Image Processing 2, Exercise 4

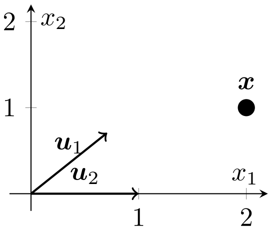
1 Bases and Frames

[refresher] In this problem, we build a geometric intuition for bases and frames, which are key concepts in understanding the transforms used in image processing.

Let the signal $\boldsymbol{x} = \begin{bmatrix}2 & 1\end{bmatrix}^T$ be a vector in $\mathbb{R}^2$ and consider the vectors $\boldsymbol{u}_1 = \frac{\sqrt{2}}{2}\begin{bmatrix}1 & 1\end{bmatrix}^T$ and $\boldsymbol{u}_2 = \begin{bmatrix}1 & 0\end{bmatrix}^T$.

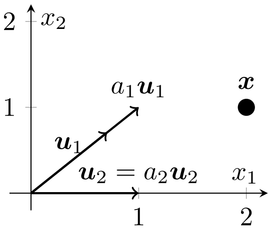
(a) Plot $\boldsymbol{x}$, $\boldsymbol{u}_1$, and $\boldsymbol{u}_2$.

Solution:



(b) Add to your plot two vectors, $a_1\boldsymbol{u}_1$ and $a_2\boldsymbol{u}_2$ such that $a_1\boldsymbol{u}_1 + a_2\boldsymbol{u}_2 = \boldsymbol{x}$. You should not need to explicitly compute a_1 and a_2!

Solution:



(c) Is $\boldsymbol{U} = \begin{bmatrix}\boldsymbol{u}_1 & \boldsymbol{u}_2\end{bmatrix}$ a basis? Justify your answer.

Solution: Yes, because $\det(\boldsymbol{U}) = -\frac{\sqrt{2}}{2} \neq 0$, meaning that $\boldsymbol{U}$ is invertible, which implies that it is a basis.

(d) Is $\boldsymbol{U}$ an orthonormal basis? Justify your answer.

Solution: No. An orthonormal basis is a basis whose vectors have unit norms and are orthogonal to each other. $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ have unit norms but they are not orthogonal because $\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle = \frac{\sqrt{2}}{2} \neq 0$.

(e) Give the dual basis, $\tilde{\boldsymbol{U}}$.

Solution: In lecture notes 8-10, the dual basis $\tilde{\boldsymbol{U}}$ of a basis $\boldsymbol{U}$ is defined as

$$\tilde{\boldsymbol{U}} = (\boldsymbol{U}^{-1})^H.$$

$$\boldsymbol{U} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 1 \\ \frac{\sqrt{2}}{2} & 0 \end{bmatrix}, \quad \boldsymbol{U}^{-1} = -\sqrt{2}\begin{bmatrix} 0 & -1 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{2} \\ 1 & -1 \end{bmatrix},$$

$$(\boldsymbol{U}^{-1})^H = (\boldsymbol{U}^{-1})^T = \begin{bmatrix} 0 & 1 \\ \sqrt{2} & -1 \end{bmatrix}.$$

(f) Using $\tilde{\boldsymbol{U}}$, compute $\boldsymbol{a}$ such that $\boldsymbol{U}\boldsymbol{a} = \boldsymbol{x}$. Do these values make sense considering the picture you drew in (b)?

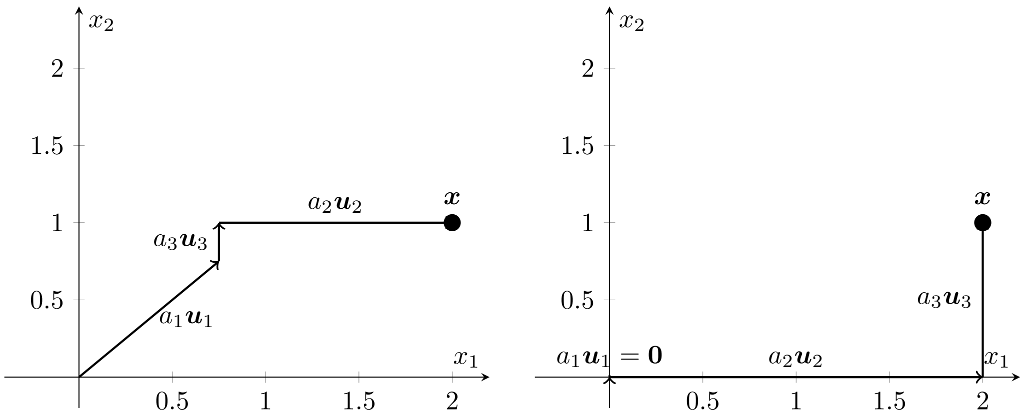
Solution:

