

Dynamic models

Examples

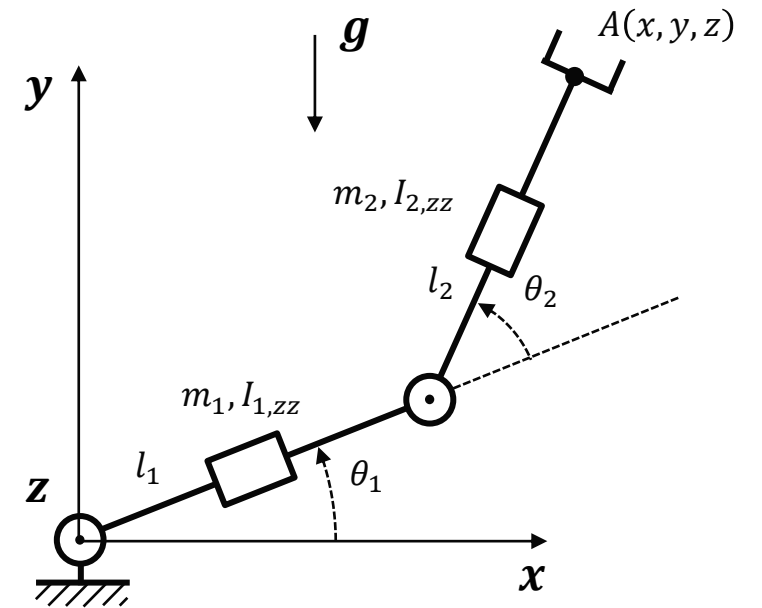
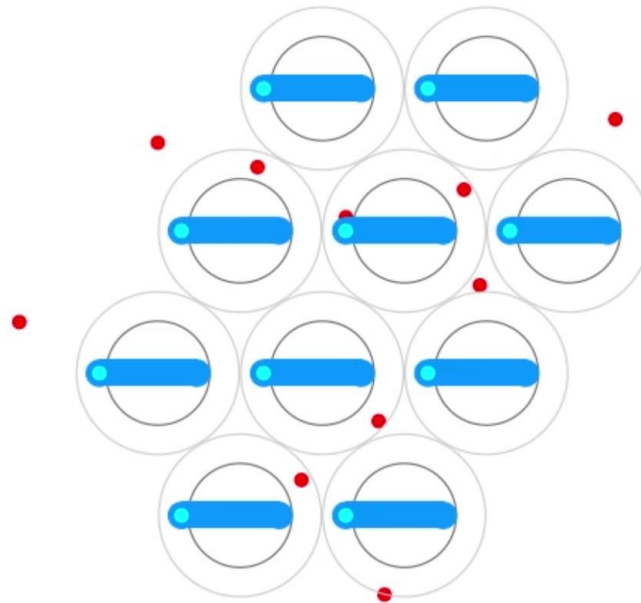
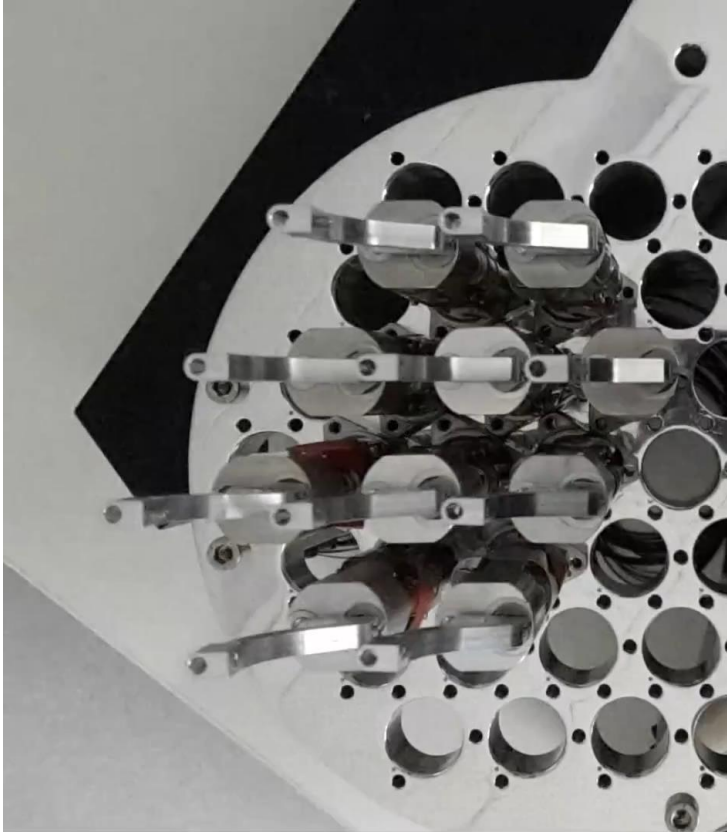
Approach of Lagrange - Summary :

1. Describe the robot-
 1. Identify the tool point (TCP) and the various joints.
 2. Identify the robot segments.
2. Define the basic frame of reference of the robot.
3. Define the positive and negative directions of each joint.
4. Write the local Jacobians related to each segment.
5. Deduce the inertia matrix $B(q)$ – slide 28
 1. From the inertial elements of each segment
 2. From the Jacobians related to the segments (pnt 4)
6. Deduce the different components of the dynamic equation (Coriolis, Centrifugal and Gravity), Slide 31

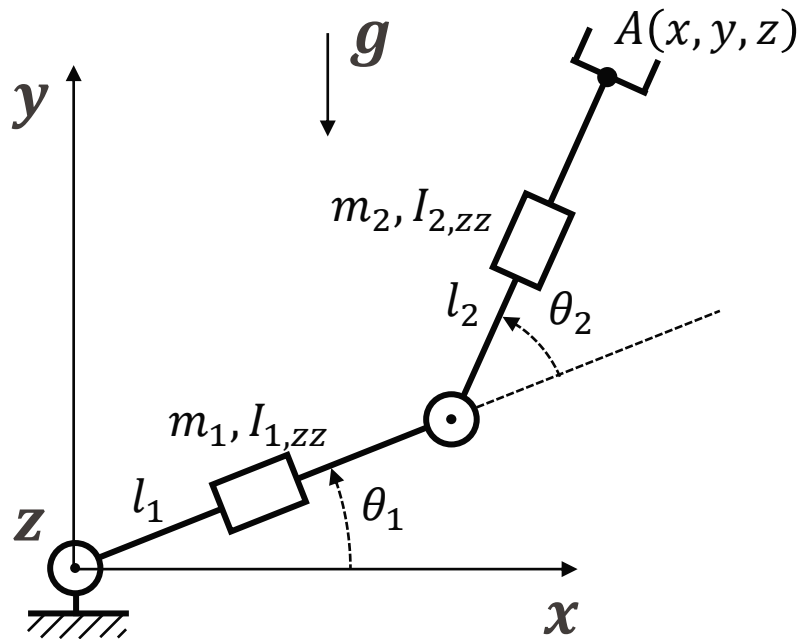
$$B(q) = \sum_{i=1}^n \left(m_i J_p^{(i)T} J_p^{(i)} + J_o^{(i)T} R_i I_i R_i^T J_o^{(i)} \right)$$

$$\Gamma_i = \underbrace{\sum_{j=1}^n b_{ij} \ddot{q}_j}_a - \underbrace{\sum_{j=1}^n \sum_{k=1}^n \left(\frac{\partial b_{ij}}{\partial q_k} - \frac{1}{2} \frac{\partial b_{jk}}{\partial q_i} \right) \dot{q}_k \dot{q}_j}_b + \underbrace{\sum_{j=1}^n m_j g J_{p,i}^j}_c$$

Example: Two-Link Planar Arm



Example: Two-Link Planar Arm RR



■ Hypotheses:

- No friction
- Neglect motors inertia
- Links are infinitely rigid
- The two arms have, respectively:
 - Lengths: l_1, l_2
 - Mass: m_1, m_2
 - Moments of inertia around the z axis relative to the arm's center of mass: $I_{1,zz}, I_{2,zz}$

■ Generalized coordinates:

$$q = [q_1 \quad q_2]^T = [\theta_1 \quad \theta_2]^T$$

■ Gravitational acceleration vector:

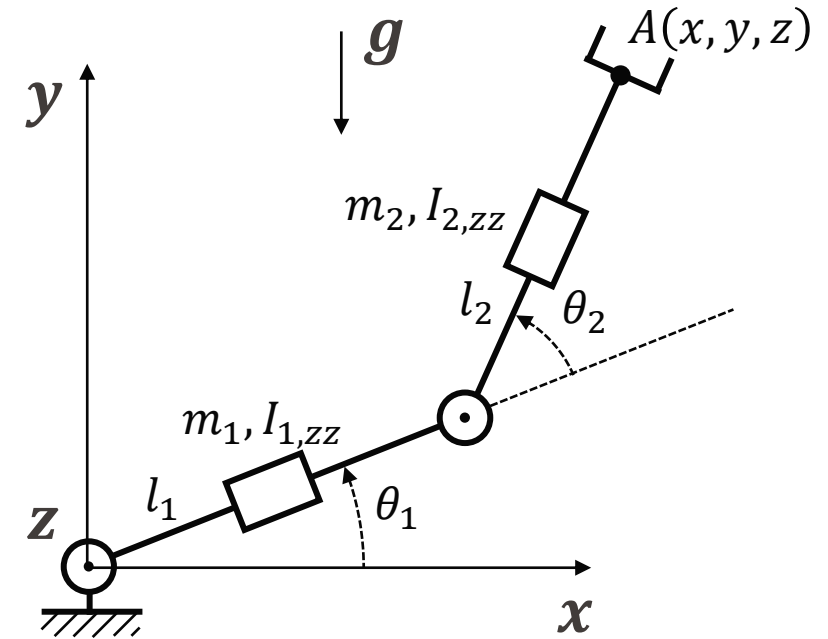
$$g = [0 \quad -g_0 \quad 0]^T$$

$$\Gamma = B(q)\ddot{q} + G(q) + C(q, \dot{q})\dot{q} + F(q, \dot{q})$$

- Γ : the generalized torque/joint torques
- $B(q)$: the inertia matrix
- $G(q)$: the gravity vector
- $C(q, \dot{q})$: the matrix of the centrifugal and Coriolis terms
- $F(q, \dot{q})$: the friction vector

Two-Link Planar Arm – Inertia matrix (1)

$$\mathbf{B}(\mathbf{q}) = \sum_{i=1}^n \left(m_i \mathbf{J}_P^{(i)T} \mathbf{J}_P^{(i)} + \mathbf{J}_O^{(i)T} \mathbf{R}_i \mathbf{I}_i^T \mathbf{R}_i \mathbf{J}_O^{(i)} \right)$$



- Local Jacobians reported to each segment:

