

Learning for Adaptive and Reactive Robot Control

Solutions for theoretical exercises of lecture 6

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Spring Semester

Introduction

INTRO

This part of the course follows *exercises 8.1 to 8.6* and *programming exercises 8.1 to 8.6* of the book "*Learning for Adaptive and Reactive Robot Control: A Dynamical Systems Approach. MIT Press, 2022*".

1 Theoretical exercises [1h]

1.1

Consider the nominal DS $\dot{x} = Ax$ with $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Construct a matrix $M(x)$ that is locally active and:

1. **Solution:** The final dynamics are given as

$$\dot{\hat{x}} = M(x)Ax$$

The system is moving away from the attractor, if any eigenvalue of $(M(x)A)$ is greater than zero. A possible choice is

$$M(x) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

2. **Solution** The attractor becomes a saddle point if one of the eigenvalues is positive and one negative, for example,

$$M(x) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. create a limit cycle around the attractor, (a) keep the system stable at the attractor or (b) make the system diverge from the attractor and converge to the limit cycle (see Fig. 2)

Solution: The limit cycle around the attractor with radius r can be formed by locally rotating the dynamical system.

$$M(x) = R(\phi) = \begin{bmatrix} \cos(\phi(x)) & -\sin(\phi(x)) \\ \sin(\phi(x)) & \cos(\phi(x)) \end{bmatrix}$$

to form a limit cycle, we need a rotation of at least $|\phi(x)| = \pi/2$, where π is the circle constant.

(a) to be stable at the attractor, we set:

$$\phi(x) = \begin{cases} \frac{\pi}{2} \sin(\|x\|\pi/2r) & \text{if } \|x\| < 2r \\ 0 & \text{otherwise} \end{cases}$$

(b) to move away from the attractor and converge to the limit cycle we set:

$$\phi(x) = \max\left(\pi\left(1 - \frac{\|x\|}{2r}\right), 0\right)$$

4. invert the direction of the initial dynamics at a fixed point x^* .

Make sure your modulation matrix is smooth.

The flow is inverted if we use a modulation matrix of the form:

$$M(x) = \begin{bmatrix} 1 - \gamma & 0 \\ 0 & 1 - \gamma \end{bmatrix} \quad \text{with } \gamma = c^1 \exp\left(-\frac{1}{\sigma^2}\|x - x^*\|\right)$$

where $c^1 = 4$. This forces the dynamical system to flip in the surrounding region.

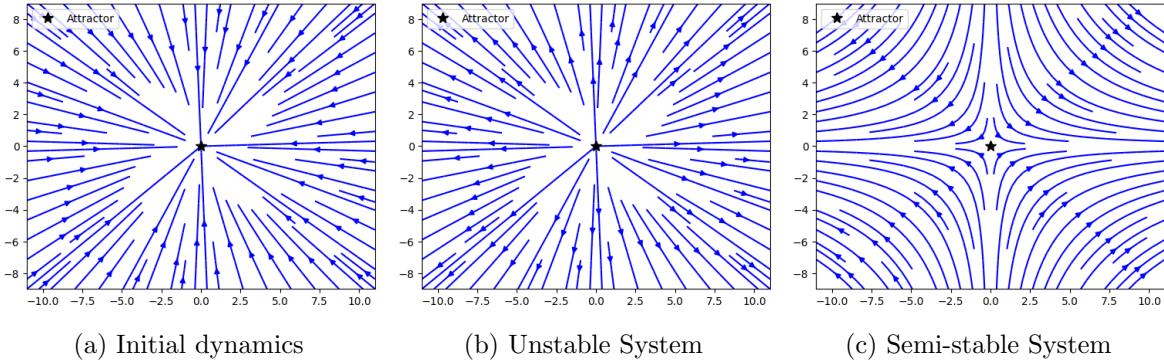


Figure 1: The different vector fields of the modulations.

