mathrm{e}^{-\mathrm{j}\,n\,\frac{2\,\pi}{N}\,N}}{1-\mathrm{e}^{-\mathrm{j}\,n\,\frac{2\,\pi}{N}}} \quad \text{(Sum of a geometric sequence)} \\
&= 0
\end{aligned}$$

$$\Rightarrow \left(\sum_{k=0}^{N-1} \mathrm{e}^{-\mathrm{j}\,n\,\frac{2\,\pi}{N}\,k} = N \sum_{m\in\mathbb{Z}} \delta[n-m\,N]\right)$$

(b) For $n \in \mathbb{Z}$, show that $\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x[k]\,\mathrm{e}^{-\mathrm{j}\,n\,\frac{2\,\pi}{N}\,k} = [\mathbf{F}_N\,\mathbf{x}]_{1+(n \bmod N)}$. (The mathematical definition of the mod operator is such that $(p \bmod q) \in [0\ldots q-1]$ for $p \in \mathbb{Z}, q \in \mathbb{N}\backslash\{0\}$.)

Solution:

$$\begin{aligned}
[\mathbf{F}_N\mathbf{x}]_n &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathbf{x}[k]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n-1)k} \quad n = 1, \cdots, N \\
[\mathbf{F}_N\mathbf{x}]_{1+(n \bmod N)} &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathbf{x}[k]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(1+(n \bmod N)-1)k} \\
&= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathbf{x}[k]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n \bmod N)k}.
\end{aligned}$$

Since $n \bmod N = n - N\left\lfloor \frac{n}{N} \right\rfloor$, we have

$$\begin{aligned}
[\mathbf{F}_N\mathbf{x}]_{1+(n \bmod N)} &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathbf{x}[k]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}\left(n-N\left\lfloor \frac{n}{N} \right\rfloor\right)k} \\
&= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathbf{x}[k]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk} \cdot \underbrace{\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}\left\lfloor \frac{n}{N} \right\rfloor k}}_{=1,\ \text{since}\ \left\lfloor \frac{n}{N} \right\rfloor \in \mathbb{Z}} \\
&= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathbf{x}[k]\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}nk}.
\end{aligned}$$

(c) Given $\mathbf{x} \in \mathbb{R}^N$, define $\mathbf{x}^{\mathrm{s}} \in \mathbb{R}^N$ such that $[\mathbf{x}^{\mathrm{s}}]_i = [\mathbf{x}]_{1+(1-i) \bmod N}$. Then, show that $\mathbf{F}_N^*\,\mathbf{x} = \mathbf{F}_N\,\mathbf{x}^{\mathrm{s}}$.

Solution:

$$
\begin{aligned}
[\mathbf{F}_N \mathbf{x}^{\mathsf{s}}]_n &= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} [\mathbf{x}^{\mathsf{s}}]_i \, \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n-1)(i-1)} \\
&= \frac{1}{\sqrt{N}} \sum_{i=1}^{N} [\mathbf{x}]_{1+(1-i) \bmod N} \, \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n-1)(i-1)}
\end{aligned}
$$

Substitute $i = N + 2 - k$,

$$
\begin{aligned}
[\mathbf{F}_N \mathbf{x}^{\mathsf{s}}]_n &= \frac{1}{\sqrt{N}} \sum_{k=2}^{N+1} [\mathbf{x}]_{(1+(1-(N+2-k)) \bmod N)} \, \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n-1)(N+2-k-1)} \\
&= \frac{1}{\sqrt{N}} \sum_{k=2}^{N+1} [\mathbf{x}]_{(1+(k-N-1) \bmod N)} \, \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n-1)(N-k+1)}
\end{aligned}
$$

Since $(k - N - 1) \bmod N = (k - 1) \bmod N$,

$$
\begin{aligned}
\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n-1)(N-k+1)} &= \mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n-1)(1-k)} \cdot \underbrace{\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}(n-1)N}}_{1} \\
&= \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}(n-1)(k-1)}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
[\mathbf{F}_N \mathbf{x}^{\mathsf{s}}]_n &= \frac{1}{\sqrt{N}} \sum_{k=2}^{N+1} \underbrace{[\mathbf{x}]_{1+(k-1) \bmod N} \cdot \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}(n-1)(k-1)}}_{\mathbf{A}_k} \\
[\mathbf{F}_N \mathbf{x}^{\mathsf{s}}]_n &= \frac{1}{\sqrt{N}} \sum_{k=2}^{N+1} \mathbf{A}_k = \frac{1}{\sqrt{N}} \left(-\mathbf{A}_1 + \sum_{k=1}^{N} \mathbf{A}_k + \mathbf{A}_{N+1} \right)
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{A}_1 = [\mathbf{x}]_1 \cdot \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}(n-1)(0)} = 1, \\
&\mathbf{A}_{N+1} = [\mathbf{x}]_1 \cdot \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}(n-1)(N)} = 1.
\end{aligned}
$$