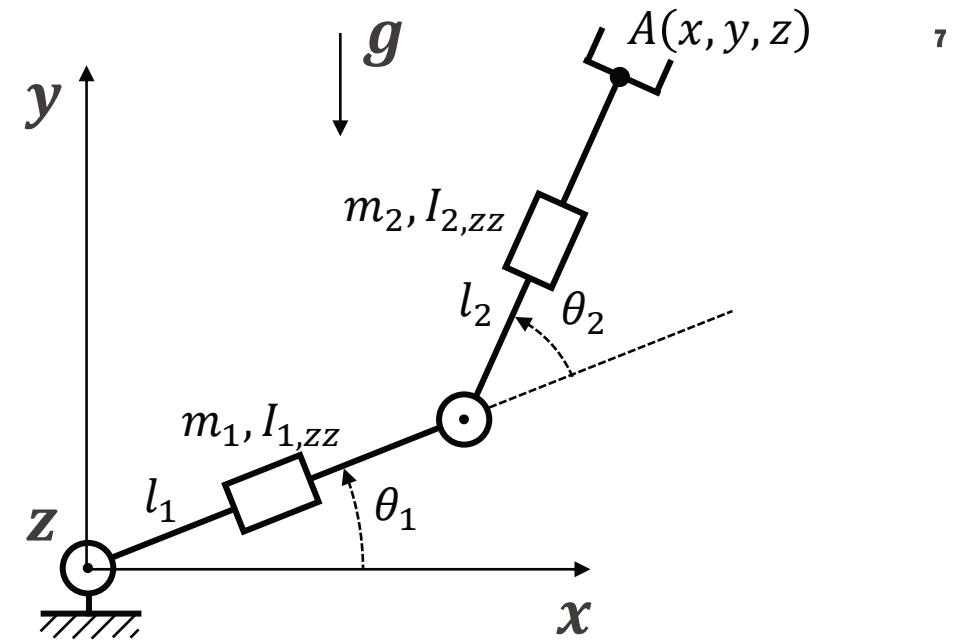
- Links in translation:

$$\mathbf{J}_P^{(1)} = \begin{bmatrix} -\frac{l_1}{2} s_1 & 0 \\ \frac{l_1}{2} c_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{J}_P^{(2)} = \begin{bmatrix} -l_1 s_1 - \frac{l_2}{2} s_{1+2} & -\frac{l_2}{2} s_{1+2} \\ l_1 c_1 + \frac{l_2}{2} c_{1+2} & \frac{l_2}{2} c_{1+2} \\ 0 & 0 \end{bmatrix}$$
- Links in rotation:

$$\mathbf{J}_O^{(1)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{J}_O^{(2)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Two-Link Planar Arm – Inertia matrix (2)

$$\mathbf{B}(\mathbf{q}) = \sum_{i=1}^n \left(m_i \mathbf{J}_P^{(i)T} \mathbf{J}_P^{(i)} + \mathbf{J}_O^{(i)T} \mathbf{R}_i \mathbf{I}_i^i \mathbf{R}_i^T \mathbf{J}_O^{(i)} \right)$$



- Link 1 for translation:

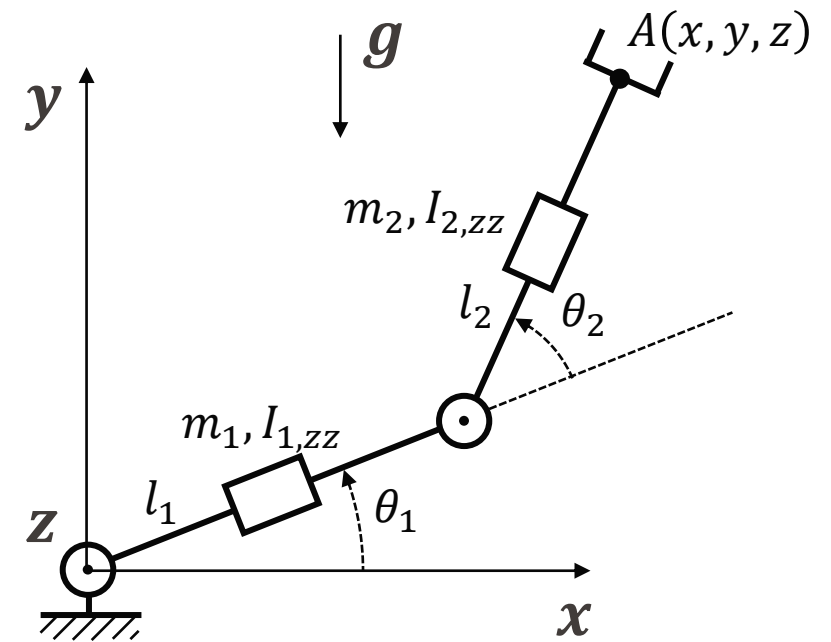
$$m_1 \mathbf{J}_P^{(1)T} \mathbf{J}_P^{(1)} = \begin{bmatrix} \frac{1}{4} m_1 l_1^2 & 0 \\ 0 & 0 \end{bmatrix}$$

- Link 2 for translation:

$$m_2 \mathbf{J}_P^{(2)T} \mathbf{J}_P^{(2)} = \frac{1}{4} m_2 \begin{bmatrix} l_2^2 + 4l_1^2 + 4l_1 l_2 c_2 & l_2^2 + 2l_1 l_2 c_2 \\ l_2^2 + 2l_1 l_2 c_2 & l_2^2 \end{bmatrix}$$

Two-Link Planar Arm – Inertia matrix (3)

$$\mathbf{B}(\mathbf{q}) = \sum_{i=1}^n \left(m_i \mathbf{J}_P^{(i)T} \mathbf{J}_P^{(i)} + \mathbf{J}_O^{(i)T} \mathbf{R}_i \mathbf{I}_i^i \mathbf{R}_i^T \mathbf{J}_O^{(i)} \right)$$



- Link 1 in rotation:

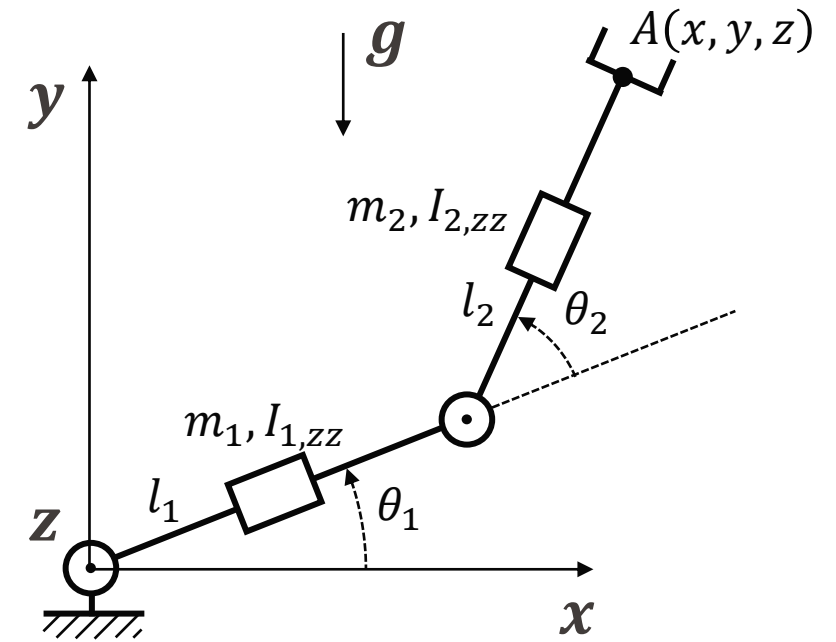
$$\mathbf{J}_O^{(1)T} \mathbf{I}_1 \mathbf{J}_O^{(1)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{1,xx} & I_{1,xy} & I_{1,xz} \\ I_{1,yx} & I_{1,yy} & I_{1,yz} \\ I_{1,zx} & I_{1,zy} & I_{1,zz} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} I_{1,zz} & 0 \\ 0 & 0 \end{bmatrix}$$

- Link 2 in rotation:

$$\mathbf{J}_O^{(2)T} \mathbf{I}_2 \mathbf{J}_O^{(2)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_{2,xx} & I_{2,xy} & I_{2,xz} \\ I_{2,yx} & I_{2,yy} & I_{2,yz} \\ I_{2,zx} & I_{2,zy} & I_{2,zz} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} I_{2,zz} & I_{2,zz} \\ I_{2,zz} & I_{2,zz} \end{bmatrix}$$

Two-Link Planar Arm – Inertia matrix (4)

$$\mathbf{B}(\mathbf{q}) = \sum_{i=1}^n \left(m_i \mathbf{J}_P^{(i)T} \mathbf{J}_P^{(i)} + \mathbf{J}_O^{(i)T} \mathbf{R}_i \mathbf{I}_i^i \mathbf{R}_i^T \mathbf{J}_O^{(i)} \right)$$



- Inertia matrix:

- $\mathbf{B}(\mathbf{q}) = m_1 \mathbf{J}_P^{(1)T} \mathbf{J}_P^{(1)} + \mathbf{J}_O^{(1)T} \mathbf{I}_1 \mathbf{J}_O^{(1)} + m_2 \mathbf{J}_P^{(2)T} \mathbf{J}_P^{(2)} + \mathbf{J}_O^{(2)T} \mathbf{I}_2 \mathbf{J}_O^{(2)} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

- $b_{11} = \frac{1}{4} m_1 l_1^2 + I_{1,zz} + \frac{1}{4} m_2 (l_2^2 + 4l_1^2 + 4l_1 l_2 c_2) + I_{2,zz}$
- $b_{12} = b_{21} = \frac{1}{4} m_2 (l_2^2 + 2l_1 l_2 c_2) + I_{2,zz}$
- $b_{22} = \frac{1}{4} m_2 l_2^2 + I_{2,zz}$

- b_{11} , b_{12} and b_{21} are configuration-dependent

Two-Link Planar Arm – Gravity vector

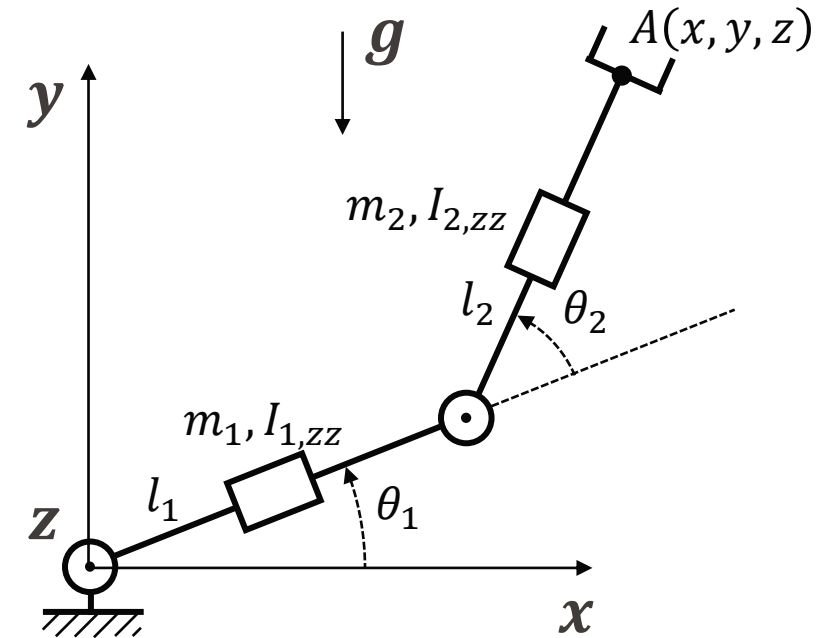
$$\Gamma = B(q)\ddot{q} + \mathbf{G}(q) + C(q, \dot{q})\dot{q} + F(q, \dot{q})$$

