Image Processing 1, Mock Midterm Exam

November 9, 2023

1 Fourier Transforms

(a) Compute the Fourier transform of

$$f(x,y) = x\operatorname{sinc}(y)\mathrm{e}^{-x^2+\mathrm{j}2\pi y}.$$

Solution: $f(x,y) = f_1(x)f_2(y)$, where $f_1(x) = x\mathrm{e}^{-x^2}$ and $f_2(y) = \operatorname{sinc}(y)\mathrm{e}^{\mathrm{j}2\pi y}$

$$\Rightarrow \hat{f}(\omega_x,\omega_y) = \hat{f}_1(\omega_x)\hat{f}_2(\omega_y)$$

Property	$g(x)$	$\hat{g}(\omega_x)$
1D-table	$\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}$	$\mathrm{e}^{-\frac{\omega_x^2}{2}}$
Linearity	$\mathrm{e}^{-\frac{x^2}{2}}$	$\sqrt{2\pi}\mathrm{e}^{-\frac{\omega_x^2}{2}}$
Scaling ($a = \frac{1}{\sqrt{2}}$)	e^{-x^2}	$\frac{\sqrt{2\pi}}{\sqrt{2}}\mathrm{e}^{-\frac{\omega_x^2}{4}}$
Multiplication by monomial	$x\mathrm{e}^{-x^2}$	$-\frac{\sqrt{\pi}}{2}\mathrm{j}\omega_x\mathrm{e}^{-\frac{\omega_x^2}{4}}$

Property	$h(y)$	$\hat{h}(\omega_y)$
1D-table	$\operatorname{sinc}(y)$	$\operatorname{rect}\left(\frac{\omega_y}{2\pi}\right)$
Modulation	$\operatorname{sinc}(y)\mathrm{e}^{\mathrm{j}2\pi y}$	$\operatorname{rect}\left(\frac{\omega_y-2\pi}{2\pi}\right)$

Hence,

$$\hat{f}(\omega_x,\omega_y) = -\frac{\sqrt{\pi}}{2}\mathrm{j}\omega_x\mathrm{e}^{-\frac{\omega_x{}^2}{4}}\operatorname{rect}\left(\frac{\omega_y-2\pi}{2\pi}\right)$$

(b) Compute the inverse Fourier transform of

$$\hat{g}(\omega_x,\omega_y) = \mathrm{e}^{-\omega_x^2}\cos(100\omega_y).$$

Solution: $\hat{g}(\omega_x,\omega_y) = \hat{g_1}(\omega_x)\hat{g_2}(\omega_y)$, where $\hat{g_1}(\omega_x) = \mathrm{e}^{-\omega_x^2}$ and $\hat{g_2} = \cos(100\omega_y)$

$$\Rightarrow g(x,y) = g_1(x)g_2(y)$$

Property	$g(x)$	$\hat{g}(\omega_x)$
1D-table	$\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}$	$\mathrm{e}^{-\frac{\omega_x^2}{2}}$
Scaling ($a = \sqrt{2}$)	$\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{4}}$	$\sqrt{2}\mathrm{e}^{-\omega_x^2}$
Linearity	$\frac{1}{\sqrt{4\pi}}\mathrm{e}^{-\frac{x^2}{4}}$	$\mathrm{e}^{-\omega_x^2}$

Property	$h(y)$	$\hat{h}(\omega_y)$
1D-table	$\cos(100y)$	$\pi\left(\delta\left(\omega_y+100\right)+\delta\left(\omega_y-100\right)\right)$
Duality	$\pi\left(\delta\left(y+100\right)+\delta\left(y-100\right)\right)$	$2\pi\cos(-100\omega_y)$
Linearity	$\frac{1}{2}\left(\delta\left(y+100\right)+\delta\left(y-100\right)\right)$	$\cos(100\omega_y)$

Hence,

$$g(x,y) = \frac{1}{4\sqrt{\pi}}\mathrm{e}^{-\frac{x^2}{4}}\left(\delta\left(y+100\right)+\delta\left(y-100\right)\right)$$

(c) Using Parseval's formula, show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-2x^2+\mathrm{j}2\pi y} x\,\mathrm{sinc}(y)\cos(100y)\mathrm{d}x\mathrm{d}y = 0.$$

