# Image Processing 2, Exercise 1

## 1 Radon transform

*[basic] Practice applying the Radon transform, which is the mathematical model for several medical imaging modalities, including X-ray CT scanners.*

(a) Determine the one-dimensional Fourier transform of the Gaussian $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$, with $x \in \mathbb{R}$ and $0 < \sigma$.

**Solution:**

$$\begin{aligned}
\mathcal{F}_x\{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}\}(\omega) &= \frac{1}{\sigma}\mathcal{F}_x\{\frac{1}{\sqrt{2\pi}} e^{-\frac{\left(\frac{x}{\sigma}\right)^2}{2}}\}(\omega) \\
&= \frac{|\sigma|}{\sigma}\mathcal{F}_y\{\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\}(\sigma\,\omega) \\
&= e^{-\frac{\sigma^2\omega^2}{2}}
\end{aligned}$$

(b) Let an image be given by $f(x_1, x_2) = \frac{\sqrt{2}}{\sigma^2\,2\pi} e^{-\frac{x_1^2+2x_2^2}{2\sigma^2}}$. Give its Radon transform. Hint: use the central-slice theorem.

**Solution:**

$$\begin{aligned}
p_\theta(t) &= \mathcal{F}_\omega^{-1}\{\hat{f}(\omega\cos(\theta), \omega\sin(\theta))\}(t) \\
&= \mathcal{F}_\omega^{-1}\{\mathcal{F}_{x_1,x_2}\{\frac{\sqrt{2}}{\sigma^2\,2\pi} e^{-\frac{x_1^2+2x_2^2}{2\sigma^2}}\}(\omega\cos(\theta), \omega\sin(\theta))\}(t) \\
&= \mathcal{F}_\omega^{-1}\{\mathcal{F}_{x_1}\{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x_1^2}{2\sigma^2}}\}(\omega\cos(\theta))\,\mathcal{F}_{x_2}\{\frac{1}{\frac{\sigma}{\sqrt{2}}\sqrt{2\pi}} e^{-\frac{x_2^2}{2\left(\frac{\sigma}{\sqrt{2}}\right)^2}}\}(\omega\sin(\theta))\}(t) \\
&= \mathcal{F}_\omega^{-1}\{e^{-\frac{\sigma^2\omega^2(\cos(\theta))^2}{2}} e^{-\frac{\frac{\sigma^2}{2}\omega^2(\sin(\theta))^2}{2}}\}(t) \\
&= \mathcal{F}_\omega^{-1}\{e^{-\frac{\left(\sigma\sqrt{(\cos(\theta))^2+\frac{1}{2}(\sin(\theta))^2}\right)^2\omega^2}{2}}\}(t) \\
&= \frac{1}{\sigma\sqrt{(\cos(\theta))^2+\frac{1}{2}(\sin(\theta))^2}\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2\left((\cos(\theta))^2+\frac{1}{2}(\sin(\theta))^2\right)}}
\end{aligned}$$

## 2 Directional Derivatives

*[refresher] Review of directional derivatives, which are useful for defining steerable filters.*

Let $f(x_1, x_2) = \cos(\pi x_1)$.

(a) Compute $D_{\boldsymbol{u}} f(\frac{1}{2}, 0)$ with $\boldsymbol{u} = (1, 0)$.

**Solution:**

$$D_{\boldsymbol{u}} f((x)) = \langle (u), f((x))\rangle = \sum_{i=1}^{2} u_i \frac{\partial f(\boldsymbol{x})}{\partial x_i} = u_1 \cdot \frac{\partial f(\boldsymbol{x})}{\partial x_1} + u_2 \cdot \frac{\partial f(\boldsymbol{x})}{\partial x_2}$$

$$D_{\boldsymbol{u}} f(\frac{1}{2}, 0) = [1 \cdot (-\sin(\pi x_1)) \cdot \pi + 0 \cdot 0]|_{x_1=\frac{1}{2}, x_2=0} = -\pi$$

(b) Compute $\mathrm{D}_{\boldsymbol{u}} f(\frac{1}{2}, 0)$ with $\boldsymbol{u} = (0, 1)$.

**Solution:** 0

(c) Compute $\mathrm{D}_{\boldsymbol{u}} f(\frac{1}{2}, \frac{1}{4})$ with $\boldsymbol{u} = \frac{1}{2}(\sqrt{3}, 1)$.

**Solution:** $-\frac{\pi\sqrt{3}}{2}$

(d) For a generic image, $f$, express $\mathrm{D}^3_{\mathbf{u}_\theta} f(x_1, x_2)$ in terms of the angle $\theta$ and the partial derivatives of $f$. *Do not* simplify the result, but do clearly specify the limits of any sums involved.