- The elements $g_i = \sum_{j=1}^n m_j \mathbf{g}^T \mathbf{J}_{P,i}^{(j)}$ of the gravity vector are:

$$\begin{aligned} \bullet \quad g_1 &= m_1 \mathbf{g}^T \mathbf{J}_{P,1}^{(1)} + m_2 \mathbf{g}^T \mathbf{J}_{P,1}^{(2)} = m_1 [0 \quad -g_0 \quad 0] \begin{bmatrix} -\frac{l_1}{2} s_1 \\ \frac{l_1}{2} c_1 \\ 0 \end{bmatrix} + \\ & m_2 [0 \quad -g_0 \quad 0] \begin{bmatrix} -l_1 s_1 - \frac{l_2}{2} s_{1+2} \\ l_1 c_1 + \frac{l_2}{2} c_{1+2} \\ 0 \end{bmatrix} = -\left(\frac{m_1}{2} + m_2\right) l_1 g_0 c_1 - \frac{1}{2} m_2 l_2 g_0 c_{1+2} \end{aligned}$$

$$\begin{aligned} \bullet \quad g_2 &= m_2 \mathbf{g}^T \mathbf{J}_{P,2}^{(1)} + m_2 \mathbf{g}^T \mathbf{J}_{P,2}^{(2)} = 0 + m_2 [0 \quad -g_0 \quad 0] \begin{bmatrix} -\frac{l_2}{2} s_{1+2} \\ \frac{l_2}{2} c_{1+2} \\ 0 \end{bmatrix} = \\ & -\frac{1}{2} m_2 l_2 g_0 c_{1+2} \end{aligned}$$

$$\mathbf{G}(q) = \begin{bmatrix} -\left(\frac{m_1}{2} + m_2\right) l_1 g_0 c_1 - \frac{1}{2} m_2 l_2 g_0 c_{1+2} \\ -\frac{1}{2} m_2 l_2 g_0 c_{1+2} \end{bmatrix}$$



Two-Link Planar Arm – Christoffel symbols (1)

$$\Gamma = B(q)\ddot{q} + G(q) + \mathbf{C}(q, \dot{q})\dot{q} + F(q, \dot{q})$$

- The elements c_{ij} of the matrix $\mathbf{C}(q, \dot{q})$ must satisfy the equation:

$$\sum_{j=1}^n c_{ij} \dot{q}_j = \sum_{j=1}^n \sum_{k=1}^n \left(\frac{\partial b_{ij}}{\partial q_k} - \frac{1}{2} \frac{\partial b_{ik}}{\partial q_i} \right) \dot{q}_k \dot{q}_j$$

- The choice of matrix $\mathbf{C}(q, \dot{q})$ is **not unique**, there exist several matrices that satisfy this equation.
- The generic element of $\mathbf{C}(q, \dot{q})$ that satisfy this equation is:

$$c_{ij} = \sum_{k=1}^n c_{ijk} \dot{q}_k$$

Where $c_{ijk} = \frac{1}{2} \left(\frac{\partial b_{ij}}{\partial q_k} + \frac{\partial b_{ik}}{\partial q_j} - \frac{\partial b_{jk}}{\partial q_i} \right)$ are called **Christoffel symbols of the first kind**. Also, as

$B(q)$ is symmetric, $c_{ijk} = c_{ikj}$.

Two-Link Planar Arm – Christoffel symbols (2)

$$\Gamma = B(q)\ddot{q} + G(q) + \mathbf{C}(q, \dot{q})\dot{q} + F(q, \dot{q})$$

- Reminder: the coefficients b_{11} , b_{12} and b_{21} of matrix $B(q)$ are configuration dependent:

- $$b_{11} = \frac{1}{4}m_1l_1^2 + I_{1,zz} + \frac{1}{4}m_2(l_2^2 + 4l_1^2 + 4l_1l_2c_2) + I_{2,zz}$$
- $$b_{12} = b_{21} = \frac{1}{4}m_2(l_2^2 + 2l_1l_2c_2) + I_{2,zz}$$

- We compute the Christoffel coefficients:

- $$c_{111} = \frac{1}{2} \left(\frac{\partial b_{11}}{\partial q_1} + \frac{\partial b_{11}}{\partial q_1} - \frac{\partial b_{11}}{\partial q_1} \right) = \frac{1}{2} \frac{\partial b_{11}}{\partial q_1} = 0$$
- $$c_{112} = c_{121} = \frac{1}{2} \frac{\partial b_{11}}{\partial q_2} = -\frac{1}{2}m_2l_1l_2s_2 = h(q_2)$$
- $$c_{122} = \frac{\partial b_{12}}{\partial q_2} - \frac{1}{2} \frac{\partial b_{22}}{\partial q_1} = -\frac{1}{2}m_2l_1l_2s_2 = h(q_2)$$
- $$c_{211} = \frac{\partial b_{21}}{\partial q_1} - \frac{1}{2} \frac{\partial b_{11}}{\partial q_2} = \frac{1}{2}m_2l_1l_2s_2 = -h(q_2)$$
- $$c_{212} = c_{221} = \frac{1}{2} \frac{\partial b_{22}}{\partial q_1} = 0$$
- $$c_{222} = \frac{1}{2} \frac{\partial b_{22}}{\partial q_2} = 0$$

Two-Link Planar Arm – C

Vector of centrifugal and Coriolis effect (2)

$$\Gamma = B(q)\ddot{q} + G(q) + \mathbf{C}(q, \dot{q})\dot{q} + F(q, \dot{q})$$

- Using the Christoffel symbols, we compute the terms of the matrix $\mathbf{C}(q, \dot{q})$:

- $c_{11} = \sum_{k=1}^2 c_{11k} \dot{q}_k = c_{111} \dot{q}_1 + c_{112} \dot{q}_2 = h(q_2) \dot{q}_2$
- $c_{12} = c_{121} \dot{q}_1 + c_{122} \dot{q}_2 = h(q_2)(\dot{q}_1 + \dot{q}_2)$
- $c_{21} = c_{211} \dot{q}_1 + c_{212} \dot{q}_2 = -h(q_2) \dot{q}_1$
- $c_{22} = c_{221} \dot{q}_1 + c_{222} \dot{q}_2 = 0$

$$\mathbf{C}(q, \dot{q}) = \begin{bmatrix} h(q_2) \dot{q}_2 & h(q_2)(\dot{q}_1 + \dot{q}_2) \\ -h(q_2) \dot{q}_1 & 0 \end{bmatrix}$$

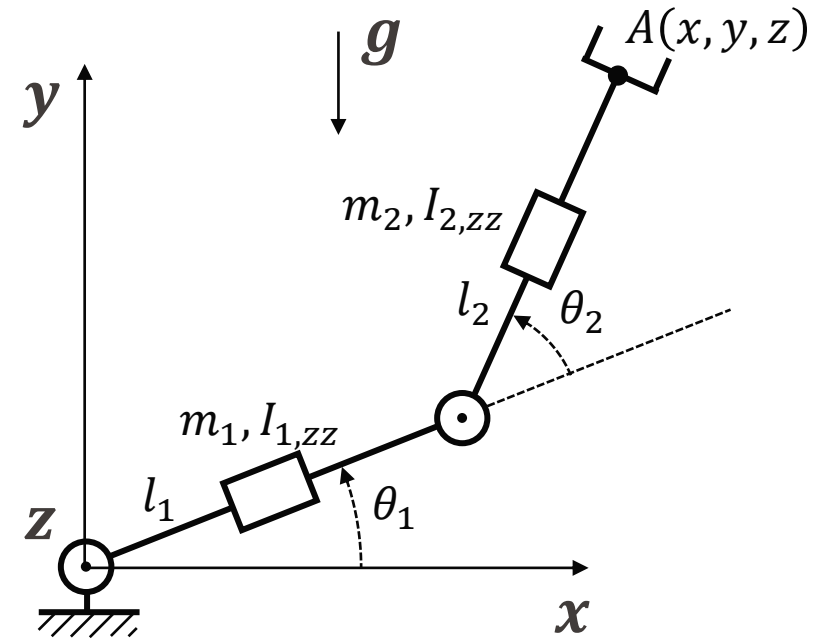
- Finally, we find the vector of centrifugal force and Coriolis terms:

$$\mathbf{C}(q, \dot{q})\dot{q} = \begin{bmatrix} h(q_2) \dot{q}_2 & h(q_2)(\dot{q}_1 + \dot{q}_2) \\ -h(q_2) \dot{q}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} h(q_2) \dot{q}_2^2 + 2h(q_2) \dot{q}_1 \dot{q}_2 \\ -h(q_2) \dot{q}_1^2 \end{bmatrix}$$

- $h(q_2) \dot{q}_2^2$, $-h(q_2) \dot{q}_1^2$ respectively represent the centrifugal effect induced on joint 1, 2 by the velocity of joint 2, 1
- $2h(q_2) \dot{q}_1 \dot{q}_2$ represent the Coriolis effect induced on joint 1 by the velocities of joints 1 and 2.