Solution:

$$A = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{\mathrm{e}^{-2x^2+\mathrm{j}2\pi y} x\,\mathrm{sinc}(y)\cos(100y)}_{=f(x,y)\hat{g}(x,y)} \mathrm{d}x\mathrm{d}y$$

where

$$\begin{cases} f(x,y) = x\,\mathrm{sinc}(y)\mathrm{e}^{-x^2+\mathrm{j}2\pi y} & \text{(Q1-a)} \\ \hat{g}(x,y) = \mathrm{e}^{-x^2}\cos(100y) & \text{(Q1-b)} \end{cases}$$

$$\Rightarrow A = \langle f, \hat{g}\rangle \overset{\text{Parseval}}{=} \frac{1}{(2\pi)^2}\left\langle \hat{f}, \mathcal{F}\{\hat{g}\}\right\rangle \overset{\text{Duality}}{=} \frac{1}{(2\pi)^2}\left\langle \hat{f}, (2\pi)^2 g(-\cdot)\right\rangle \overset{g \text{ is even}}{=} \left\langle \hat{f}, g\right\rangle$$

$$\Rightarrow A = \left(\int_{-\infty}^{\infty} -\frac{\sqrt{\pi}}{2}\mathrm{j}x\mathrm{e}^{-\frac{x^2}{4}}\frac{1}{4\sqrt{\pi}}\mathrm{e}^{-\frac{x^2}{4}}\mathrm{d}x\right)\left(\int_{-\infty}^{\infty}\mathrm{rect}\left(\frac{y-2\pi}{2\pi}\right)\left(\delta\left(y+100\right)+\delta\left(y-100\right)\right)\mathrm{d}y\right)$$

$$= \left(-\frac{\mathrm{j}}{8}\int_{-\infty}^{\infty} x\mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x\right)\cdot\left(\underbrace{\mathrm{rect}\left(\frac{-100-2\pi}{2\pi}\right)}_{=0}+\underbrace{\mathrm{rect}\left(\frac{100-2\pi}{2\pi}\right)}_{=0}\right) = 0$$

2 Image Acquisition

A camera's acquisition system is modeled by the following block-diagram:

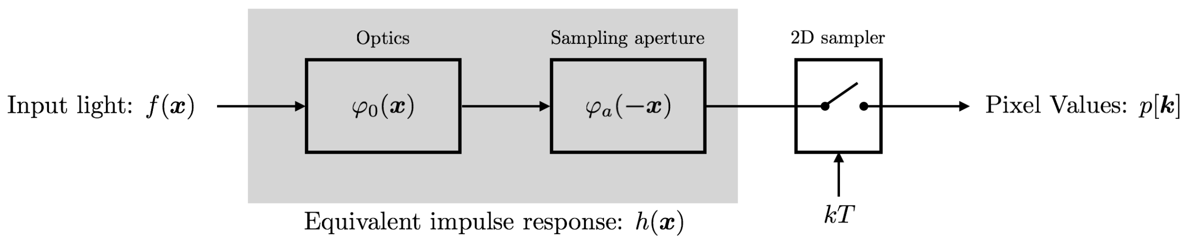


Figure 1: An image-acquisition system

In this system, we set

$$\widehat{\varphi_0}(\omega_x,\omega_y) = \mathrm{tri}\left(\frac{\omega_x}{\omega_0}\right)\mathrm{tri}\left(\frac{\omega_y}{\omega_0}\right),$$

and

$$\varphi_a(x,y) = \frac{\omega_0^2}{4\pi^2}\,\mathrm{sinc}\left(\frac{\omega_0 x}{2\pi}\right)\mathrm{sinc}\left(\frac{\omega_0 y}{2\pi}\right).$$

(a) Compute the Fourier transform of the equivalent impulse response $h(x,y)$ of the camera.

