Image Processing 2, Exercise 2

1 Structure tensor

[intermediate] An example of the power of the structure tensor to encode local gradient information about an image.

(a) Consider computing over an image f the delocalized structure tensor $\mathbf{J}$ characterized by the constant-valued observation window $w = 1$. Use Parseval to give an expression of $\mathbf{J}$ where $\hat{f}$ appears instead of f.

Solution:

$$\begin{aligned}
\mathbf{J} &= \int_{\mathbb{R}^2} \boldsymbol{\nabla} f(\mathbf{x})\,(\boldsymbol{\nabla} f(\mathbf{x}))^{\mathsf{H}}\,\mathrm{d}\mathbf{x} \\
\mathbf{J} &= \int_{\mathbb{R}^2} \begin{pmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1}\left(\frac{\partial f(\mathbf{x})}{\partial x_1}\right)^* & \frac{\partial f(\mathbf{x})}{\partial x_1}\left(\frac{\partial f(\mathbf{x})}{\partial x_2}\right)^* \\ \frac{\partial f(\mathbf{x})}{\partial x_2}\left(\frac{\partial f(\mathbf{x})}{\partial x_1}\right)^* & \frac{\partial f(\mathbf{x})}{\partial x_2}\left(\frac{\partial f(\mathbf{x})}{\partial x_2}\right)^* \end{pmatrix} \mathrm{d}x_1 \mathrm{d}x_2 \\
&\overset{(1)}{=} \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \begin{pmatrix} \left(\mathrm{j}\,\omega_1\,\hat{f}(\boldsymbol{\omega})\right)\left(\mathrm{j}\,\omega_1\,\hat{f}(\boldsymbol{\omega})\right)^* & \left(\mathrm{j}\,\omega_1\,\hat{f}(\boldsymbol{\omega})\right)\left(\mathrm{j}\,\omega_2\,\hat{f}(\boldsymbol{\omega})\right)^* \\ \left(\mathrm{j}\,\omega_2\,\hat{f}(\boldsymbol{\omega})\right)\left(\mathrm{j}\,\omega_1\,\hat{f}(\boldsymbol{\omega})\right)^* & \left(\mathrm{j}\,\omega_2\,\hat{f}(\boldsymbol{\omega})\right)\left(\mathrm{j}\,\omega_2\,\hat{f}(\boldsymbol{\omega})\right)^* \end{pmatrix} \mathrm{d}\omega_1 \mathrm{d}\omega_2 \\
&\overset{(2)}{=} \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \begin{pmatrix} \omega_1^2 & \omega_1\,\omega_2 \\ \omega_2\,\omega_1 & \omega_2^2 \end{pmatrix} \left|\hat{f}(\boldsymbol{\omega})\right|^2 \mathrm{d}\omega_1 \mathrm{d}\omega_2
\end{aligned}$$

where (1) uses the differentiation property of Fourier transform; (2) uses the basic properties of complex numbers.

(b) Let an image be $f(\mathbf{x}) = \mathrm{sinc}(2\,x_1 + 3\,x_2)\,\frac{1}{\sqrt{2\,\pi}}\,\mathrm{e}^{-\frac{1}{2}\left(x_1^2 + 4\,x_1\,x_2 + 4\,x_2^2\right)}$. Determine its gradient $\boldsymbol{\nabla} f$. Hint: To avoid direct calculation of $\boldsymbol{\nabla} f$ which is complex, notice that image f can be written as applying an affine transformation $\mathbf{A}$ to another image g that has a much simpler form, try to find out $\mathbf{A}$ and g.

