Image Processing 2, Exercise 3

1 Linear Interpolation

[basic] Using interpolation to shift a sequence.

(a) Let a sequence of samples be given by $f = \{\ldots, -2, -2, -5, \boxed{1}, 3, 5, 5, \ldots\}$. This sequence is linearly interpolated to produce the continuously defined function $f_1(x) = \sum_{k\in\mathbb{Z}} c[k]\,\beta^1(x-k)$ with $x \in \mathbb{R}$ such that $f_1(x)|_{x=k} = f[k]$ for $k \in \mathbb{Z}$. Find the coefficients $c[k]$ for $k \in \mathbb{Z}$.

> **Solution:** Since β^1 is interpolating, it is easy to see that $c[k] = f[k]$.

(b) By translating f_1, one creates $g_1(x) = f_1(x - \frac{4}{3})$. Then, the sequence $g_1[k]$ satisfies $g_1[k] = g_1(x)|_{x=k}$ for $k \in \mathbb{Z}$. Find the filter w such that $g_1 = w * f$.

> **Solution:**
>
> $$\begin{aligned} g_1(n) &= f_1(n - \frac{4}{3}) \\ &= \sum_{k\in\mathbb{Z}} f[k]\beta^1(n - \frac{4}{3} - k), \end{aligned}$$
>
> where $\beta^1(x) = \begin{cases} 1 - |x|, & |x| \leqslant 1, \\ 0, & \text{else.} \end{cases}$
>
> The center of this filter is at: $k = n \Rightarrow \beta^1(n - \frac{4}{3} - n) = \beta^1(-\frac{4}{3}) = 0.$
>
> Nonzero entries of this filter only happen at locations:
>
> $k = n - 1 \Rightarrow \beta^1(n - \frac{4}{3} - (n-1)) = \beta^1(-\frac{1}{3}) = \frac{2}{3}$
>
> $k = n - 2 \Rightarrow \beta^1(n - \frac{4}{3} - (n-2)) = \beta^1(\frac{2}{3}) = \frac{1}{3}$
>
> Hence,
>
> $$\begin{aligned} g_1[n] &= f[n]\cdot 0 + f[n-1]\cdot\frac{2}{3} + f[n-2]\cdot\frac{1}{3} \\ &= \left(f * \underbrace{\left[\ldots, 0, \boxed{0}, \frac{2}{3}, \frac{1}{3}, 0, \ldots\right]}_{w} \right)[n] \end{aligned}$$

2 B-spline derivatives

[basic] B-splines representation of a function can be extremely useful when dealing with the continuous-domain quantities related to the function, including its derivatives.

Compute the Fourier transform of the function $\left(\beta^{n-1}(x + \frac{1}{2}) - \beta^{n-1}(x - \frac{1}{2})\right)$, with $x \in \mathbb{R}$ and $n \in \mathbb{N} \setminus \{0\}$. Verify that it is equal to the Fourier transform of the function $\frac{\mathrm{d}\beta^n(x)}{\mathrm{d}x}$ (the derivative is sometimes also termed as $\dot{\beta}^n(x)$).

> **Solution:** Fourier transform (FT) of a B-spline $\beta^n(x)$ is $\hat{\beta}^n(\omega) = \left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^{n+1}$.

Shift property of the FT: $f(x-x_0) \leftrightarrow \mathrm{e}^{-\mathrm{j}\omega x_0}\hat{f}(\omega)$ gives

$$\begin{aligned}\beta^{n-1}(x+\frac{1}{2}) &\leftrightarrow \mathrm{e}^{\frac{\mathrm{j}\omega}{2}}\hat{\beta}^{n-1}(\omega)\\ \beta^{n-1}(x-\frac{1}{2}) &\leftrightarrow \mathrm{e}^{-\frac{\mathrm{j}\omega}{2}}\hat{\beta}^{n-1}(\omega)\end{aligned}$$

Hence, the FT of the function $\left(\beta^{n-1}(x+\frac{1}{2})-\beta^{n-1}(x-\frac{1}{2})\right)$ is

$$\begin{aligned}\hat{\beta}^{n-1}(\omega)\left(\mathrm{e}^{\frac{\mathrm{j}\omega}{2}}-\mathrm{e}^{-\frac{\mathrm{j}\omega}{2}}\right) &= \left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^n\left(\cos\frac{\omega}{2}+\mathrm{j}\sin\frac{\omega}{2}-\left(\cos\frac{\omega}{2}-\mathrm{j}\sin\frac{\omega}{2}\right)\right)\\ &= \left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^n 2\mathrm{j}\sin\frac{\omega}{2}\\ &= 2\mathrm{j}\frac{(\sin\frac{\omega}{2})^{n+1}}{(\frac{\omega}{2})^n}\end{aligned}$$

On the other hand, the differentiation property of FT: $\frac{\partial^n f(x)}{\partial x^n} \leftrightarrow (\mathrm{j}\omega_i)^n\hat{f}(\omega)$ gives $\frac{\mathrm{d}\beta^n(x)}{\mathrm{d}x} \leftrightarrow \mathrm{j}\omega\hat{\beta}^n(\omega)$.