Two-Link Planar Arm – Generalized torque (IDM)

- Friction vector:
 - $F(q, \dot{q}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, we considered no friction.
- Generalized torque:



$$\Gamma = B(q)\ddot{q} + G(q) + C(q, \dot{q})\dot{q} + F(q, \dot{q})$$

- $$\Gamma_1 = b_{11}\ddot{q}_1 + b_{12}\ddot{q}_2 + g_1 + h(q_2)\dot{q}_2^2 + 2h(q_2)\dot{q}_1\dot{q}_2 = \left(\frac{1}{4}m_1l_1^2 + I_{1,zz} + \frac{1}{4}m_2(l_2^2 + 4l_1^2 + 4l_1l_2c_2) + I_{2,zz}\right)\ddot{q}_1 + \left(\frac{1}{4}m_2(l_2^2 + 2l_1l_2c_2) + I_{2,zz}\right)\ddot{q}_2 - \left(\frac{m_1}{2} + m_2\right)l_1g_0c_1 - \frac{1}{2}m_2l_2g_0c_{1+2} - m_2l_1l_2s_2\dot{q}_1\dot{q}_2 - \frac{1}{2}m_2l_1l_2s_2\dot{q}_2^2$$
- $$\Gamma_2 = b_{22}\ddot{q}_2 + b_{21}\ddot{q}_1 + g_2 - h(q_2)\dot{q}_1^2 = \left(\frac{1}{4}m_2l_2^2 + I_{2,zz}\right)\ddot{q}_2 + \left(\frac{1}{4}m_2(l_2^2 + 2l_1l_2c_2) + I_{2,zz}\right)\ddot{q}_1 - \frac{1}{2}m_2l_2g_0c_{1+2} + \frac{1}{2}m_2l_1l_2s_2\dot{q}_1^2$$

Two-Link Planar Arm – Generalized torque (IDM)

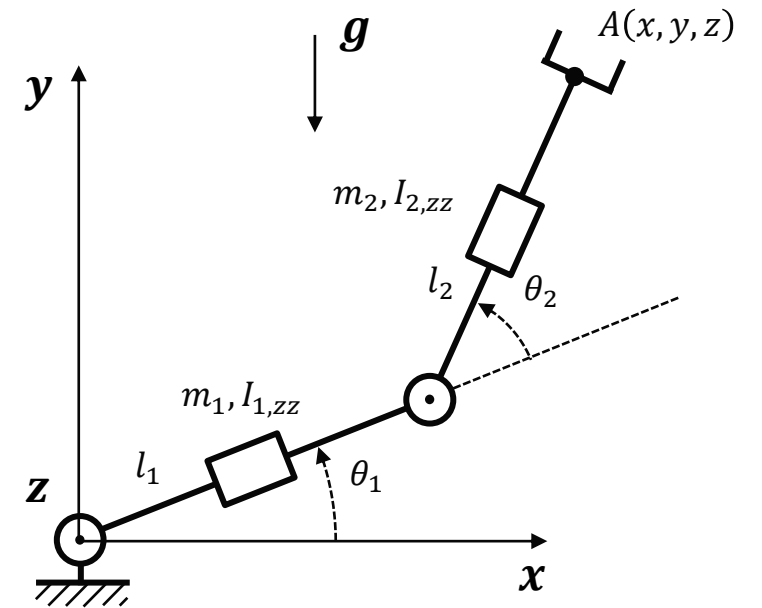
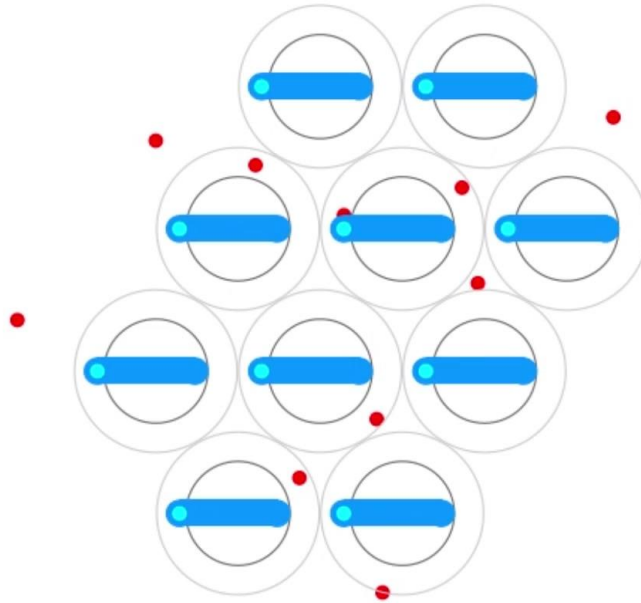
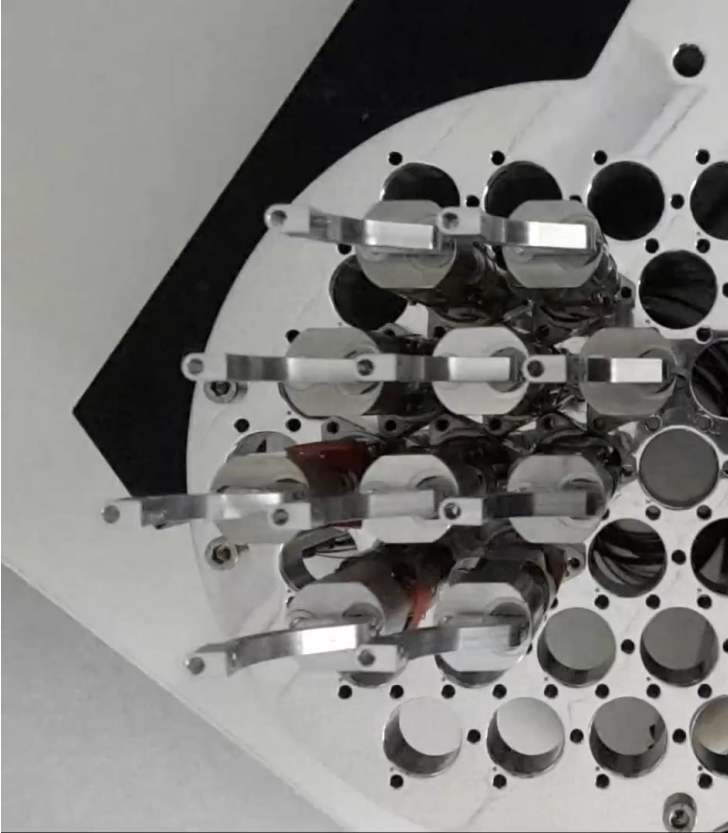
- $$\Gamma_1 = \left(\frac{1}{4} m_2 (l_2^2 + 2l_1 l_2 c_2) + I_{2,zz} \right) \ddot{q}_2 - \left(\frac{m_1}{2} + m_2 \right) l_1 g_0 c_1 - \frac{1}{2} m_2 l_2 g_0 c_{1+2} - \frac{1}{2} m_2 l_1 l_2 s_2 \dot{q}_2^2$$

$$c_{11} = \sum_{k=1}^2 c_{11k} \dot{q}_k = c_{111} \dot{q}_1 + c_{112} \dot{q}_2 = \mathbf{h}(\mathbf{q}_2) \dot{q}_2$$

$$c_{12} = c_{121} \dot{q}_1 + c_{122} \dot{q}_2 = \mathbf{h}(\mathbf{q}_2) (\dot{q}_2)$$

How to proceed with simulation ?