**Solution:** Applying the directional derivative equation from the notes directly,

$$\mathrm{D}^3_{\mathbf{u}_\theta} f(x_1, x_2) = \sum_{k_1,k_2} \binom{3}{k_1, k_2} u_1^{k_1} u_2^{k_2} \frac{\partial^3 f(\mathbf{x})}{\partial x_1^{k_1} \partial x_2^{k_2}}.$$

The problem is that the limits on the sum are unspecified and the components of $\boldsymbol{u}$ appear rather than $\theta$. Because $k_1 + k_2 = 3$, we can make the substitutions $k_1 = k$ and $k_2 = 3 - k$ and specify limits on $k$,

$$\mathrm{D}^3_{\mathbf{u}_\theta} f(x_1, x_2) = \sum_{k=0}^{3} \binom{3}{k, 3-k} u_1^{k} u_2^{3-k} \frac{\partial^3 f(\mathbf{x})}{\partial x_1^{k} \partial x_2^{3-k}}.$$

Finally, we need to write $\boldsymbol{u}$ in explicitly,

$$\mathrm{D}^3_{\mathbf{u}_\theta} f(x_1, x_2) = \sum_{k=0}^{3} \binom{3}{k, 3-k} (\cos(\theta))^k (\sin(\theta))^{3-k} \frac{\partial^3 f(\mathbf{x})}{\partial x_1^{k} \partial x_2^{3-k}}.$$

# 3 Third Order Detector

*[intermediate] The course notes show how to realize steerable edge (gradient-based) detectors and ridge (Hessian-based) detectors. For this exercise, we want you to go beyond edges and ridges by considering directional derivatives of third order.*

(a) In two dimensions, how many filters do you need to compute all third order derivatives?

**Solution:** Four filters are needed, which should produce

$$\left\{\frac{\partial^3 (\varphi * f)(\mathbf{x})}{\partial x_1^3}, \frac{\partial^3 (\varphi * f)(\mathbf{x})}{\partial x_1^2\, \partial x_2}, \frac{\partial^3 (\varphi * f)(\mathbf{x})}{\partial x_1\, \partial x_2^2}, \frac{\partial^3 (\varphi * f)(\mathbf{x})}{\partial x_2^3}\right\}$$

(b) Give the impulse response of each filter, assuming that the isotropic lowpass function is the Gaussian $\varphi(\mathbf{x}) = \frac{1}{\sigma^2\, 2\,\pi}\, \mathrm{e}^{-\frac{\|\mathbf{x}\|^2}{2\,\sigma^2}}$.

**Solution:**

$$\frac{\partial \varphi(\mathbf{x})}{\partial x_1} = -\frac{x_1}{2\,\pi\,\sigma^4}\, \mathrm{e}^{-\frac{\|\mathbf{x}\|^2}{2\,\sigma^2}}$$

$$\frac{\partial^2 \varphi(\mathbf{x})}{\partial x_1^2} = \frac{x_1^2 - \sigma^2}{2\,\pi\,\sigma^6}\, \mathrm{e}^{-\frac{\|\mathbf{x}\|^2}{2\,\sigma^2}}$$

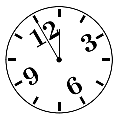
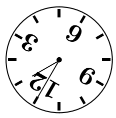
$$h_{3,0}(\mathbf{x}) = \frac{\partial^3 \varphi(\mathbf{x})}{\partial x_1^3} = -\frac{x_1\,\left(x_1^2 - 3\,\sigma^2\right)}{2\,\pi\,\sigma^8}\, \mathrm{e}^{-\frac{\|\mathbf{x}\|^2}{2\,\sigma^2}} \longleftrightarrow \frac{\partial^3 (\varphi * f)(\mathbf{x})}{\partial x_1^3}$$