Hence, the FT of $\frac{\mathrm{d}\beta^n(x)}{\mathrm{d}x}$ is

$$\mathrm{j}\omega\hat{\beta}^n(\omega) = \mathrm{j}\omega\left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^{n+1} = 2\mathrm{j}\frac{(\sin\frac{\omega}{2})^{n+1}}{(\frac{\omega}{2})^n}.$$

Remark: This example shows that if

$$f(x) = \sum_{k\in\mathbb{Z}} c[k]\beta^n(x-k),$$

then

$$\begin{aligned}\dot{f}(x) &= \sum_{k\in\mathbb{Z}} c[k]\dot{\beta}^n(x-k),\\ &= \sum_{k\in\mathbb{Z}} c[k]\left(\beta^{n-1}\left(x-k+\frac{1}{2}\right)-\beta^{n-1}\left(x-k-\frac{1}{2}\right)\right),\end{aligned}$$

3 B-spline properties

[intermediate] Practice with the definition and properties of B-splines.

For nonnegative arguments, show that polynomial B-splines are non-increasing functions. Suggestion: Establish a recurrence relation. You may want to partition the set of nonnegative numbers as $\mathbb{R}_+ = [0,\frac{1}{2}) \cup [\frac{1}{2},\infty)$.

Solution:

We distinguish 3 cases:

1. $n=0: \beta^0(x) = \mathrm{rect}(x) = \begin{cases} 1, & 0 \leqslant x \leq \frac{1}{2} \\ 0, & x \geq \frac{1}{2} \end{cases}$, it is easy to see that

   $\beta^n(x) \geqslant \beta^n(y), \quad 0 \leqslant x \leqslant y;$

2. $n = 1 : \beta^1(x) = \text{tri}(x) = \begin{cases} 1 - x, & 0 \leqslant x \leq 1 \\ 0, & x \geq 1 \end{cases}$, it is easy to see that

$\beta^n(x) \geqslant \beta^n(y), \quad 0 \leqslant x \leqslant y;$

3. $n \geqslant 1$: from question 2 we have that $\dot{\beta}^{n+1}(x) = \beta^{n-1}(x + \frac{1}{2}) - \beta^{n-1}(x - \frac{1}{2})$, and $\beta^{n+1}(x)$ is continuous on $[0, +\infty)$ when $n \geqslant 1$.

   We further distinguish 2 cases:

   (a) $0 \leqslant x < \frac{1}{2}$:

$$\begin{aligned} \dot{\beta}^{n+1}(x) &= \beta^{n-1} \underbrace{(x + \frac{1}{2})}_{>0} - \beta^{n-1} \underbrace{(x - \frac{1}{2})}_{<0} \\ &\overset{(1)}{=} \beta^{n-1}(x + \frac{1}{2}) - \beta^{n-1}(\frac{1}{2} - x) \\ &\overset{(2)}{\leqslant} 0, \end{aligned}$$

   (1) is due to the fact that B-splines are even functions, i.e. $\beta^{n-1}(x+\frac{1}{2}) = \beta^{n-1}(\frac{1}{2}-x)$ and (2) is because $x + \frac{1}{2} \geqslant \frac{1}{2} - x > 0$.

   (b) $x \geqslant \frac{1}{2}$:

$$\begin{aligned} \dot{\beta}^{n+1}(x) &= \beta^{n-1} \underbrace{(x + \frac{1}{2})}_{>0} - \beta^{n-1} \underbrace{(x - \frac{1}{2})}_{>0} \\ &\overset{(3)}{\leqslant} 0, \end{aligned}$$

   (3) is due to $x + \frac{1}{2} \geqslant x - \frac{1}{2} \geqslant 0$.

Hence, B-splines are non-increasing functions.

4 Two-scale relation

[intermediate] The two-scale relation is a key ingredient of wavelets, which we study in-depth in Chapter 8.

For some special φ, it is possible to write the relation $\forall x \in \mathbb{R} : \varphi(\frac{x}{2}) = \sum_{k\in\mathbb{Z}} h[k]\,\varphi(x - k)$ for a well-chosen sequence h which is called the refinement filter. The relation itself is called the two-scale relation.