Simulation, without the analytical model



Simmechanics

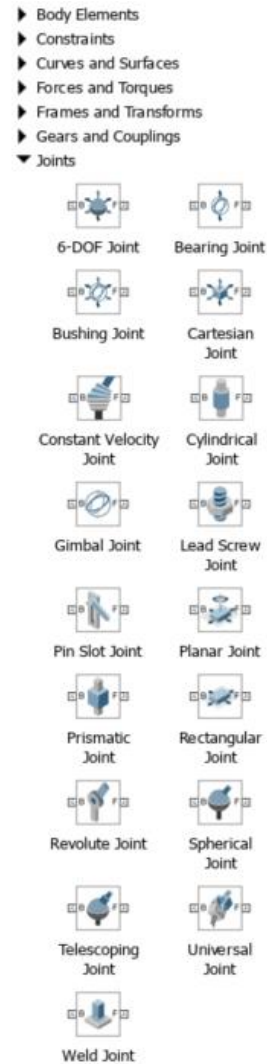
Or SimScape is a toolbox of matlab,

To model Dynamics

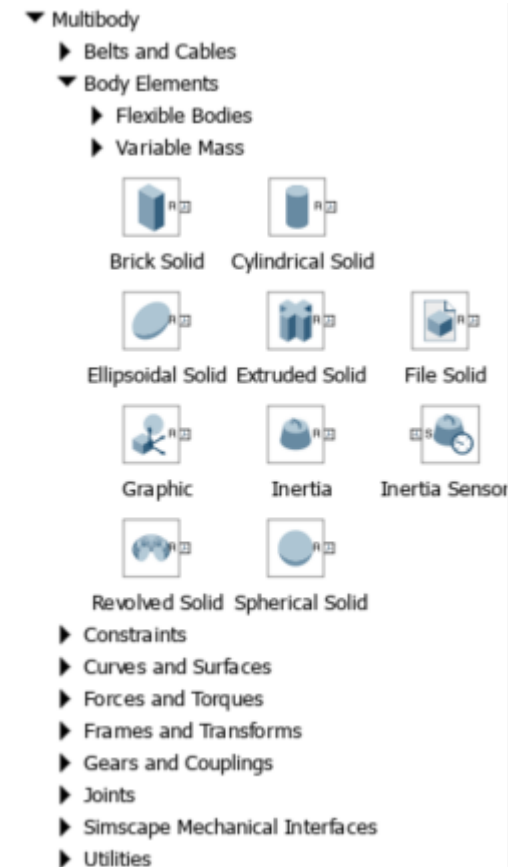
Robots are sets of

- **Joints**
- **Bodies**
- **Frames**

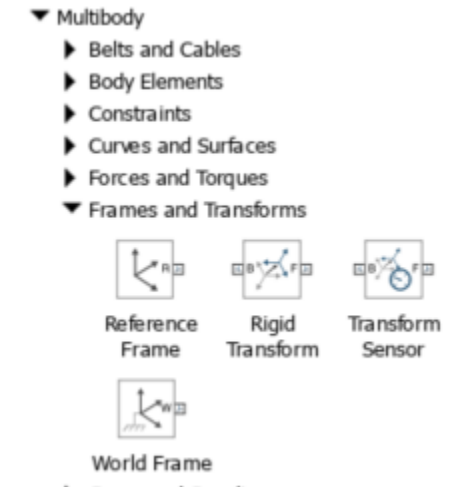
Joints



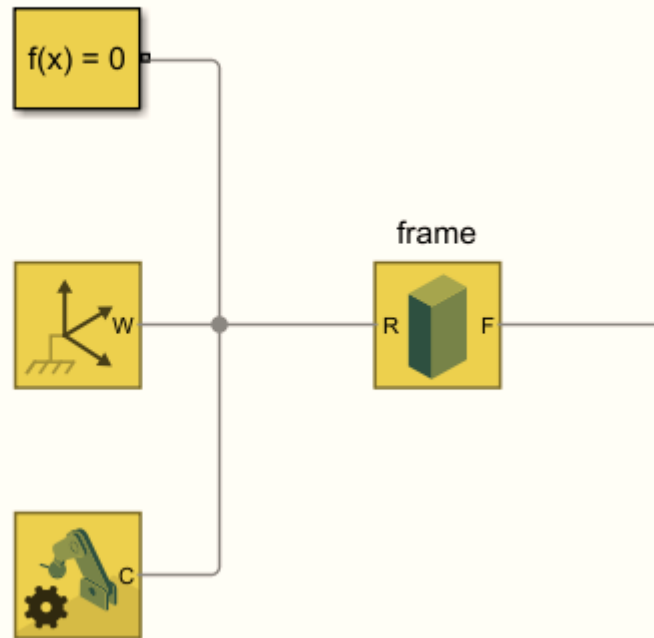
Bodies



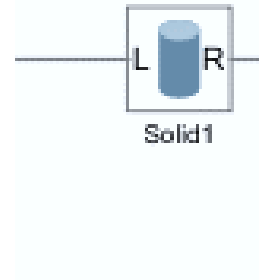
Frames



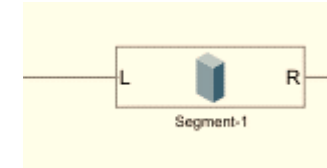
Frame



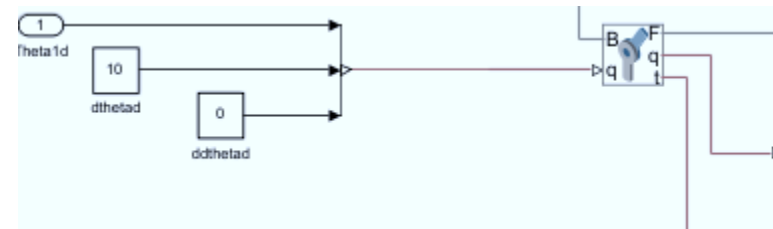
Visualization

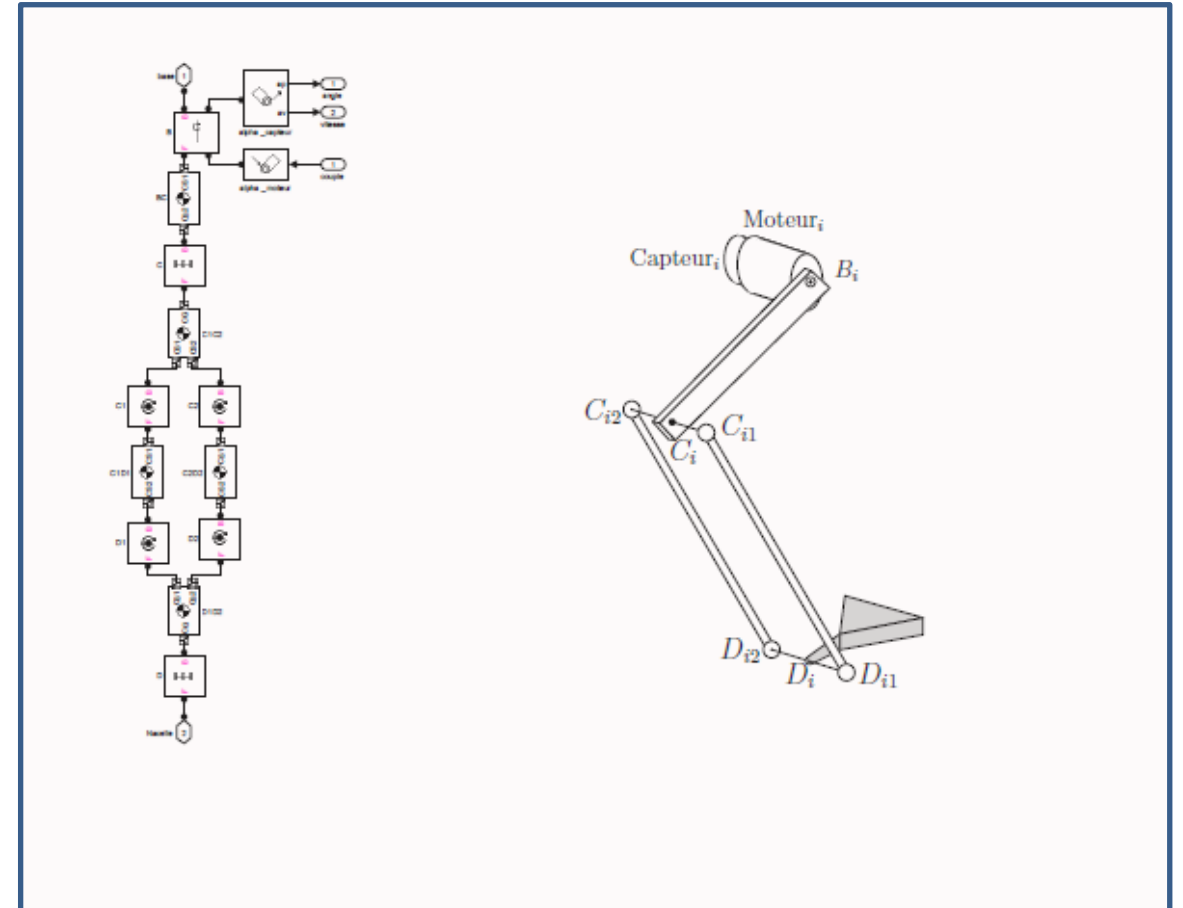
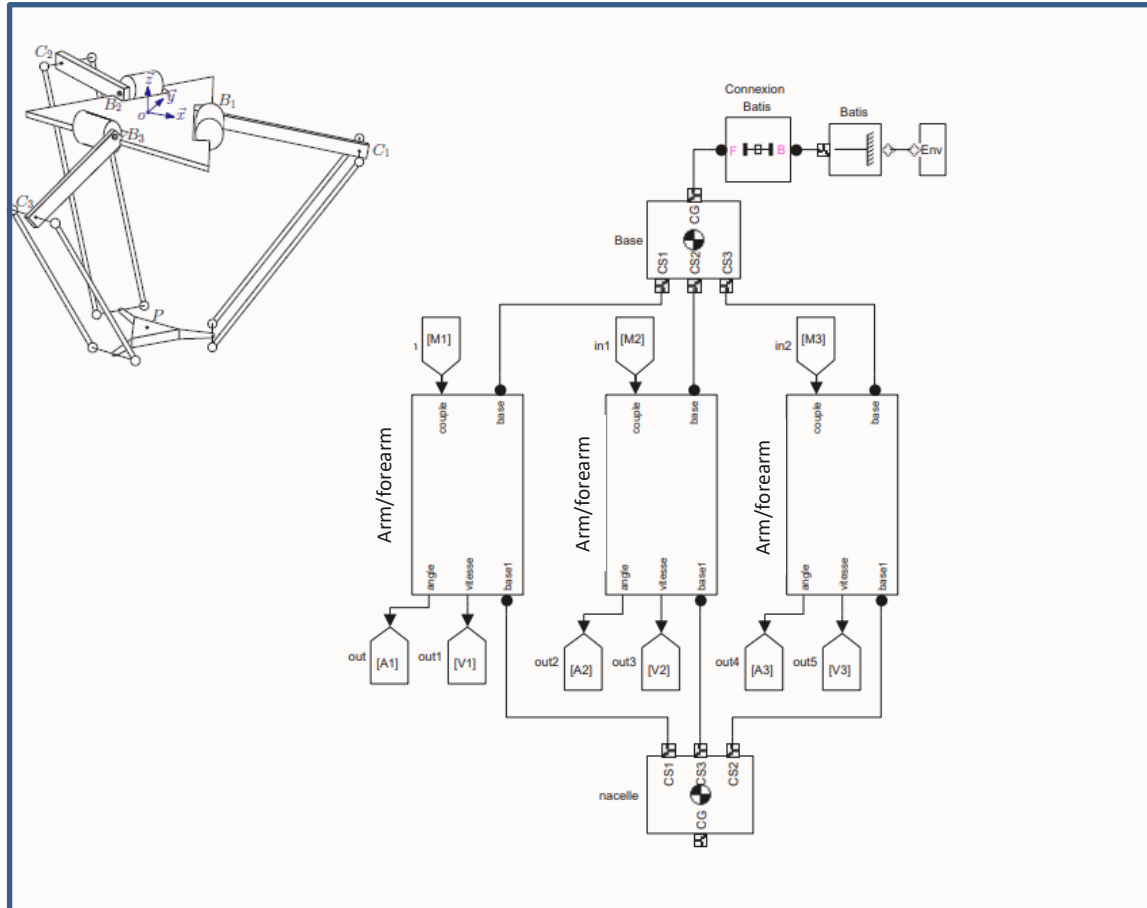
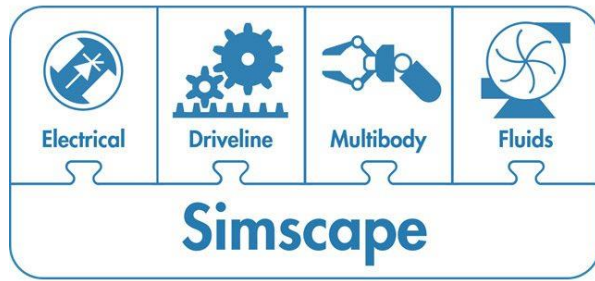