$$
\begin{aligned}
h_{2,1}(\mathbf{x}) &= \frac{\partial^3 \varphi(\mathbf{x})}{\partial x_1^2\, \partial x_2} = -\frac{x_2\, \left(x_1^2 - \sigma^2\right)}{2\,\pi\,\sigma^8}\, \mathrm{e}^{-\frac{\|\mathbf{x}\|^2}{2\,\sigma^2}} \longleftrightarrow \frac{\partial^3 \left(\varphi * f\right)(\mathbf{x})}{\partial x_1^2\, \partial x_2} \\
h_{1,2}(\mathbf{x}) &= \frac{\partial^3 \varphi(\mathbf{x})}{\partial x_1\, \partial x_2^2} = -\frac{x_1\, \left(x_2^2 - \sigma^2\right)}{2\,\pi\,\sigma^8}\, \mathrm{e}^{-\frac{\|\mathbf{x}\|^2}{2\,\sigma^2}} \longleftrightarrow \frac{\partial^3 \left(\varphi * f\right)(\mathbf{x})}{\partial x_1\, \partial x_2^2} \\
h_{0,3}(\mathbf{x}) &= \frac{\partial^3 \varphi(\mathbf{x})}{\partial x_2^3} = -\frac{x_2\, \left(x_2^2 - 3\,\sigma^2\right)}{2\,\pi\,\sigma^8}\, \mathrm{e}^{-\frac{\|\mathbf{x}\|^2}{2\,\sigma^2}} \longleftrightarrow \frac{\partial^3 \left(\varphi * f\right)(\mathbf{x})}{\partial x_2^3}
\end{aligned}
$$

# 4 Identifying a rotation matrix

*[basic] An example of determining a transform matrix directly from a pair of images.*

Determine the matrix $\mathbf{G}$ such that $g(\mathbf{x}) = f(\mathbf{G}\,\mathbf{x})$. The coordinate $\mathbf{x} = \mathbf{0}$ is the center of the clock face in the images $f$ and $g$. Pay attention to the image-processing conventions for the orientation of the axes and explain your answer.

$f =$  $g =$ 

**Solution:** Define the unit-length vector $\mathbf{u}_\phi = (\cos(\phi), \sin(\phi))$ and the rotation matrix

$$
\mathbf{R}_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{1}
$$

Then, the transformation $\mathbf{G}$ is a simple rotation $\mathbf{R}_\theta$, without reflection.

$$
\begin{aligned}
&\left(g(\mathbf{u}_{\frac{5\,\pi}{6}}) = f(\mathbf{u}_{-\frac{3\,\pi}{6}}) = f(\mathbf{R}_\theta\, \mathbf{u}_{\frac{5\,\pi}{6}}) = f(\mathbf{u}_{\theta + \frac{5\,\pi}{6}})\right) \\
&\Rightarrow \left(\theta = \frac{4\,\pi}{6}, \mathbf{G} = \mathbf{R}_{\frac{4\,\pi}{6}} = \frac{1}{2} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\right)
\end{aligned} \tag{2}
$$

# 5 Steerable filter

*[intermediate] Design and analysis of a steerable filter, which is useful for rotation-invariant detection, e.g., in analysis of microscopy images.*

(a) Let the impulse response of a two-dimensional filter be $h(\mathbf{x}) = \frac{\partial g(\mathbf{x})}{\partial x_1} + \frac{\partial g(\mathbf{x})}{\partial x_2}$ with $g(\mathbf{x}) = \frac{1}{2\,\pi}\, \mathrm{e}^{-\frac{\|\mathbf{x}\|^2}{2}}$. This filter is steerable of order $M = 2$. Give all nontrivial expansion coefficients $\alpha_{m,n}$ that are needed to expand $h$ in terms of the $(m-n, n)$th partial derivative of the Gaussian $g$.

**Solution:**

$$
\begin{aligned}
h(\mathbf{x}) &= \sum_{m=1}^{2} \sum_{n=0}^{m} \alpha_{m,n}\, \frac{\partial^m}{\partial x_1^{m-n}\, \partial x_2^n} g(\mathbf{x}) \\
&= \underbrace{1}_{\alpha_{1,0}} \frac{\partial g(\mathbf{x})}{\partial x_1} + \underbrace{1}_{\alpha_{1,1}} \frac{\partial g(\mathbf{x})}{\partial x_2} + \underbrace{0}_{\alpha_{2,0}} \frac{\partial^2 g(\mathbf{x})}{\partial x_1^2} + \underbrace{0}_{\alpha_{2,1}} \frac{\partial^2 g(\mathbf{x})}{\partial x_1\, \partial x_2} + \underbrace{0}_{\alpha_{2,2}} \frac{\partial^2 g(\mathbf{x})}{\partial x_2^2}
\end{aligned}
$$

| $n$ / $m$ $\alpha_{m,n}$ | 0 | 1 | 2 |
|---|---|---|---|
| 1 | 1 | 1 | |
| 2 | 0 | 0 | 0 |

(b) Give all nontrivial coefficients $a_{m,n}(\theta)$ that must be applied in order to steer $h$.