Solution: Observe that $f(\mathbf{x})$ can be written as

$$f(\mathbf{x}) = \mathrm{sinc}(2x_1 + 3x_2)\,\frac{1}{\sqrt{2\,\pi}}\,\mathrm{e}^{-\frac{(x_1+2x_2)^2}{2}}.$$

Let $y_1 = 2x_1 + 3x_2$, $y_2 = x_1 + 2x_2$, and $g(y_1, y_2) = \mathrm{sinc}(y_1)\,\frac{1}{\sqrt{2\,\pi}}\,\mathrm{e}^{-\frac{y_2^2}{2}}$, then $f(\mathbf{x}) = g(\mathbf{y}) = g(\mathbf{A}\,\mathbf{x})$, where $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$.

Hence,

$$\begin{aligned}
\boldsymbol{\nabla} f(\mathbf{x}) &= \boldsymbol{\nabla} g(\mathbf{A}\,\mathbf{x}) \\
&= (\mathbf{A}\,\mathbf{x})'^{\mathsf{T}} g'(\mathbf{A}\,\mathbf{x}) \\
&= \mathbf{A}^{\mathsf{T}}\,g'(\mathbf{A}\,\mathbf{x}) \\
&= \mathbf{A}^{\mathsf{T}}\,g'(\mathbf{y})|_{\mathbf{y}=\mathbf{A}\,\mathbf{x}}.
\end{aligned}$$

The gradient of g is

$$\begin{aligned}
g'(\mathbf{y}) &= \boldsymbol{\nabla}\left(\mathrm{sinc}(y_1)\,\frac{1}{\sqrt{2\,\pi}}\,\mathrm{e}^{-\frac{y_2^2}{2}}\right) \\
&= \frac{1}{\sqrt{2\,\pi}} \begin{pmatrix} \frac{\cos(\pi\,y_1)\,\pi\,\pi\,y_1 - \sin(\pi\,y_1)\,\pi}{(\pi\,y_1)^2} \\ \mathrm{sinc}(y_1)\,(-y_2) \end{pmatrix} \mathrm{e}^{-\frac{y_2^2}{2}}.
\end{aligned}$$

Combining all the information above, we obtain

$$\begin{aligned} \boldsymbol{\nabla} f(\mathbf{x}) &= \mathbf{A}^\mathsf{T} g'(\mathbf{y})|_{\mathbf{y}=\mathbf{A}\,\mathbf{x}} \\ &= \frac{1}{\sqrt{2\,\pi}} \begin{pmatrix} 2\,\frac{\cos(\pi\,(2\,x_1+3\,x_2))-\mathrm{sinc}(2\,x_1+3\,x_2)}{2\,x_1+3\,x_2} - \mathrm{sinc}(2\,x_1+3\,x_2)\,(x_1+2\,x_2) \\ 3\,\frac{\cos(\pi\,(2\,x_1+3\,x_2))-\mathrm{sinc}(2\,x_1+3\,x_2)}{2\,x_1+3\,x_2} - 2\,\mathrm{sinc}(2\,x_1+3\,x_2)\,(x_1+2\,x_2) \end{pmatrix} \end{aligned}$$

$\times \mathrm{e}^{-\frac{(x_1+2x_2)^2}{2}}$.

(c) Determine the Fourier transform $\hat{f}$ of the image f.

Solution:

$$\begin{aligned} \hat{g}(\boldsymbol{\omega}) &= \mathcal{F}_{\mathbf{x}}\{\mathrm{sinc}(x_1)\,\frac{1}{\sqrt{2\,\pi}}\,\mathrm{e}^{-\frac{x_2^2}{2}}\}(\boldsymbol{\omega}) \\ &= \mathcal{F}_{x_1}\{\mathrm{sinc}(x_1)\}(\omega_1)\,\mathcal{F}_{x_2}\{\frac{1}{\sqrt{2\,\pi}}\,\mathrm{e}^{-\frac{x_2^2}{2}}\}(\omega_2) \\ &= \mathrm{rect}(\frac{\omega_1}{2\,\pi})\,\mathrm{e}^{-\frac{\omega_2^2}{2}} \end{aligned}$$

$$\begin{aligned} \hat{f}(\boldsymbol{\omega}) &\overset{(1)}{=} |\det(\mathbf{A})|^{-1}\,\hat{g}(\mathbf{A}^{-\mathsf{T}}\,\boldsymbol{\omega}) \\ &= \mathrm{rect}(\frac{2\,\omega_1-\omega_2}{2\,\pi})\,\mathrm{e}^{-\frac{1}{2}\left(9\,\omega_1^2-12\,\omega_1\,\omega_2+4\,\omega_2^2\right)} \end{aligned}$$

where (1) uses the affine transformation property of Fourier transfrom in the appendix.