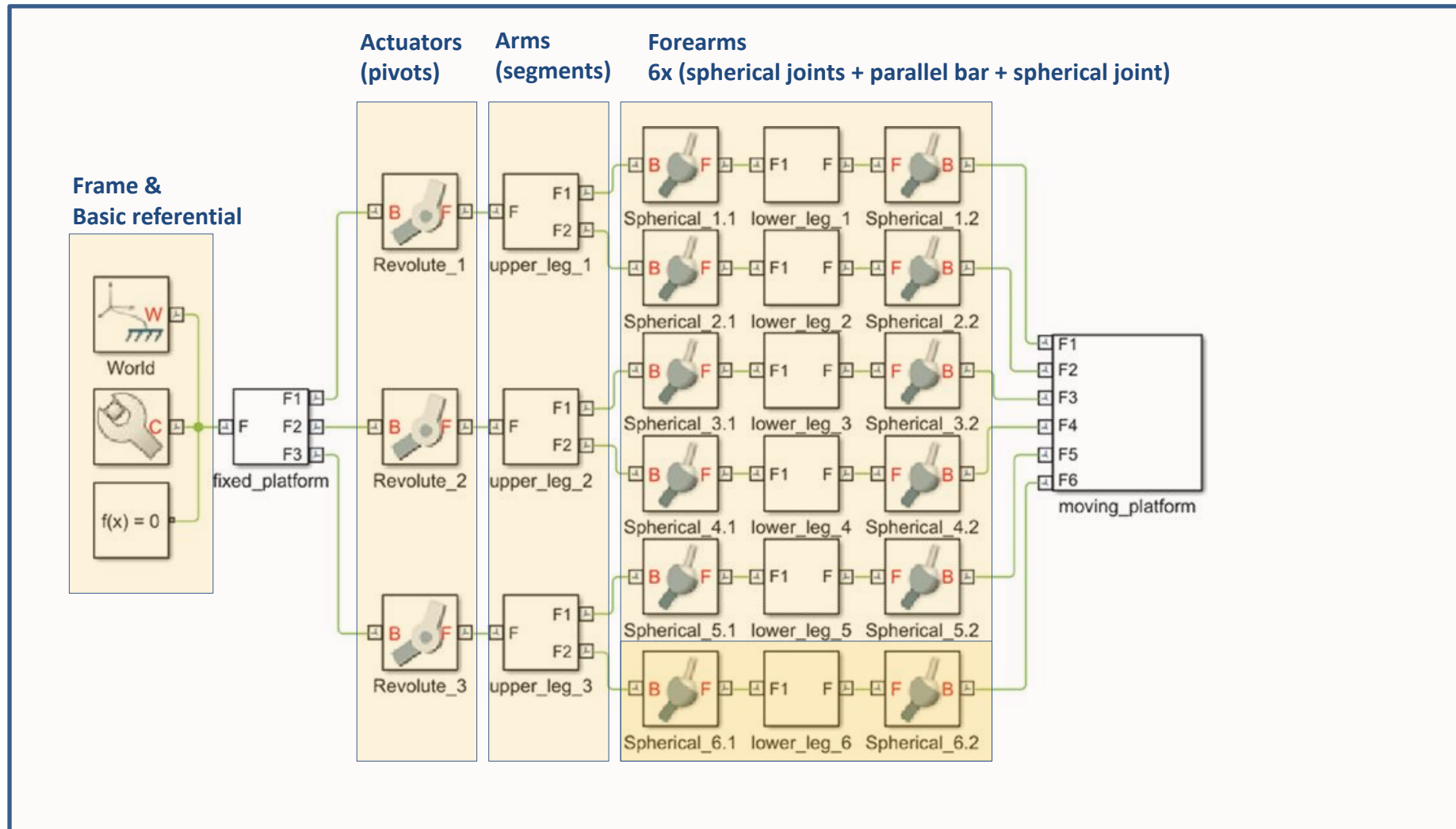
Body segment 1



Joint, controlled by motion

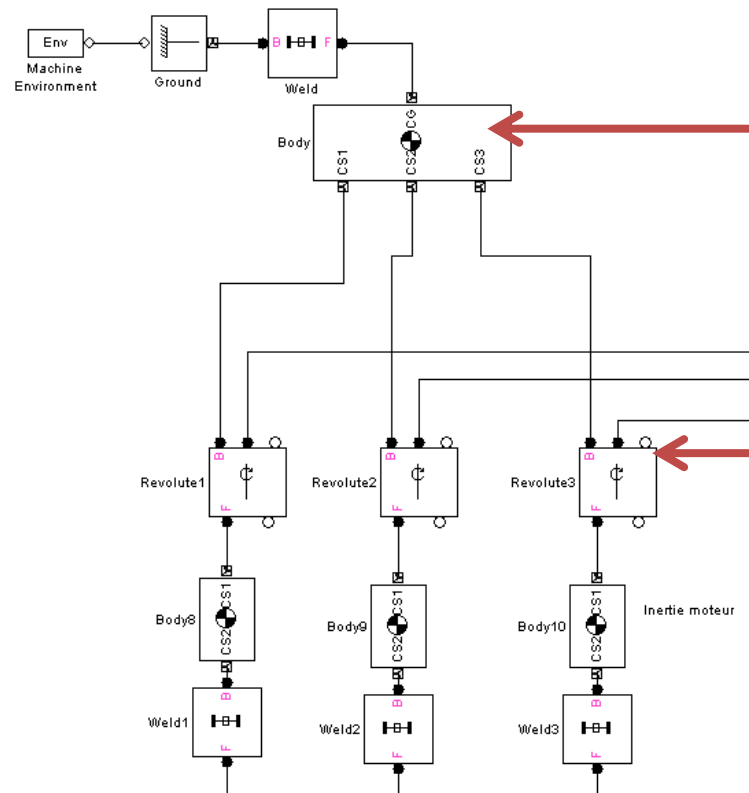






Le, H. N., & Vo, N. T. (2024). Modeling and analysis of an RUU Delta Robot using SolidWorks and SimMechanics. *International Journal of Dynamics and Control*, 1-13.

Simscape



Bodies:

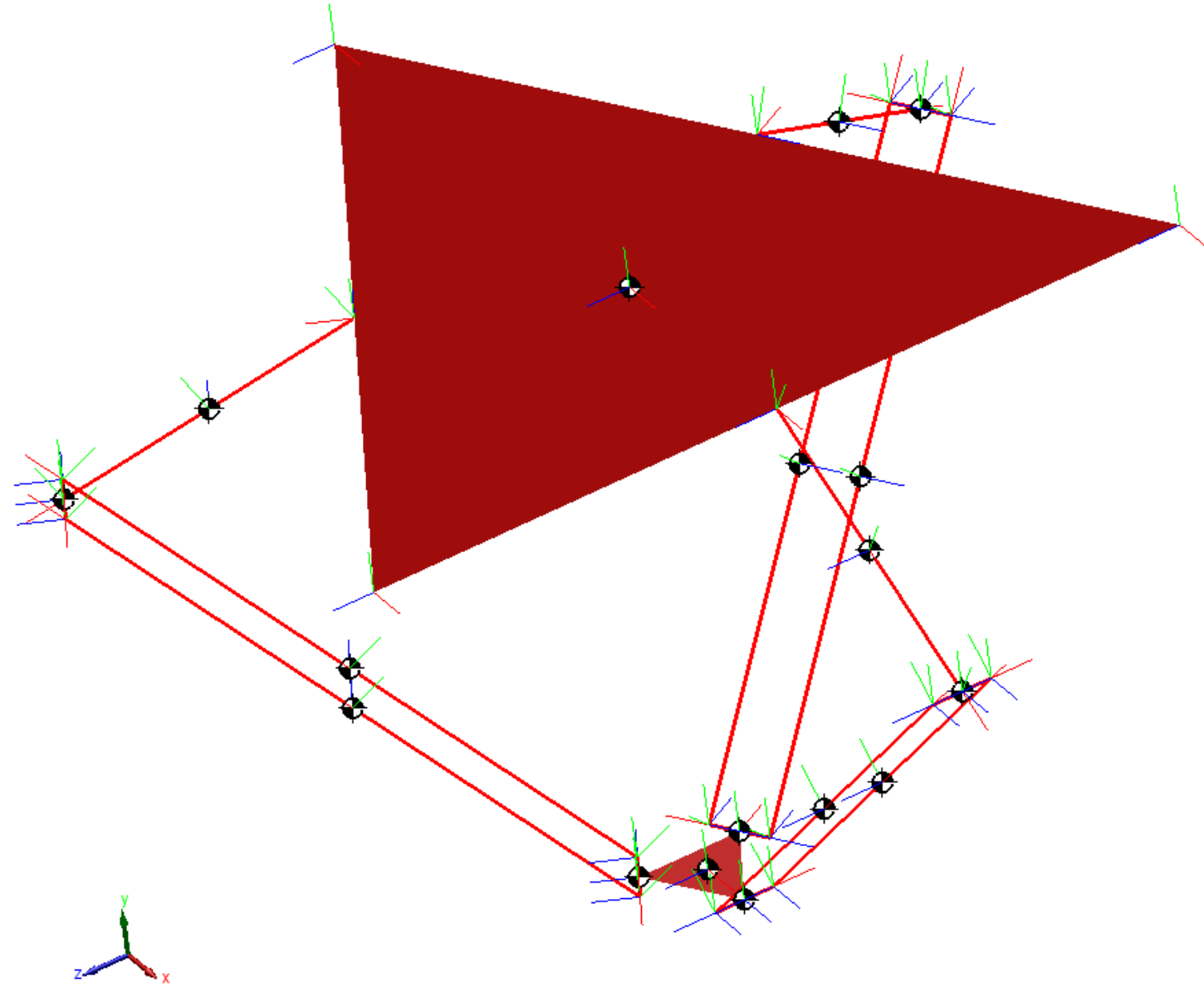
- Weights
- Lengths
- Inertia: uniform, ...

Joints:

- Rotation axis
- Dimension: 0

Such as actuation:

- Motors (joints)



Required specifications

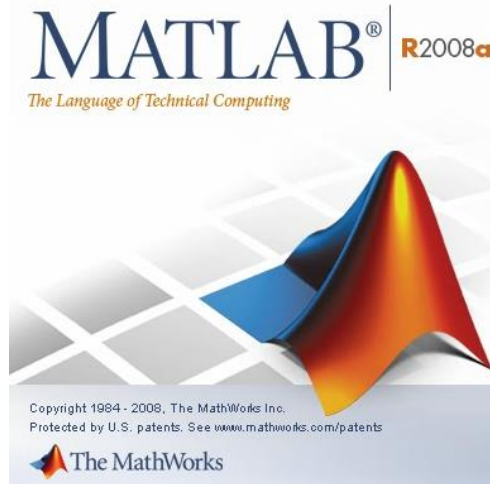
- Pick-up and place of a 1 kg load
- 15g acceleration when (loaded) and 30g (unloaded)
- A rate of 3 to 4 Hz, 300 mm / h25mm travel

Models – Sizing and validation approach

Different models

- Analytic model
 - known

- Simscape
Simulink



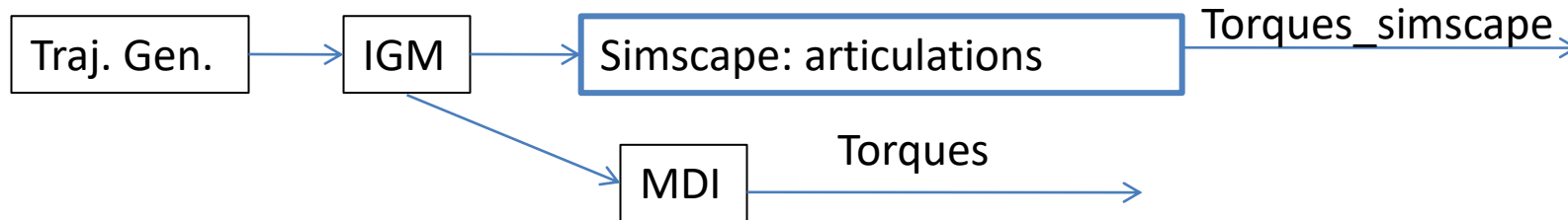
- ProEngineer
(Creo)



