Image Processing 2, Exercise 4

1 Bases and Frames

[refresher] In this problem, we build a geometric intuition for bases and frames, which are key concepts in understanding the transforms used in image processing.

Let the signal $\boldsymbol{x} = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$ be a vector in $\mathbb{R}^2$ and consider the vectors $\boldsymbol{u}_1 = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $\boldsymbol{u}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$.

(a) Plot $\boldsymbol{x}$, $\boldsymbol{u}_1$, and $\boldsymbol{u}_2$.

(b) Add to your plot two vectors, $a_1\boldsymbol{u}_1$ and $a_2\boldsymbol{u}_2$ such that $a_1\boldsymbol{u}_1 + a_2\boldsymbol{u}_2 = \boldsymbol{x}$. You should not need to explicitly compute a_1 and a_2!

(c) Is $\boldsymbol{U} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix}$ a basis? Justify your answer.

(d) Is $\boldsymbol{U}$ an orthonormal basis? Justify your answer.

(e) Give the dual basis, $\tilde{\boldsymbol{U}}$.

(f) Using $\tilde{\boldsymbol{U}}$, compute $\boldsymbol{a}$ such that $\boldsymbol{U}\boldsymbol{a} = \boldsymbol{x}$. Do these values make sense considering the picture you drew in (b)?

(g) Pick a $\boldsymbol{u}_3$ different from $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ and make two new plots graphically expressing $\boldsymbol{x}$ as $a_1\boldsymbol{u}_1 + a_2\boldsymbol{u}_2 + a_3\boldsymbol{u}_3$ in two different ways. Which property of frames does this demonstrate?

2 Karhunen-Loeve Transform

[intermediate] A concrete example of computing the KLT.

Consider two Gaussian independent random variables x_1 and x_2 with zero mean and unit variance. Define the random variables $y_1 = \frac{3x_1 - x_2}{2}$ and $y_2 = \frac{-x_1 + 3x_2}{2}$.

(a) Let $\boldsymbol{x} = (x_1, x_2)$. Compute the covariance matrix of $\boldsymbol{x}$.

(b) Identify the matrix $\boldsymbol{A}$ such that $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, where $\boldsymbol{y} = (y_1, y_2)$.

(c) Compute the covariance matrix of $\boldsymbol{y}$.

(d) Compute the KLT of $\boldsymbol{y}$, $\boldsymbol{A}_{\mathrm{KLT}}$. Give $\boldsymbol{z} = \boldsymbol{A}_{\mathrm{KLT}}\boldsymbol{y}$ in terms of $\boldsymbol{x}$.

(e) Compute the covariance matrix of $\boldsymbol{z}$.

(f) Based on your analysis, is there a unique representation of $\boldsymbol{y}$ of the form $\boldsymbol{B}\boldsymbol{q}$, where q_1 and q_2 are decorrelated?

3 LSI Systems and Circulant Matrices

[intermediate] Because the DFT diagonalizes circulant matrices, it can be an efficient way to implement digital LSI systems. Here, we explore the connection between LSI systems, circulant matrices, and the DFT.

(a) Give the result of the application of an LSI system with impulse response $h = \begin{bmatrix} \dots & 0 & \boxed{a} & b & c & 0 & \dots \end{bmatrix}$ to the signal $\begin{bmatrix} \dots & 0 & \boxed{x} & y & z & 0 & \dots \end{bmatrix}$.

(b) Write the nonzero part of the computation from part (a) as a matrix-vector product, i.e.

$$\begin{bmatrix} \quad & \quad & \quad \\ \\ \\ \\ \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \\ \\ \end{bmatrix}.$$

Note whether the matrix is Toeplitz, circulant, or neither.

(c) Give the result of the application of an LSI system with impulse response $h = \begin{bmatrix} \boxed{a} & b & c \end{bmatrix}$ to the signal $\begin{bmatrix} \boxed{x} & y & z \end{bmatrix}$, this time assuming periodic boundary conditions. Is the result the same as the nonzero part of part (a)?

(d) Write the computation in part (c) in the form of a matrix-vector product. Note whether the matrix is Toeplitz, circulant, or neither.

(e) Find the 5×5 circulant matrix, $\boldsymbol{H}$, such that $\boldsymbol{H} \begin{bmatrix} \boxed{x} & y & z & 0 & 0 \end{bmatrix}^T$ matches the nonzero part of the result from part (a).