(d) Determine the value of the delocalized structure tensor associated to f. You may want to take advantage of $\int_{\mathbb{R}} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$ and $\int_{\mathbb{R}} x^2\,\mathrm{e}^{-x^2}\,\mathrm{d}x = \frac{1}{2}\,\sqrt{\pi}$.

Solution: Let

$$\boldsymbol{\varphi} = \mathbf{A}^{-\mathsf{T}}\,\boldsymbol{\omega} \qquad\qquad \boldsymbol{\omega} = \mathbf{A}^\mathsf{T}\,\boldsymbol{\varphi}$$

Combining the result of part (a) and (c),

$$\begin{aligned} \mathbf{J} &= \frac{1}{4\,\pi^2}\int_{\mathbb{R}^2}\begin{pmatrix} \omega_1^2 & \omega_1\,\omega_2 \\ \omega_1\,\omega_2 & \omega_2^2 \end{pmatrix} \left|\mathrm{rect}(\frac{2\,\omega_1-\omega_2}{2\,\pi})\,\mathrm{e}^{-\frac{1}{2}\left(9\,\omega_1^2-12\,\omega_1\,\omega_2+4\,\omega_2^2\right)}\right|^2 \mathrm{d}\omega_1\mathrm{d}\omega_2 \\ &\overset{(1)}{=} \frac{1}{4\,\pi^2}\int_{\mathbb{R}^2}\begin{pmatrix} (2\,\varphi_1+\varphi_2)^2 & (2\,\varphi_1+\varphi_2)\,(3\,\varphi_1+2\,\varphi_2) \\ (3\,\varphi_1+2\,\varphi_2)\,(2\,\varphi_1+\varphi_2) & (3\,\varphi_1+2\,\varphi_2)^2 \end{pmatrix} \mathrm{rect}(\frac{\varphi_1}{2\,\pi})\,\mathrm{e}^{-\varphi_2^2}\,\mathrm{d}\varphi_1\mathrm{d}\varphi_2 \\ &\overset{(2)}{=} \frac{1}{4\,\pi^2}\int_{-\infty}^{\infty}\int_{-\pi}^{\pi}\begin{pmatrix} 4\,\varphi_1^2+4\,\varphi_1\,\varphi_2+\varphi_2^2 & 6\,\varphi_1^2+7\,\varphi_1\,\varphi_2+2\,\varphi_2^2 \\ 6\,\varphi_1^2+7\,\varphi_1\,\varphi_2+2\,\varphi_2^2 & 9\,\varphi_1^2+12\,\varphi_1\,\varphi_2+4\,\varphi_2^2 \end{pmatrix} \mathrm{d}\varphi_1\,\mathrm{e}^{-\varphi_2^2}\,\mathrm{d}\varphi_2 \\ &= \frac{1}{4\,\pi^2}\int_{-\infty}^{\infty}\begin{pmatrix} \frac{8}{3}\,\pi^3+2\,\pi\,\varphi_2^2 & 4\,\pi^3+4\,\pi\,\varphi_2^2 \\ 4\,\pi^3+4\,\pi\,\varphi_2^2 & 6\,\pi^3+8\,\pi\,\varphi_2^2 \end{pmatrix} \mathrm{e}^{-\varphi_2^2}\,\mathrm{d}\varphi_2 \\ &= \frac{1}{12\,\sqrt{\pi}}\begin{pmatrix} 3+8\,\pi^2 & 6+12\,\pi^2 \\ 6+12\,\pi^2 & 12+18\,\pi^2 \end{pmatrix}. \end{aligned}$$