Models – Sizing and validation approach

Simmechanics

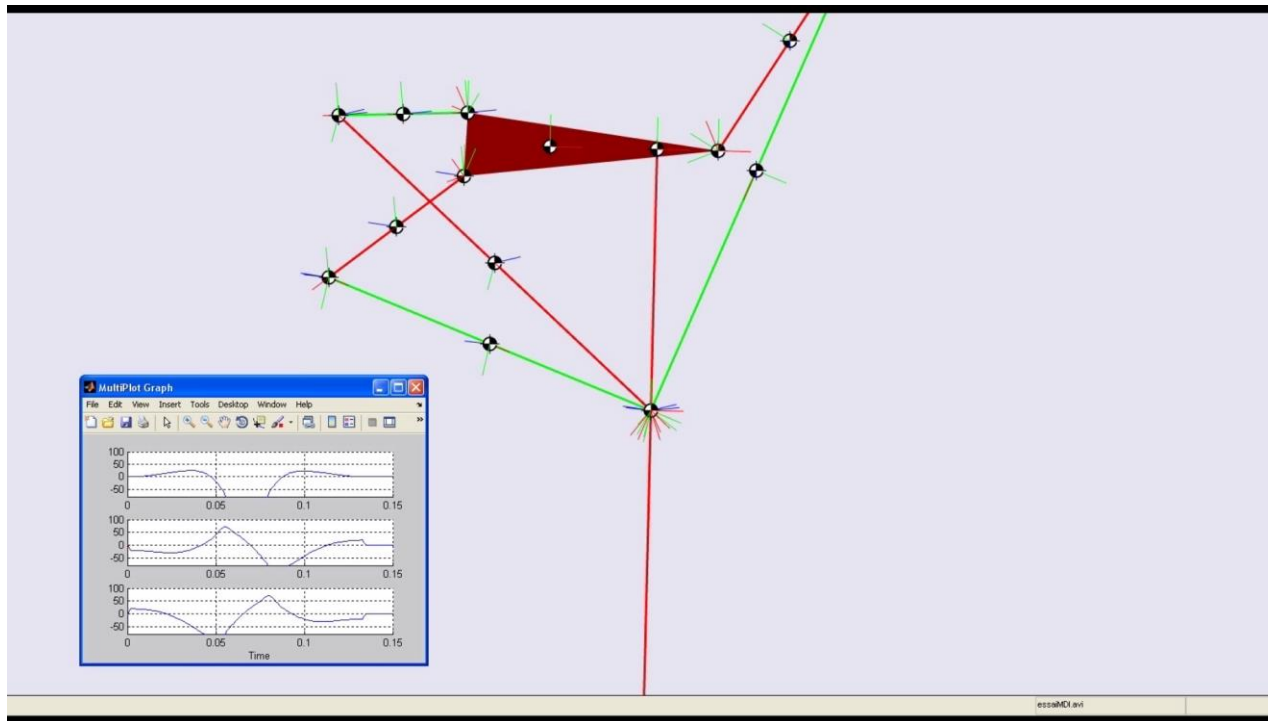
- **Scenario 1, IGM available**



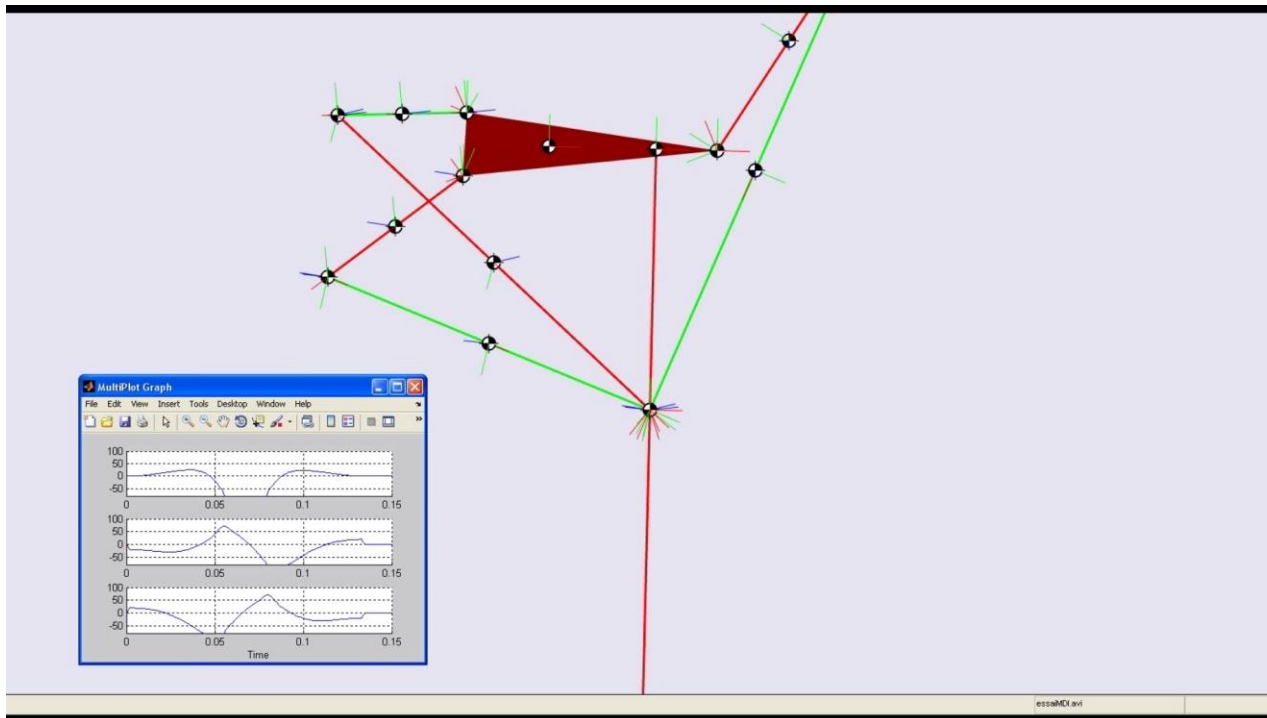
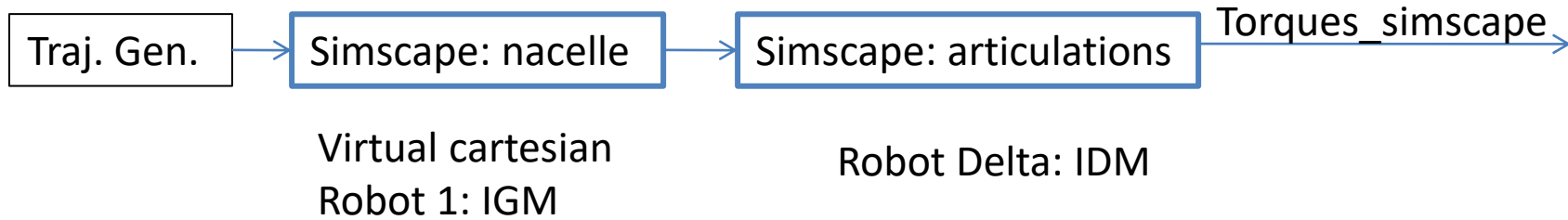
Models – Sizing and validation approach

Simscape

- Visualisation

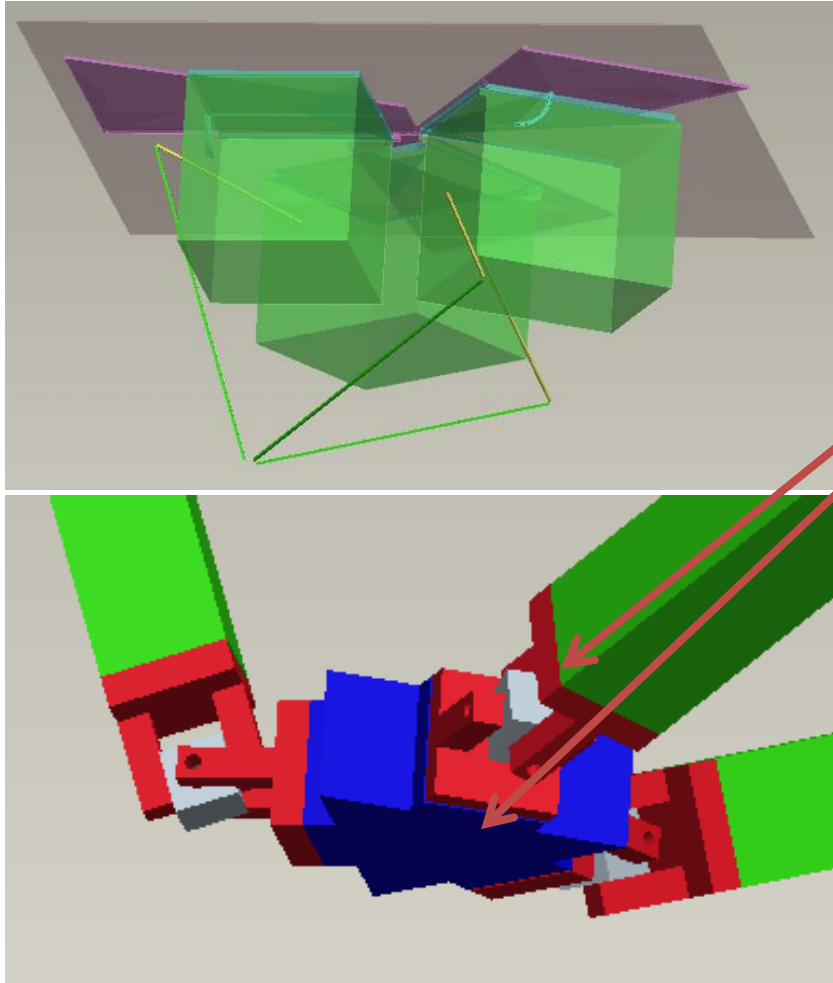


- **Scenario 2, IGM not available**



Models – Sizing and validation approach

ProEngineer (Creo)

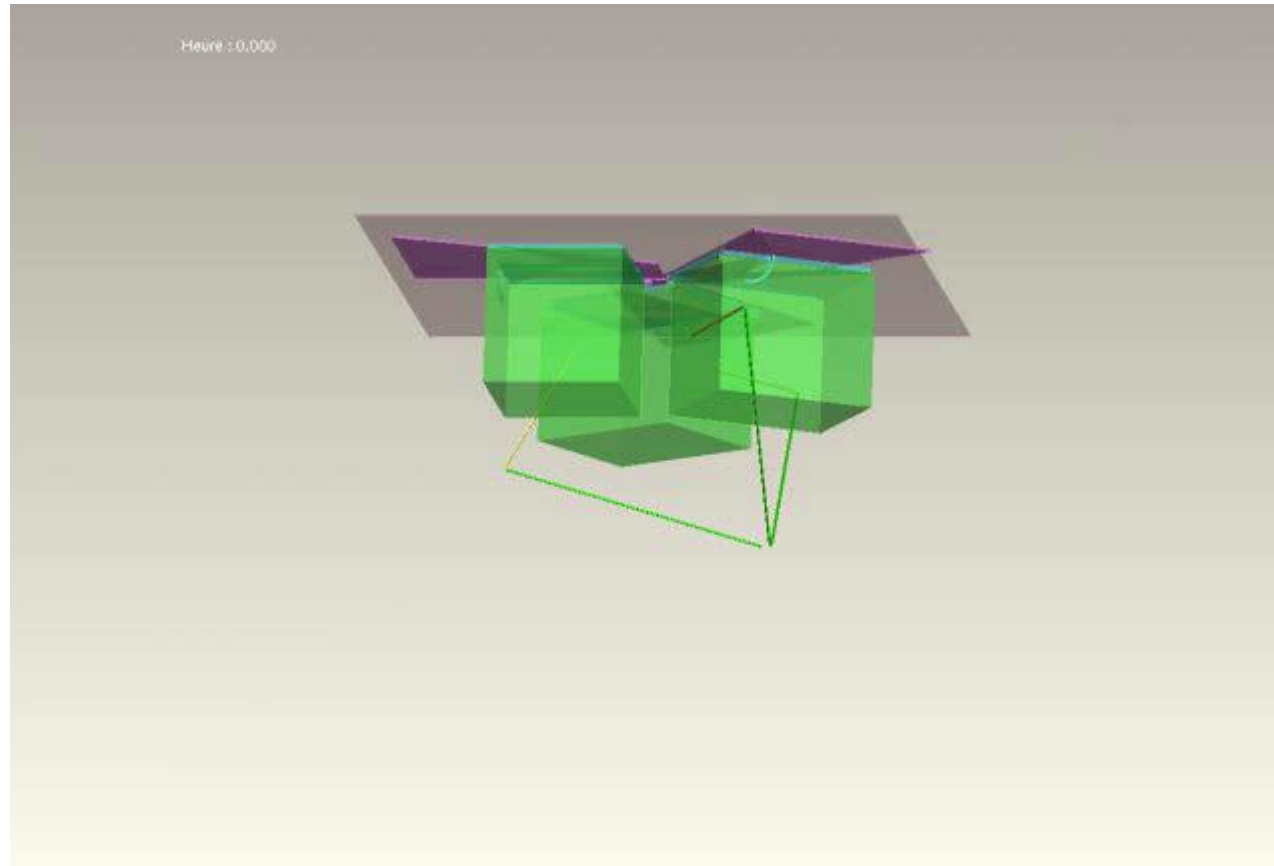


– Cardans

Models – Sizing and validation approach

ProEngineer (Creo)