$$\boldsymbol{U}\boldsymbol{a} = \boldsymbol{x} \quad \Rightarrow \quad \boldsymbol{a} = \boldsymbol{U}^{-1}\boldsymbol{x} = \tilde{\boldsymbol{U}}^H\boldsymbol{x} = \begin{bmatrix} 0 & \sqrt{2} \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}.$$

Yes, the value of $\boldsymbol{a}$ makes sense.

(g) Pick a $\boldsymbol{u}_3$ different from $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ and make two new plots graphically expressing $\boldsymbol{x}$ as $a_1\boldsymbol{u}_1 + a_2\boldsymbol{u}_2 + a_3\boldsymbol{u}_3$ in two different ways. Which property of frames does this demonstrate?

Solution:



This demonstrates the nonuniqueness of the dual frame.

2 Karhunen-Loeve Transform

[intermediate] A concrete example of computing the KLT.

Consider two Gaussian independent random variables x_1 and x_2 with zero mean and unit variance. Define the random variables $y_1 = \frac{3x_1 - x_2}{2}$ and $y_2 = \frac{-x_1 + 3x_2}{2}$.

(a) Let $\boldsymbol{x} = (x_1, x_2)$. Compute the covariance matrix of $\boldsymbol{x}$.

Solution:

$$\begin{aligned} \boldsymbol{C}_{\boldsymbol{x}} &= E\{(\boldsymbol{x} - E\{\boldsymbol{x}\})(\boldsymbol{x} - E\{\boldsymbol{x}\})^T\} \\ &= \begin{bmatrix} E\{(x_1 - E\{x_1\})^2\} & E\{(x_1 - E\{x_1\})(x_2 - E\{x_2\})\} \\ E\{(x_1 - E\{x_1\})(x_2 - E\{x_2\})\} & E\{(x_2 - E\{x_2\})^2\} \end{bmatrix} \\ &= \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

where $\sigma^2 = 1$ because x_1 and x_2 have unit variance, and the covariances are zero because they are independent.

(b) Identify the matrix $\boldsymbol{A}$ such that $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, where $\boldsymbol{y} = (y_1, y_2)$.

Solution:

$$\boldsymbol{A} = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}.$$

(c) Compute the covariance matrix of $\boldsymbol{y}$.

Solution:

$$\boldsymbol{C}_{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{C}_{\boldsymbol{x}}\boldsymbol{A}^T = \boldsymbol{A}\boldsymbol{A}^T = \begin{bmatrix} \frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{5}{2} \end{bmatrix}.$$

(d) Compute the KLT of $\boldsymbol{y}$, $\boldsymbol{A}_{\mathrm{KLT}}$. Give $\boldsymbol{z} = \boldsymbol{A}_{\mathrm{KLT}}\boldsymbol{y}$ in terms of $\boldsymbol{x}$.

Solution: First, we need to compute the eigenvalues of $C_{\boldsymbol{y}}$. To do so, we need to solve the equation

$$\det(\boldsymbol{C}_{\boldsymbol{y}} - \lambda \boldsymbol{I}) = 0.$$

It has two solutions, $\lambda_1 = 4$ and $\lambda_2 = 1$. Their corresponding normalized eigenvectors are $\boldsymbol{u}_1 = \frac{\sqrt{2}}{2}(1, -1)^T$ and $\boldsymbol{u}_2 = \frac{\sqrt{2}}{2}(1, 1)^T$. Hence, we have the decomposition

$$\boldsymbol{C}_{\boldsymbol{y}} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1^T \\ \boldsymbol{u}_2^T \end{bmatrix}.$$

Therefore

$$\boldsymbol{A}_{\mathrm{KLT}} = \begin{bmatrix} \boldsymbol{u}_1^T \\ \boldsymbol{u}_2^T \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Finally, by computing $\boldsymbol{z} = \boldsymbol{A}_{\mathrm{KLT}}\boldsymbol{A}\boldsymbol{x}$, we have

$$\boldsymbol{z} = \frac{\sqrt{2}}{2} \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \boldsymbol{x}.$$

(e) Compute the covariance matrix of $\boldsymbol{z}$.