(a) Given that the Fourier transform of φ is $\hat{\varphi}$ and that the z-transform of h is H, express the two-scale relation in the Fourier domain.

Solution:
Scaling property of the Fourier transform (FT): $f(\frac{x}{a}) \leftrightarrow |a|\hat{f}(a\omega)$, the FT of $\varphi(\frac{x}{2})$ is $2\varphi(2\omega)$.
Shift property of the FT: $f(x - x_0) \leftrightarrow \mathrm{e}^{-\mathrm{j}\omega x_0}\hat{f}(\omega)$, the FT of $\varphi(x - k)$ is $\mathrm{e}^{-\mathrm{j}\omega k}\hat{\varphi}(\omega)$.
Hence, applying the FT on both sides of the two-scale relation gives

$$2\hat{\varphi}(2\omega) = \underbrace{\left(\sum_{k\in\mathbb{Z}} h[k]\mathrm{e}^{-\mathrm{j}\omega k}\right)}_{H(z)|_{z=\mathrm{e}^{\mathrm{j}\omega}}} \hat{\varphi}(\omega),$$

In other words,

$$\left(H(\mathrm{e}^{\mathrm{j}\,\omega}) = \frac{2\,\hat{\varphi}(2\,\omega)}{\hat{\varphi}(\omega)}\right)$$

is the two-scale relation in the Fourier domain.

(b) Show how to use the previous result to determine $H_2^n(\mathrm{e}^{\mathrm{j}\,\omega})$ for all $\omega \in \mathbb{R}$ when $\varphi = \beta_+^n$.

Solution:
On lecture slide 7-81, B-spline dilated by an integer factor m is defined as

$$\beta_+^n(\frac{x}{m}) = \sum_{k\in\mathbb{Z}} h_m^n[k]\beta_+^n(x-k), \quad H_m^n(z) = \frac{1}{m^n}\left(\sum_{k=0}^{m-1} z^{-k}\right)^{n+1},$$

hence, when $m = 2$ and $\varphi = \beta_+^n$, we have

$$H_2^n(\mathrm{e}^{\mathrm{j}\omega}) = \frac{1}{2^n}\left(\sum_{k=0}^{1} \mathrm{e}^{-\mathrm{j}\omega k}\right)^{n+1}. \tag{1}$$

Now we use the result in part (a) to derive this formula.
According to the result in part (a), in order to compute $H(\mathrm{e}^{\mathrm{j}\omega})$, we need to compute $\hat{\beta}_+^n$ first.
$\beta_+^n(x) := \beta^n(x - \frac{n+1}{2})$, using the shifting property of the FT and the fact that

$$\hat{\beta}^n(\omega) = \left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^{n+1} = \left(\frac{\mathrm{e}^{\frac{\mathrm{j}\omega}{2}} - \mathrm{e}^{-\frac{\mathrm{j}\omega}{2}}}{\mathrm{j}\omega}\right)^{n+1}$$

we have

$$\begin{aligned}
\hat{\beta}_+^n(\omega) &= \mathrm{e}^{-\frac{\mathrm{j}\omega}{2}(n+1)}\hat{\beta}(\omega) \\
&= \mathrm{e}^{-\frac{\mathrm{j}\omega}{2}(n+1)}\left(\frac{\mathrm{e}^{\frac{\mathrm{j}\omega}{2}} - \mathrm{e}^{-\frac{\mathrm{j}\omega}{2}}}{\mathrm{j}\omega}\right)^{n+1} \\
&= \frac{(1-\mathrm{e}^{-\mathrm{j}\omega})^{n+1}}{(\mathrm{j}\omega)^{n+1}}
\end{aligned}$$

Hence,

$$\begin{aligned}
H(\mathrm{e}^{\mathrm{j}\omega}) &= \frac{2\hat{\beta}_+^n(2\omega)}{\hat{\beta}_+^n(\omega)} \\
&= \frac{2(1-\mathrm{e}^{-2\mathrm{j}\omega})^{n+1}}{(2\mathrm{j}\omega)^{n+1}} \cdot \frac{(\mathrm{j}\omega)^{n+1}}{(1-\mathrm{e}^{-\mathrm{j}\omega})^{n+1}} \\
&= \frac{1}{2^n} \cdot \frac{(1-\mathrm{e}^{-2\mathrm{j}\omega})^{n+1}}{(1-\mathrm{e}^{-\mathrm{j}\omega})^{n+1}} \\
&= \frac{1}{2^n}\left(1+\mathrm{e}^{-\mathrm{j}\omega}\right)^{n+1}
\end{aligned}$$

which is exactly as (1).