**Solution:**

$$\left(f * h(\mathbf{R}_\theta \cdot)\right)(\mathbf{x}) = \sum_{m=1}^{2} \sum_{n=0}^{m} a_{m,n}(\theta) \left(f * \frac{\partial^m}{\partial x_1^{m-n}\, \partial x_2^n} g\right)(\mathbf{x})$$

$$\underbrace{\hat{f}(\boldsymbol{\omega})}_{\neq 0}\, \hat{h}(\mathbf{R}_\theta\, \boldsymbol{\omega}) = \sum_{m=1}^{2} \sum_{n=0}^{m} a_{m,n}(\theta)\, \underbrace{\hat{f}(\boldsymbol{\omega})}_{\neq 0}\, (\mathrm{j}\,\omega_1)^{m-n}\, (\mathrm{j}\,\omega_2)^n\, \hat{g}(\boldsymbol{\omega})$$

$$\begin{aligned}
\hat{h}(\mathbf{R}_\theta\, \boldsymbol{\omega}) &= \hat{h}(\omega_1 \cos(\theta) - \omega_2 \sin(\theta), \omega_1 \sin(\theta) + \omega_2 \cos(\theta)) \\
&= \mathrm{j}\, (\omega_1 \cos(\theta) - \omega_2 \sin(\theta))\, \underbrace{\hat{g}(\mathbf{R}_\theta\, \boldsymbol{\omega})}_{\hat{g}(\boldsymbol{\omega})} + \mathrm{j}\, (\omega_1 \sin(\theta) + \omega_2 \cos(\theta))\, \underbrace{\hat{g}(\mathbf{R}_\theta\, \boldsymbol{\omega})}_{\hat{g}(\boldsymbol{\omega})} \\
&= \underbrace{(\cos(\theta) + \sin(\theta))}_{\sqrt{2}\, \sin(\frac{\pi}{4} + \theta)}\, (\mathrm{j}\,\omega_1)\, \hat{g}(\boldsymbol{\omega}) + \underbrace{(-\sin(\theta) + \cos(\theta))}_{\sqrt{2}\, \sin(\frac{\pi}{4} - \theta)}\, (\mathrm{j}\,\omega_2)\, \hat{g}(\boldsymbol{\omega}) \\
&\overset{\hat{f}(\boldsymbol{\omega}) \neq 0}{=} \sum_{m=1}^{2} \sum_{n=0}^{m} a_{m,n}(\theta)\, (\mathrm{j}\,\omega_1)^{m-n}\, (\mathrm{j}\,\omega_2)^n\, \hat{g}(\boldsymbol{\omega}) \\
&= \underbrace{a_{1,0}(\theta)}_{\sqrt{2}\, \sin(\frac{\pi}{4} + \theta)}\, (\mathrm{j}\,\omega_1)\, \hat{g}(\boldsymbol{\omega}) + \underbrace{a_{1,1}(\theta)}_{\sqrt{2}\, \sin(\frac{\pi}{4} - \theta)}\, (\mathrm{j}\,\omega_2)\, \hat{g}(\boldsymbol{\omega}) \\
&\quad + \underbrace{a_{2,0}(\theta)}_{0}\, (\mathrm{j}\,\omega_1)^2\, \hat{g}(\boldsymbol{\omega}) + \underbrace{a_{2,1}(\theta)}_{0}\, (\mathrm{j}\,\omega_1)\, (\mathrm{j}\,\omega_2)\, \hat{g}(\boldsymbol{\omega}) + \underbrace{a_{2,2}(\theta)}_{0}\, (\mathrm{j}\,\omega_2)^2\, \hat{g}(\boldsymbol{\omega})
\end{aligned}$$

| $n$ / $m$ $a_{m,n}(\theta)$ | 0 | 1 | 2 |
|---|---|---|---|
| 1 | $\sqrt{2}\, \sin(\frac{\pi}{4} + \theta)$ | $\sqrt{2}\, \sin(\frac{\pi}{4} - \theta)$ | |
| 2 | 0 | 0 | 0 |

(c) Take advantage of steering to summarize the spatial-domain convolutions needed to compute $(h_\theta * f)(\mathbf{x})$. The purpose of this summary is to highlight how many convolutions must be performed by the algorithm, and with precisely which filters $g_{m,n}$. It is not necessary that you further develop the expressions once that goal is reached.

**Solution:**

$$(h_\theta * f)(\mathbf{x}) = a_{1,0}(\theta)\, (g_{1,0} * f)(\mathbf{x}) + a_{1,1}(\theta)\, (g_{1,1} * f)(\mathbf{x})$$