(1) applies the change of variables, $\mathrm{d}\omega_1\mathrm{d}\omega_2 = \left|\det(\mathbf{A}^T)\right|\mathrm{d}\varphi_1\mathrm{d}\varphi_2$. (2) uses the property that function $\mathrm{rect}(\frac{\varphi_1}{2\,\pi})$ is supported on $[-\pi,\pi]$.

(e) Give the delocalized gradient energy of f.

Solution:

$$\begin{aligned} E &= \operatorname{tr}(\mathbf{J}) \\ &= [\mathbf{J}]_{2,2} + [\mathbf{J}]_{1,1} \\ &= \frac{15 + 26\,\pi^2}{12\,\sqrt{\pi}} \end{aligned}$$

(f) Give the delocalized coherency of f.

Solution:

$$\begin{aligned} C &= \frac{1}{E}\sqrt{\Big([\mathbf{J}]_{2,2} - [\mathbf{J}]_{1,1}\Big)^2 + 4\,[\mathbf{J}]_{1,2}^2} \\ &= \frac{\sqrt{225 + 756\,\pi^2 + 676\,\pi^4}}{15 + 26\,\pi^2} \end{aligned}$$

2 Spline Interpolation 1D

[basic] Interpolating the samples of a function using B-splines. Interpolation is fundamental in image processing, because we often want to move back and forth between continuous-domain signals and their discrete-domain representations.

Assume that $f(x) = (10 - |6x + 3|)\,\mathrm{rect}(\frac{x}{4} + \frac{1}{8})$. Find the quadratic spline coefficients $\{c[m]\}_{m\in\mathbb{Z}}$ such that $s(x) = \sum_{m\in\mathbb{Z}} c[m]\beta^2(x - m)$ satisfies the interpolation condition $s[k] = f[k]$ for all $k \in \mathbb{Z}$.

Solution: Let $F(z)$ be the z-transform of the sequence $f[k]$ for $k \in \mathbb{Z}$. It is given as

$$F(z) = z^2 + 7z + 7 + z^{-1},$$

because $f[k]$ is nonzero only at $k = 0 \Rightarrow f[0] = 7$, $k = 1 \Rightarrow f[1] = 1$, $k = -1 \Rightarrow f[-1] = 7$, $k = -2 \Rightarrow f[-2] = 1$.

We now want that $s[k] = f[k]$ for $k \in \mathbb{Z}$. This means their z-transform are equal too *i.e.*,

$$\forall k \in \mathbb{Z} \quad f[k] = s[k] \Longrightarrow F(z) = S(z).$$

The z-transform of $s[k]$ is given by

$$\begin{aligned} S(z) &= C(z)B(z) \\ S(z) &= C(z)\frac{z + 6 + z^{-1}}{8}. \end{aligned}$$

From the previous equalities we get,

$$\begin{aligned} C(z) &= \frac{8S(z)}{z + 6 + z^{-1}}, \\ &= \frac{8F(z)}{z + 6 + z^{-1}}, \\ &= 8\frac{z(z + 6 + z^{-1}) + (z + 6 + z^{-1})}{z + 6 + z^{-1}}, \\ &= 8z + 8. \end{aligned}$$

This gives

$$c[m] = 8\delta[m] + 8\delta[m+1].$$

This example shows that given a sample of a function (in this case of $f(x)$), we can create a new function $s(x)$ using splines such that its samples are the same as of $f(x)$.

Alternative solution:

Here, we present an alternative solution based on the implementation presented in slide 7-24 and the table of transform function of B-splines in slide 7-34. Because we chose quadratic splines, $n = 2$ and $a = -3 + 2\sqrt{2}$.

