Image Processing 2, Exercise 3

1 Linear Interpolation

[basic] Using interpolation to shift a sequence.

(a) Let a sequence of samples be given by $f = \{\ldots, -2, -2, -5, \boxed{1}, 3, 5, 5, \ldots\}$. This sequence is linearly interpolated to produce the continuously defined function $f_1(x) = \sum_{k\in\mathbb{Z}} c[k]\, \beta^1(x-k)$ with $x \in \mathbb{R}$ such that $f_1(x)|_{x=k} = f[k]$ for $k \in \mathbb{Z}$. Find the coefficients $c[k]$ for $k \in \mathbb{Z}$.

(b) By translating f_1, one creates $g_1(x) = f_1(x - \frac{4}{3})$. Then, the sequence $g_1[k]$ satisfies $g_1[k] = g_1(x)|_{x=k}$ for $k \in \mathbb{Z}$. Find the filter w such that $g_1 = w * f$.

2 B-spline derivatives

[basic] B-splines representation of a function can be extremely useful when dealing with the continuous-domain quantities related to the function, including its derivatives.

Compute the Fourier transform of the function $\left(\beta^{n-1}(x+\frac{1}{2}) - \beta^{n-1}(x-\frac{1}{2})\right)$, with $x \in \mathbb{R}$ and $n \in \mathbb{N} \setminus \{0\}$. Verify that it is equal to the Fourier transform of the function $\frac{\mathrm{d}\beta^n(x)}{\mathrm{d}x}$ (the derivative is sometimes also termed as $\dot{\beta}^n(x)$).

3 B-spline properties

[intermediate] Practice with the definition and properties of B-splines.

For nonnegative arguments, show that polynomial B-splines are non-increasing functions. Suggestion: Establish a recurrence relation. You may want to partition the set of nonnegative numbers as $\mathbb{R}_+ = [0, \frac{1}{2}) \cup [\frac{1}{2}, \infty)$.

4 Two-scale relation

[intermediate] The two-scale relation is a key ingredient of wavelets, which we study in-depth in Chapter 8.

For some special φ, it is possible to write the relation $\forall x \in \mathbb{R} : \varphi(\frac{x}{2}) = \sum_{k\in\mathbb{Z}} h[k]\, \varphi(x-k)$ for a well-chosen sequence h which is called the refinement filter. The relation itself is called the two-scale relation.

(a) Given that the Fourier transform of φ is $\hat{\varphi}$ and that the z-transform of h is H, express the two-scale relation in the Fourier domain.

(b) Show how to use the previous result to determine $H_2^n(\mathrm{e}^{\mathrm{j}\,\omega})$ for all $\omega \in \mathbb{R}$ when $\varphi = \beta_+^n$.