Solution:

$$\begin{aligned} \boldsymbol{C}_{\boldsymbol{z}} &= (\boldsymbol{A}_{\mathrm{KLT}}\boldsymbol{A})\boldsymbol{C}_{\boldsymbol{x}}(\boldsymbol{A}_{\mathrm{KLT}}\boldsymbol{A})^T \\ &= (\boldsymbol{A}_{\mathrm{KLT}}\boldsymbol{A})(\boldsymbol{A}_{\mathrm{KLT}}\boldsymbol{A})^T \\ &= \frac{\sqrt{2}}{2} \begin{bmatrix} 2 & -2 \\ 2 & 1 \end{bmatrix} \cdot \frac{\sqrt{2}}{2} \begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

(f) Based on your analysis, is there a unique representation of $\boldsymbol{y}$ of the form $\boldsymbol{B}\boldsymbol{q}$, where q_1 and q_2 are decorrelated?

Solution: No: $\boldsymbol{A}\boldsymbol{x}$ is one such model and $\boldsymbol{A}_{\mathrm{KLT}}^{-1}\boldsymbol{z}$ is another.

3 LSI Systems and Circulant Matrices

[intermediate] Because the DFT diagonalizes circulant matrices, it can be an efficient way to implement digital LSI systems. Here, we explore the connection between LSI systems, circulant matrices, and the DFT.

(a) Give the result of the application of an LSI system with impulse response $h = \begin{bmatrix} \dots & 0 & \boxed{a} & b & c & 0 & \dots \end{bmatrix}$ to the signal $\begin{bmatrix} \dots & 0 & \boxed{x} & y & z & 0 & \dots \end{bmatrix}$.

Solution: $\begin{bmatrix} \dots & 0 & \boxed{ax} & bx+ay & cx+by+az & cy+bz & cz & 0 & \dots \end{bmatrix}$

(b) Write the nonzero part of the computation from part (a) as a matrix-vector product, i.e.

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}.$$

Note whether the matrix is Toeplitz, circulant, or neither.

Solution:

$$\begin{bmatrix} a & 0 & 0 \\ b & a & 0 \\ c & b & a \\ 0 & c & b \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax \\ bx+ay \\ cx+by+az \\ cy+bz \\ cz \end{bmatrix}.$$

The matrix is Toeplitz, but not circulant.

(c) Give the result of the application of an LSI system with impulse response $h = [\![\boxed{a} \quad b \quad c]\!]$ to the signal $[\![\boxed{x} \quad y \quad z]\!]$, this time assuming periodic boundary conditions. Is the result the same as the nonzero part of part (a)?

Solution:

$$[\![\boxed{ax+cy+bz} \quad bx+ay+cz \quad cx+by+az]\!].$$

Not the same as (a).

(d) Write the computation in part (c) in the form of a matrix-vector product. Note whether the matrix is Toeplitz, circulant, or neither.

Solution:

$$\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax+cy+bz \\ bx+ay+cz \\ cx+by+az \end{bmatrix}.$$

The matrix is Toeplitz, and circulant.

(e) Find the 5×5 circulant matrix, $\boldsymbol{H}$, such that $\boldsymbol{H} [\![\boxed{x} \quad y \quad z \quad 0 \quad 0]\!]^T$ matches the nonzero part of the result from part (a).

Solution:

$$\begin{bmatrix} a & 0 & 0 & c & b \\ b & a & 0 & 0 & c \\ c & b & a & 0 & 0 \\ 0 & c & b & a & 0 \\ 0 & 0 & c & b & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} ax \\ bx+ay \\ cx+by+az \\ cy+bz \\ cz \end{bmatrix}$$