Learning for Adaptive and Reactive Robot Control
Solutions for theoretical exercises of lecture 4

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Introduction

INTRO

This part of the course follows *exercises* 3.2*,* 3.3 *and* 3.5 and *programming exercises* 3.1 *to* 3.4 of the book "*Learning for Adaptive and Reactive Robot Control: A Dynamical Systems Approach. MIT Press, 2022*".

1 Theoretical exercises [1h]

1.1

Design a matrix $A \in \mathbb{R}^{2\times 2}$ and Lyapunov function shaping matrix $P \in \mathbb{R}^{2\times 2}$ to ensure that a linear dynamical system (DS),

$$\dot{x} = f(x) = A(x - x^*)$$

with another attractor at the origin $x^* = [0\ 0]^T$ to be globally, asymptotically stable (GAS) with respect to the conditions stated using either:

(a) A matrix $Q \in \mathbb{R}^{2\times 2}$ with the following form:

$$Q = q\,\mathbb{I}_2, \quad q \in \mathbb{R}$$

Solution: Remember that, the system is GAS if

$$A^T P + PA = Q, \quad P = P^T \succ 0, \quad Q = Q^T \prec 0$$

with matrix A and P given as:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix},$$

The Q matrix is evaluated as

$$Q = \begin{bmatrix} 2a_{11}p_{11} + 2a_{21}p_{12} & a_{11}p_{12} + a_{12}p_{11} + a_{21}p_{22} + a_{22}p_{12} \\ a_{11}p_{12} + a_{12}p_{11} + a_{21}p_{22} + a_{22}p_{12} & 2a_{12}p_{12} + 2a_{22}p_{22} \end{bmatrix} \tag{1}$$

The diagonal Q matrix is negative definite if $q < 0$. The following the choice of values can achieve this:

$$p_{11} > 0,\ p_{22} = p_{11}\frac{a_{11}}{a_{22}},\ p_{12} = 0, \quad \text{and} \quad a_{11} < 0,\ a_{22} < 0,\ a_{12} = (-1)a_{21}\frac{p_{11}}{p_{22}}$$

We obtain:

$$Q = \begin{bmatrix} 2a_{11}p_{11} & 0 \\ 0 & 2a_{11}p_{11} \end{bmatrix}$$

(b) *Optional* A matrix $Q \in \mathbb{R}^{2\times 2}$ with the following form:

$$Q = \begin{bmatrix} q_1 & q_2 \\ q_2 & q_1 \end{bmatrix}, \quad q_1, q_2 \in \mathbb{R}$$

Solution: The matrix Q is negative definite if the eigenvalues are smaller than zero, i.e.,

$$\lambda_{1,2} = q_1 \pm q_2 < 0 \quad \Rightarrow \quad q_1 < 0,\ |q_2| > q_1$$

We propose similar values, but only soften the constraints for a_{12}, a_{21}:

$$|a_{12}p_{11} + a_{21}p_{22}| > 2a_{11}p_{11}$$

1.2

Consider a DS of the form

$$\dot{x} = \sum_{k=1}^{K} \gamma_k(x)(A^k x + b^k)$$

with $N = 2$, $K = 2$, and $\gamma_1(x) = 0.7$ and $\gamma_2(x) = 0.3$. Design nondiagonal matrices $A^1 \in \mathbb{R}^{2\times 2}$ and $A^2 \in \mathbb{R}^{2\times 2}$ such that the DS is GAS at the origin - that is, $x^* = [0, 0]^T$, with a quadratic Lyapunov function (QLF).

Solution: The condition to have an attractor at $x = [0, 0]^T$ is that $\dot{x} = \sum_{k=1} \gamma_k(x) b^k = 0$. It is sufficient to have

$$A^k x + b^k = 0\, \forall\, k \Rightarrow b^k = 0$$

Since the sum of two negative definite matrices, leads to a negative definite matrix. Any choice of two matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

fulfills the condition as long as the eigenvalues are negative, i.e.,

$$\lambda_{1,2} = \frac{a_{11} + a_{22}}{2} \pm \frac{\sqrt{a_{11}^2 - 2a_{11}a_{22} + 4a_{12}a_{21} + a_{22}^2}}{2} > 0$$

1.3

Optional: To be done at home.

Consider the nonlinear DS

$$\dot{x} = \mathbb{E}\{p(\dot{x}|x)\} = \sum_{k=1}^{K} \gamma_k(x)(A^k x + b^k)$$

with $N = 1$, $K = 4$. Design the Gaussian mixture regression (GMR) parameters $\Theta_{GMR} = \{\pi_k, \mu^k, \Sigma^k\}_{k=1}^{2}$ that will produce a GAS DS at the target $x^* = 0$ such that:

- If $x = -1$, then $\dot{x} = 0.5$; and when $x = -2$, then $\dot{x} = 2$.
- If $x = +1$, then $\dot{x} = -0.5$; and when $x = +2$, then $\dot{x} = -2$.

Solution: Remember that the GMR uses multivariate Gaussian (or normal) distribution $\mathbb{N}(\cdot)$ to evaluate the probability and hence its weight:

$$\gamma_k(x) = \frac{\pi_k p(x|\mu_x^k, \Sigma_x^k)}{\sum_{i=1}^{K} \pi_i p(x|\mu_x^i, \Sigma_x^i)} \qquad \text{with} \quad p(x|\mu_x^k, \Sigma_x^k) = \mathbb{N}(\cdot|\mu^k, \Sigma^k)$$

Our system has two dimension, position x and velocity $\dot{x}$, hence we have the state vector

$$X = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}.$$

Since we have four data points and four Gaussians, we propose to center one Gaussian on each data point, such that it dominates the evaluation there. The resulting is mean (center) of the Gaussians are chosen as:

$$\mu^1 = \begin{bmatrix} -1 \\ 0.5 \end{bmatrix}, \ \mu^2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \ \mu^3 = \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}, \ \mu^4 = \begin{bmatrix} 2 \\ -2 \end{bmatrix},$$

Conversely, the Covariance matrices Σ^k are chosen uniform and with a small variance in all directions. They are selected to be equal for all Gaussians.

$$\Sigma^1 = \Sigma^2 = \Sigma^3 = \Sigma^4 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

Similarly, for the priors π^k. Moreover, they have to sum up to 1:

$$\pi^1 = \pi^2 = \pi^3 = \pi^4 = 1/4$$