# Image Processing 2, Exercise 1

## 1 Radon transform

*[basic] Practice applying the Radon transform, which is the mathematical model for several medical imaging modalities, including X-ray CT scanners.*

(a) Determine the one-dimensional Fourier transform of the Gaussian $f(x) = \frac{1}{\sigma\sqrt{2\pi}}\,\mathrm{e}^{-\frac{x^2}{2\sigma^2}}$, with $x \in \mathbb{R}$ and $0 < \sigma$.

(b) Let an image be given by $f(x_1, x_2) = \frac{\sqrt{2}}{\sigma^2\,2\pi}\,\mathrm{e}^{-\frac{x_1^2+2x_2^2}{2\sigma^2}}$. Give its Radon transform. Hint: use the central-slice theorem.

## 2 Directional Derivatives

*[refresher] Review of directional derivatives, which are useful for defining steerable filters.*

Let $f(x_1, x_2) = \cos(\pi x_1)$.

(a) Compute $\mathrm{D}_{\boldsymbol{u}} f(\frac{1}{2}, 0)$ with $\boldsymbol{u} = (1, 0)$.

(b) Compute $\mathrm{D}_{\boldsymbol{u}} f(\frac{1}{2}, 0)$ with $\boldsymbol{u} = (0, 1)$.

(c) Compute $\mathrm{D}_{\boldsymbol{u}} f(\frac{1}{2}, \frac{1}{4})$ with $\boldsymbol{u} = \frac{1}{2}(\sqrt{3}, 1)$.

(d) For a generic image, $f$, express $\mathrm{D}^3_{\mathbf{u}_\theta} f(x_1, x_2)$ in terms of the angle $\theta$ and the partial derivatives of $f$. *Do not* simplify the result, but do clearly specify the limits of any sums involved.

## 3 Third Order Detector

*[intermediate] The course notes show how to realize steerable edge (gradient-based) detectors and ridge (Hessian-based) detectors. For this exercise, we want you to go beyond edges and ridges by considering directional derivatives of third order.*

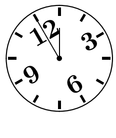
(a) In two dimensions, how many filters do you need to compute all third order derivatives?

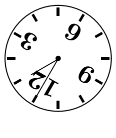
(b) Give the impulse response of each filter, assuming that the isotropic lowpass function is the Gaussian $\varphi(\mathbf{x}) = \frac{1}{\sigma^2\,2\pi}\,\mathrm{e}^{-\frac{\|\mathbf{x}\|^2}{2\sigma^2}}$.

## 4 Identifying a rotation matrix

*[basic] An example of determining a transform matrix directly from a pair of images.*

Determine the matrix $\mathbf{G}$ such that $g(\mathbf{x}) = f(\mathbf{G}\,\mathbf{x})$. The coordinate $\mathbf{x} = \mathbf{0}$ is the center of the clock face in the images $f$ and $g$. Pay attention to the image-processing conventions for the orientation of the axes and explain your answer.

$f =$ 


$g =$ 

## 5 Steerable filter

*[intermediate] Design and analysis of a steerable filter, which is useful for rotation-invariant detection, e.g., in analysis of microscopy images.*

(a) Let the impulse response of a two-dimensional filter be $h(\mathbf{x}) = \frac{\partial g(\mathbf{x})}{\partial x_1} + \frac{\partial g(\mathbf{x})}{\partial x_2}$ with $g(\mathbf{x}) = \frac{1}{2\pi}\,\mathrm{e}^{-\frac{\|\mathbf{x}\|^2}{2}}$. This filter is steerable of order $M = 2$. Give all nontrivial expansion coefficients $\alpha_{m,n}$ that are needed to expand $h$ in terms of the $(m-n, n)$th partial derivative of the Gaussian $g$.

(b) Give all nontrivial coefficients $a_{m,n}(\theta)$ that must be applied in order to steer $h$.

(c) Take advantage of steering to summarize the spatial-domain convolutions needed to compute $(h_\theta * f)(\mathbf{x})$. The purpose of this summary is to highlight how many convolutions must be performed by the algorithm, and with precisely which filters $g_{m,n}$. It is not necessary that you further develop the expressions once that goal is reached.