Solution:

$$\widehat{h}(\omega_x,\omega_y) = \mathcal{F}\{h(x,y)\} = \mathcal{F}\{\varphi_0 * \varphi_a(-\cdot)\} = \widehat{\varphi_0}(\omega_x,\omega_y)\widehat{\varphi_a}(-\omega_x,-\omega_y)$$

$$\left.\begin{array}{l}\mathrm{sinc}(x) \xrightarrow{\mathcal{F}} \mathrm{rect}\left(\frac{\omega_x}{2\pi}\right)\\ \mathrm{sinc}\left(\frac{\omega_0}{2\pi}x\right) \xrightarrow{\mathcal{F}} \frac{2\pi}{\omega_0}\mathrm{rect}\left(\frac{\omega_x}{\omega_0}\right)\end{array}\right\} \Rightarrow \widehat{\varphi_a}(\omega_x,\omega_y) = \frac{\omega_0^2}{4\pi^2}\left(\frac{2\pi}{\omega_0}\mathrm{rect}\left(\frac{\omega_x}{\omega_0}\right)\right)\left(\frac{2\pi}{\omega_0}\mathrm{rect}\left(\frac{\omega_y}{\omega_0}\right)\right)$$

$$\Rightarrow \widehat{h}(\omega_x,\omega_y) = \mathrm{tri}\left(\frac{\omega_x}{\omega_0}\right)\mathrm{tri}\left(\frac{\omega_y}{\omega_0}\right)\mathrm{rect}\left(\frac{\omega_x}{\omega_0}\right)\mathrm{rect}\left(\frac{\omega_y}{\omega_0}\right)$$

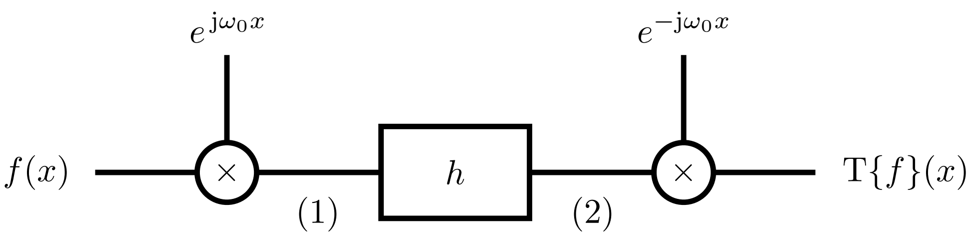
(b) Given a signal that has passed through the camera, what is the maximum step T that can be used to sample it without aliasing.

Solution:

$$\omega_{\max} = \frac{\omega_0}{2} \Rightarrow T_{\max} = \frac{\pi}{\omega_{\max}} = \frac{2\pi}{\omega_0}$$

3 Modulation

Consider the system T specified by the following block diagram.



In this system, we set $\hat{h}(\omega) = \mathrm{tri}(\omega/\pi)$ and $\omega_0 = \pi$. Furthermore, suppose that the input $f(x)$ to the system has Fourier transform $\hat{f}$ that is supported on $[-\pi, \pi]$.

(a) On what frequencies is the system supported on at (1) with input f?

(b) On what frequencies is the system supported on at (2) with input f?

(c) On what frequencies is the output of the system $\mathrm{T}\{f\}$ supported on?

(d) Can you propose an equivalent implementation of T as an LSI filter? If yes, (i) give an explicit formula for the frequency response of the corresponding filter (ii) draw a picture of the frequency response and (iii) specify if the filter is all-pass (AP), low-pass (LP), band-pass (BP), high-pass (HP), band-cut (BC), or all-cut (AC). If no, explain why.

Solution:

(a) Since $\hat{f}$ is supported on $[-\pi, \pi]$, after the modulation $e^{\mathrm{j}\pi x}$, the frequency support will be $[0, 2\pi]$.

(b) Since $\hat{h}$ is supported on $[-\pi, \pi]$, after the convolution with h, the frequency support will be $[0, \pi]$.

(c) After the (de)modulation $e^{-\mathrm{j}\pi x}$, the output of the system will be supported on $[-\pi, 0]$.

(d) Notice that in the Fourier domain, the system is implemented as

$$\hat{h}(\omega + \pi)\hat{f}(\omega)$$

Therefore, T is specified by the frequency response $\mathrm{tri}(\omega/\pi + 1)$. The picture would be a shifted triangle. Thus, this is a bandpass filter.