For $1 \leq k \leq N$: $1 + (k - 1) \bmod N = 1 + (k - 1) = k$, therefore,

$$
\begin{aligned}
[\mathbf{F}_N \mathbf{x}^{\mathsf{s}}]_n &= \frac{1}{\sqrt{N}} \sum_{k=1}^{N} [\mathbf{x}]_k \, \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}(n-1)(k-1)} \\
&= [\mathbf{F}_N^* \, \mathbf{x}]_n \, .
\end{aligned}
$$

3 Multirate operations

[basic] The goal of this exercise is to study the combination of linear systems with down/up sampling operators

(a) Develop the operation $-\!\!\left(\uparrow 2\right)\!\!-\!\!\left(\downarrow 2\right)\!\!-$ in the z domain.

Solution:
Let $X(z)$ be the input, $U(z)$ the output after upsampling, $Y(z)$ the output after downsampling.

$$U(z) = X(z^2)$$

$$\begin{aligned}
Y(z) &= \frac{1}{2}\left(U\left(z^{\frac{1}{2}}\right) + U\left(-z^{\frac{1}{2}}\right)\right) \\
&= \frac{1}{2}\left(X\left(\left(z^{\frac{1}{2}}\right)^2\right) + X\left(\left(-z^{\frac{1}{2}}\right)^2\right)\right) \\
&= \frac{1}{2}\left(X\left(z\right) + X\left(z\right)\right) \\
&= X(z)
\end{aligned}$$

(b) Show that $-\!\!\!\!\bigcirc\!\!\!\!\downarrow 2 - \boxed{H(z)} - \quad \Leftrightarrow \quad - \boxed{H(z^2)} -\!\!\!\!\bigcirc\!\!\!\!\downarrow 2 -$.

Solution:
LHS: Let $X(z)$ be the input, $U'(z)$ the output after downsampling, $Y'(z)$ the output after filtering.
RHS: Let $X(z)$ be the input, $Y''(z)$ the output after downsampling, $U''(z)$ the output after filtering.
LHS:

$$U'(z) = \frac{1}{2}\left(X\left(z^{\frac{1}{2}}\right) + X\left(-z^{\frac{1}{2}}\right)\right)$$

$$\begin{aligned}
Y'(z) &= H(z)\,U'(z) \\
&= H(z)\,\frac{1}{2}\left(X\left(z^{\frac{1}{2}}\right) + X\left(-z^{\frac{1}{2}}\right)\right)
\end{aligned}$$

RHS:

$$U''(z) = H(z^2)\,X(z)$$

$$\begin{aligned}
Y''(z) &= \frac{1}{2}\left(U''\left(z^{\frac{1}{2}}\right) + U''\left(-z^{\frac{1}{2}}\right)\right) \\
&= \frac{1}{2}\left(H\left(\left(z^{\frac{1}{2}}\right)^2\right) X\left(z^{\frac{1}{2}}\right) + H\left(\left(-z^{\frac{1}{2}}\right)^2\right) X\left(-z^{\frac{1}{2}}\right)\right) \\
&= \frac{1}{2}\left(H(z)\,X\left(z^{\frac{1}{2}}\right) + H(z)\,X\left(-z^{\frac{1}{2}}\right)\right) \\
&= Y'(z)
\end{aligned}$$

(c) Show that $-\!\!\!\!\bigcirc\!\!\!\!\uparrow 2 - \boxed{H(z^2)} - \quad \Leftrightarrow \quad - \boxed{H(z)} -\!\!\!\!\bigcirc\!\!\!\!\uparrow 2 -$.

Solution:
LHS: Let $X(z)$ be the input, $U'(z)$ the output after upsampling. $Y'(z)$ the output after filtering.
RHS: Let $X(z)$ be the input, $Y''(z)$ the output after upsampling. $U''(z)$ the output after filtering.
LHS:

$$U'(z) = X(z^2)$$

$$
\begin{aligned}
Y'(z) &= H(z^2)\, U'(z) \\
&= H(z^2)\, X(z^2)
\end{aligned}
$$

RHS:

$$U''(z) = H(z)\, X(z)$$

$$
\begin{aligned}
Y''(z) &= U''(z^2) \\
&= H(z^2)\, X(z^2) \\
&= Y'(z)
\end{aligned}
$$