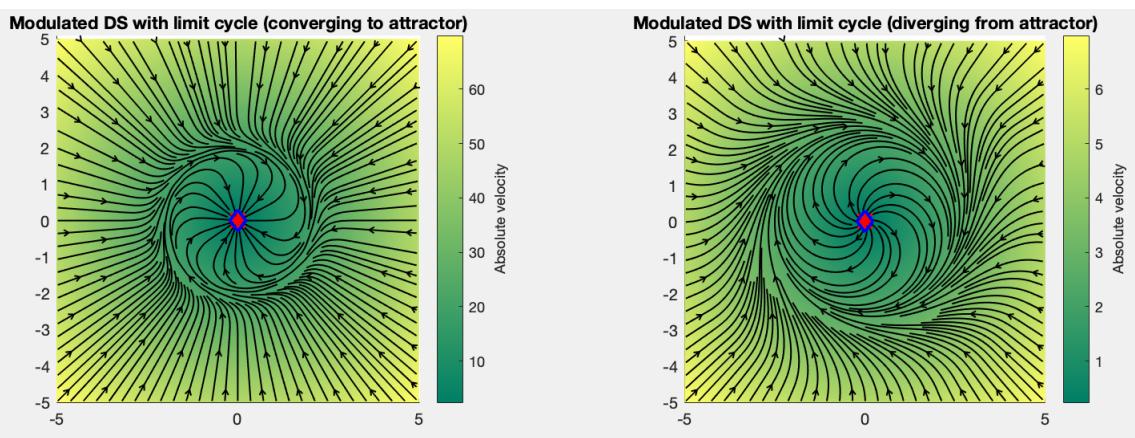


Figure 2: Two limit cycles at the attractor

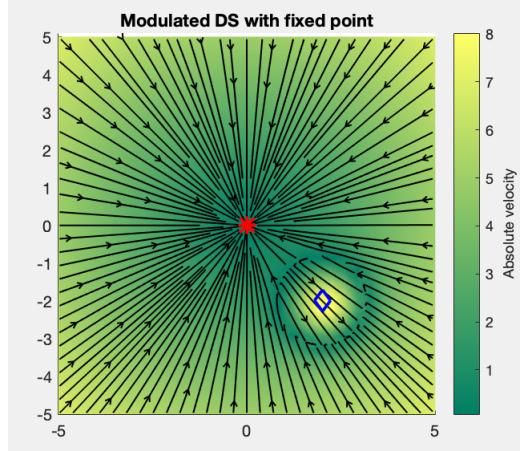


Figure 3: Locally inverted flow.

1.2

Show that if $M(x)$ is full rank for all x , the modulated dynamics has the same equilibrium point as the nominal dynamics.

Solution: If $M(x)$ has full rank, it has an empty null-space, and hence

$$x = g(x) = M(x)f(x)$$

is zero iff $f(x) = 0$.

1.3

(Optional)

Show that if the nominal dynamics is bounded and $M(x)$ is locally active in a compact-set $\chi \subset \mathbb{R}^N$. Then the modulated dynamics is bounded.

Solution:

Let B_R be a ball centered at the origin of radius R in \mathbb{R}^N . Let R be chosen such that χ lies entirely in B_R . Since χ is a compact set in \mathbb{R}^N , it is always possible to find such a R . For each $\delta > 0$, let $\epsilon(\delta) > 0$ be an associated boundary for the original dynamics. Define $\hat{\epsilon}(\delta)$ as a boundary for the reshaped dynamics as follows: $\hat{\epsilon} = \epsilon(R)$ for $\delta < R$ and $\hat{\epsilon} = \epsilon(\delta)$ for $\delta \geq R$.

1.4

(Optional)

Consider a system $\dot{x} = f(x)$ that has a single equilibrium point. Without loss of generality, let this equilibrium point be placed at the origin. Assume further that the equilibrium point is stable and the modulated dynamics are bounded and have the same equilibrium point as the nominal dynamics. Show that if χ does not include the origin, the modulated system is stable at the origin.

Solution: The reshaped dynamics has a single equilibrium point at the origin. Let B_r be a ball centered at the origin with a radius r small enough that B_r does not include any point in χ . Hence, inside B_r , we have $g(x) = f(x)$. By the stability of f , there exists for all $0 < \epsilon \leq r$ a $\delta(\epsilon)$ such that

$$\|x(0)\| < \delta(\epsilon) \Rightarrow \|x(t)\| < \epsilon \quad \forall t > 0$$

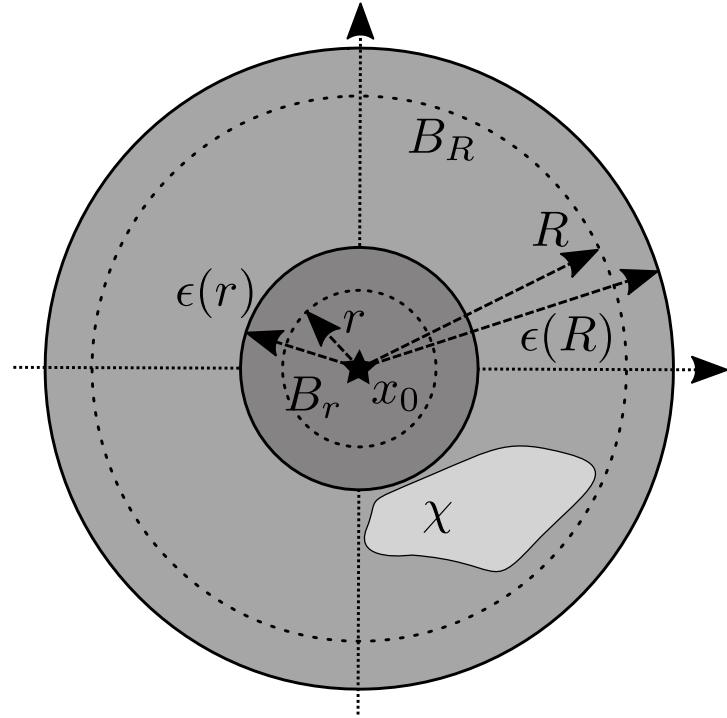


Figure 4: Sets of stability.

For any $\epsilon > r$, let $\delta(\epsilon) = \delta(r)$. Then, by the stability of f ,

$$\|x(0)\| < \delta(\epsilon) = \delta(r) \Rightarrow \|x(t)\| < r < \epsilon$$

References

[1] Aude Billard, Sina Mirrazavi, and Nadia Figueroa. *Learning for Adaptive and Reactive Robot Control: A Dynamical Systems Approach*. MIT press, 2022.