We consider the discrete signal

$$f[k] = \begin{bmatrix} \cdots & 0 & 0 & 1 & 7 & \boxed{7} & 1 & 0 & 0 & \cdots \end{bmatrix} \tag{1}$$

and

$$f[k] = b * c[k],$$

where $b[k] = \begin{bmatrix} \cdots & \frac{1}{8} & \boxed{\frac{6}{8}} & \frac{1}{8} \cdots \end{bmatrix}$.

Let y_1 be the result of the causal filter on f, let y_2 be the result of the anti-causal filter on y_1.

Be very careful! They are the results of the filtering, not the filters themselves! Using the cascade of a causal and anti-causal filter, we obtain the following partial signals.

Causal filter (recall $y_1[k] = f[k] + ay_1[k-1]$ from the causal filter cascade)

$$\begin{aligned}
y_1[k] &= 0, \quad k \leq -3 \text{ since filter is causal and } f[k] = 0 \text{ for } k \leq -3 \\
y_1[-2] &= f[-2] + z_1 y_1[-3] \\
&= 1 \\
y_1[-1] &= f[-1] + ay_1[-2] \\
&= 7 - 3 + 2\sqrt{2} \\
&= 4 + 2\sqrt{2} \\
y_1[0] &= f[0] + ay_1[-1] \\
&= 3 + 2\sqrt{2} \\
y_1[1] &= f[1] + ay_1[0] \\
&= 0 \\
y_1[k] &= 0 \quad \text{for } k > 1
\end{aligned}$$

and we have

$$y_1[k] = \begin{bmatrix} \cdots & 0 & 1 & 4 + 2\sqrt{2} & \boxed{3 + 2\sqrt{2}} & 0 & \cdots \end{bmatrix}.$$

Anti-causal filter (recall $y_2[k] = y_1[k] + ay_2[k+1]$ from the filter cascade)

$$
\begin{aligned}
y_2[k] &= 0 \quad k \geq 1 \text{ since filter is anti-causal and } y_1[k] = 0 \text{ for } k \geq 1 \\
y_2[0] &= y_1[0] + ay_2[1] \\
&= 3 + 2\sqrt{2} \\
y_2[-1] &= y_2[-1] + ay_2[0] \\
&= 4 + 2\sqrt{2} + (-3 + 2\sqrt{2})(3 + 2\sqrt{2}) \\
&= 4 + 2\sqrt{2} + 8 - 9 \\
&= 3 + 2\sqrt{2} \\
y_2[-2] &= y_1[-2] + ay_2[-1] \\
&= 1 + (-3 + 2\sqrt{2})(3 + 2\sqrt{2}) \\
&= 1 - 1 \\
&= 0 \\
y_2[k] &= 0 \quad \text{for } k < -2
\end{aligned}
$$

and we have

$$y_2[k] = \begin{bmatrix} \cdots 0 & 0 & 3+2\sqrt{2} & \boxed{3+2\sqrt{2}} & 0 & \cdots \end{bmatrix}.$$

Gain ($G = \frac{8}{3+2\sqrt{2}}$)

$$c[k] = \frac{8}{3+2\sqrt{2}} y_2[k] = \begin{bmatrix} \cdots 0 & 0 & 8 & \boxed{8} & 0 & \cdots \end{bmatrix}$$

We can finally verify that

$$f[k] = b * c[k]$$

by computing the FIR convolution:

	0	0	0	8	$\boxed{8}$	0	0	0
$\frac{1}{8}$	1	$\boxed{6}$	1					
$\frac{1}{8}$		1	$\boxed{6}$	1				
$\frac{1}{8}$			1	$\boxed{6}$	1			
$\frac{1}{8}$				1	$\boxed{6}$	1		
$\frac{1}{8}$					1	$\boxed{6}$	1	
$\frac{1}{8}$						1	$\boxed{6}$	1
	0	0	1	7	$\boxed{7}$	1	0	0