Learning for Adaptive and Reactive Robot Control
Instructions for exercises of lecture 3

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1 Exercise 1 - Dynamical Systems and Stability

1.1 Exercise 1.1

Consider a 2 dimensional linear DS, $\dot{x} = Ax$, with $x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We wish to introduce a modulation matrix M to modify the dynamics as follows: $\dot{x} = MAx$. Given $M = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$,

1. Find a diagonal matrix $A = \textbf{diag}(a_1, a_2)$, with $a_1 \neq a_2$ for which the system converges to $x^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
2. Compute the path integral of the modulated DS

1.2 Exercise 1.2

Consider two variables x and y coupled with the following dynamics

$$\dot{x} = \beta x, \quad \beta \in \mathbb{R}$$
$$\dot{y} = -y + \alpha x, \quad \alpha \in \mathbb{R}$$

Answer the following:

1. Does this system have a fixed point? What is it?
2. For what values of α and β is the system stable at the fixed point?
3. For what values of α and β is the system unstable at the fixed point?

1.3 Exercise 1.3

Consider a Lyapunov function $V(x) = x_1^2 + x_2^2$ for the following DS

$$\dot{x}_1 = -x_1 + x_1 x_2, \quad \dot{x}_2 = -x_2$$

1. Find the fixed point
2. Find a region of attraction and show that the fixed point is asymptotically stable

1.4 Exercise 1.4 (Bonus)

Consider the pendulum DS without friction

$$\ddot{\theta} = -g\sin(\theta)$$

1. Write down a state space representation using variable $x = (x_1, x_2)$.
2. As $x = (0,0)$ is a Lyapunov-stable fixed point, there exists a $V(x)$ such that:

$$V(0,0) = 0$$
$$V(x) > 0, \dot{V}(x) \leq 0 \quad \forall x \neq (0,0)$$

   Furthermore, from mechanical intuition, we knew that the pendulum without friction is energy conservative, therefore, we hypothesise that there exists $V(x)$ with $\dot{V} = 0$.

   (a) Expand $\dot{V}(x(t))$ and obtain a partial differential equation (PDE) in x_1 and x_2 that satisfies $\dot{V}(x) = 0$,

   (b) Solve the PDE to find $V(x)$.

Now consider the pendulum with friction

$$\ddot{\theta} = -g\sin(\theta) - \dot{\theta}$$

1. Conclude that $x = (0,0)$ is stable with the previously obtained $V(x)$.
2. Show that the only solution of the DS in the set $S = \{x : \dot{V}(x) = 0\}$ is $x(t) = (0,0)\ \forall t$ and conclude that $x = (0,0)$ is asymptotically stable by La Salle's Invariance principle.

2 Exercise 2: Dynamical System Implementation

Book correspondence: exercise 1.3, and Appendix A
The aim of this exercise is to familiarize the reader with dynamical systems. Open `appA_ex1.m` *and edit the code to do the following:*

Implement three dynamical systems:

1. A linear DS such as given in Book exercise 1.3

$$\dot{x} = A(x - x^*)$$

   Experiment with different x^* and A matrices ranging from stable to unstable behaviour. Modulate the DS according to the values in pen and paper exercise 1.1.

2. one limit cycle (for example, Van der Pol oscillator, see lecture slide 13)

$$\dot{x_1} = x_2$$
$$\dot{x_2} = \mu(1 - x_1^2)x_2 - x_1, \quad \mu > 0$$

   (Optional) Find a limit cycle in polar coordinates for stable motion on a unit circle.

3. the nonlinear DS given in Equation A.6 of the book.

$$\dot{x} = e^{(-\|x\|)}cos(x)$$

For each of these DS, do the following:

1. **Evaluate DS on grid**: Go over each point $\begin{pmatrix} x \\ y \end{pmatrix}$ to compute the velocity vector $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ and the velocity magnitude `abs_vel`. Each of these variable should be a 2D matrix of size `nb_gridpoints` by `nb_gridpoints`.

2. **Start from a random point and calculate path integral**: Compute the path integral using Euler integration method to fill the array `path_integral`.

3. **Plot DS and path integral**: Use the provided function `plot_ds` to compute your velocity field and path integral. Refer to the function description at the end of the file to use the right parameters.

References

[1] Aude Billard, Sina Mirrazavi, and Nadia Figueroa. *Learning for Adaptive and Reactive Robot Control: A Dynamical Systems Approach.* MIT press, 2022.