- Visualisation

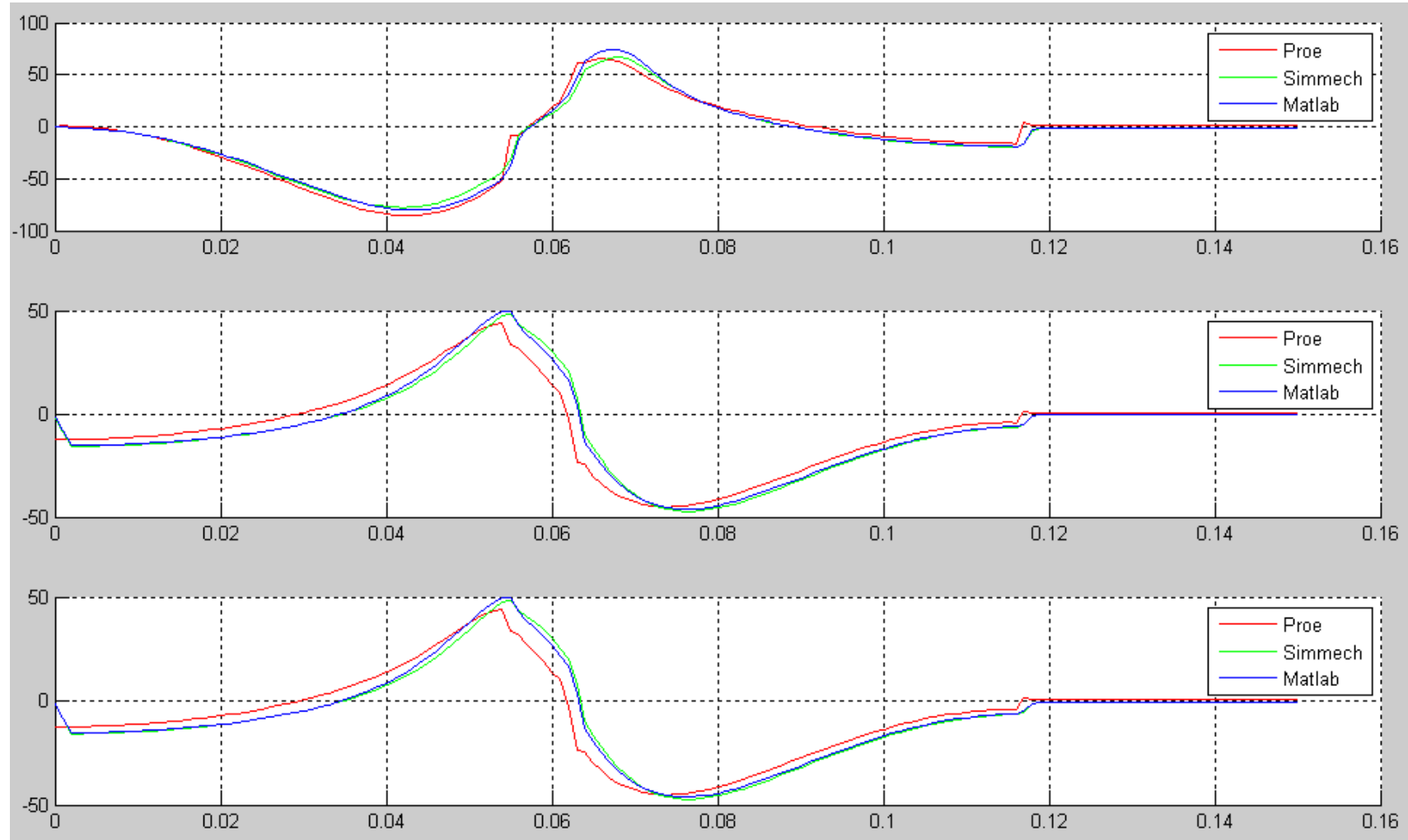


Models – Sizing and validation approach

Torques for an elliptic pick and place

Acceleration: 15g

Load : 1kg



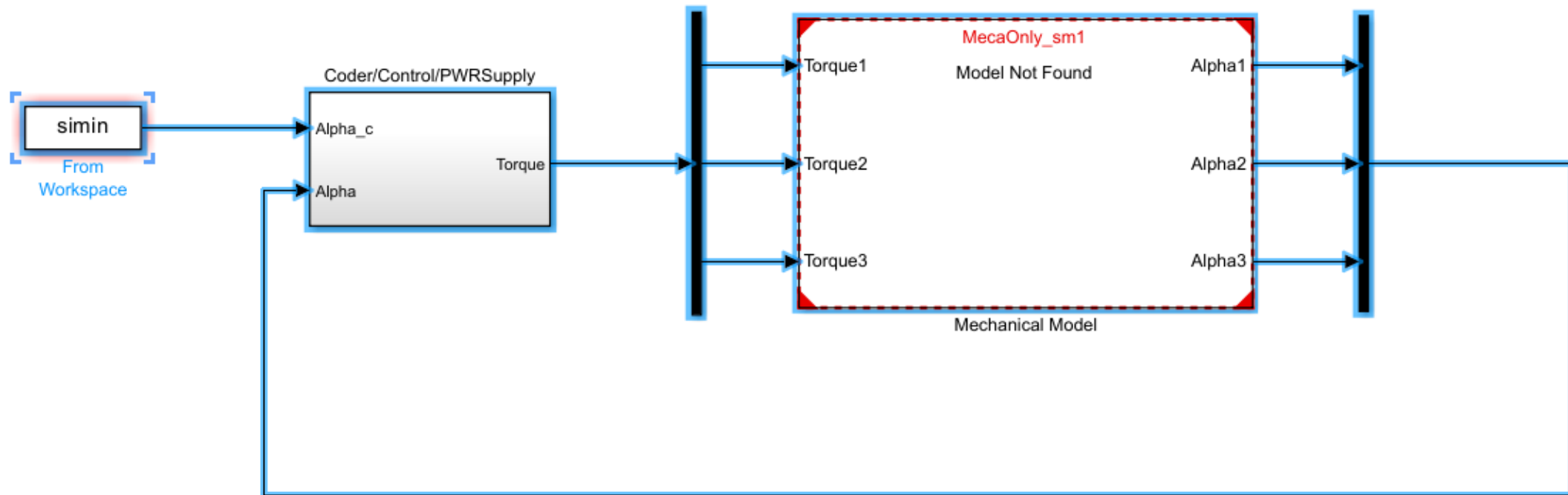
Moteur 1
[Nm]

Moteur 2
[Nm]

Moteur 3
[Nm]

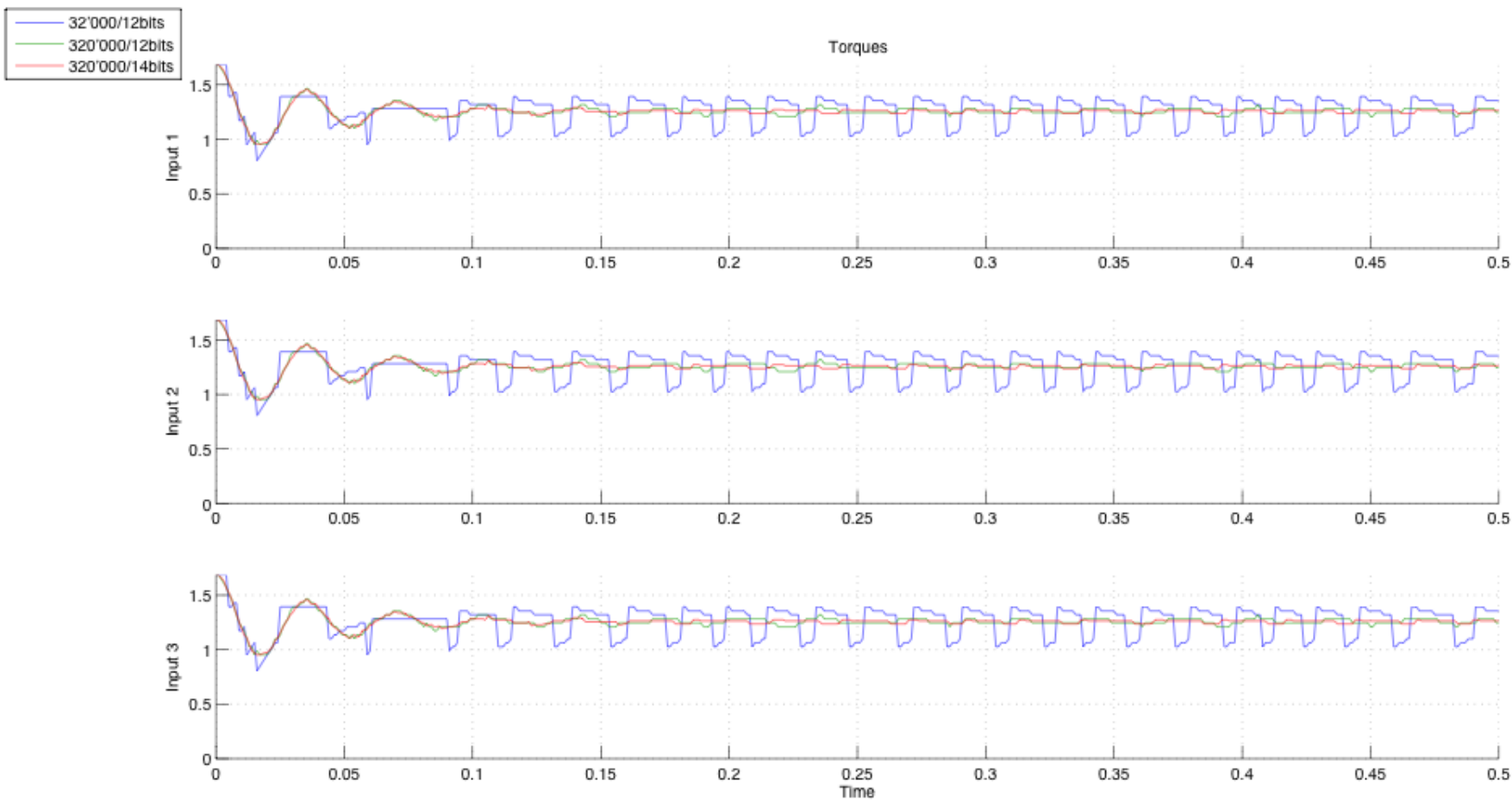
Context of Simulation : control

Simulation of a PID control with an a priori

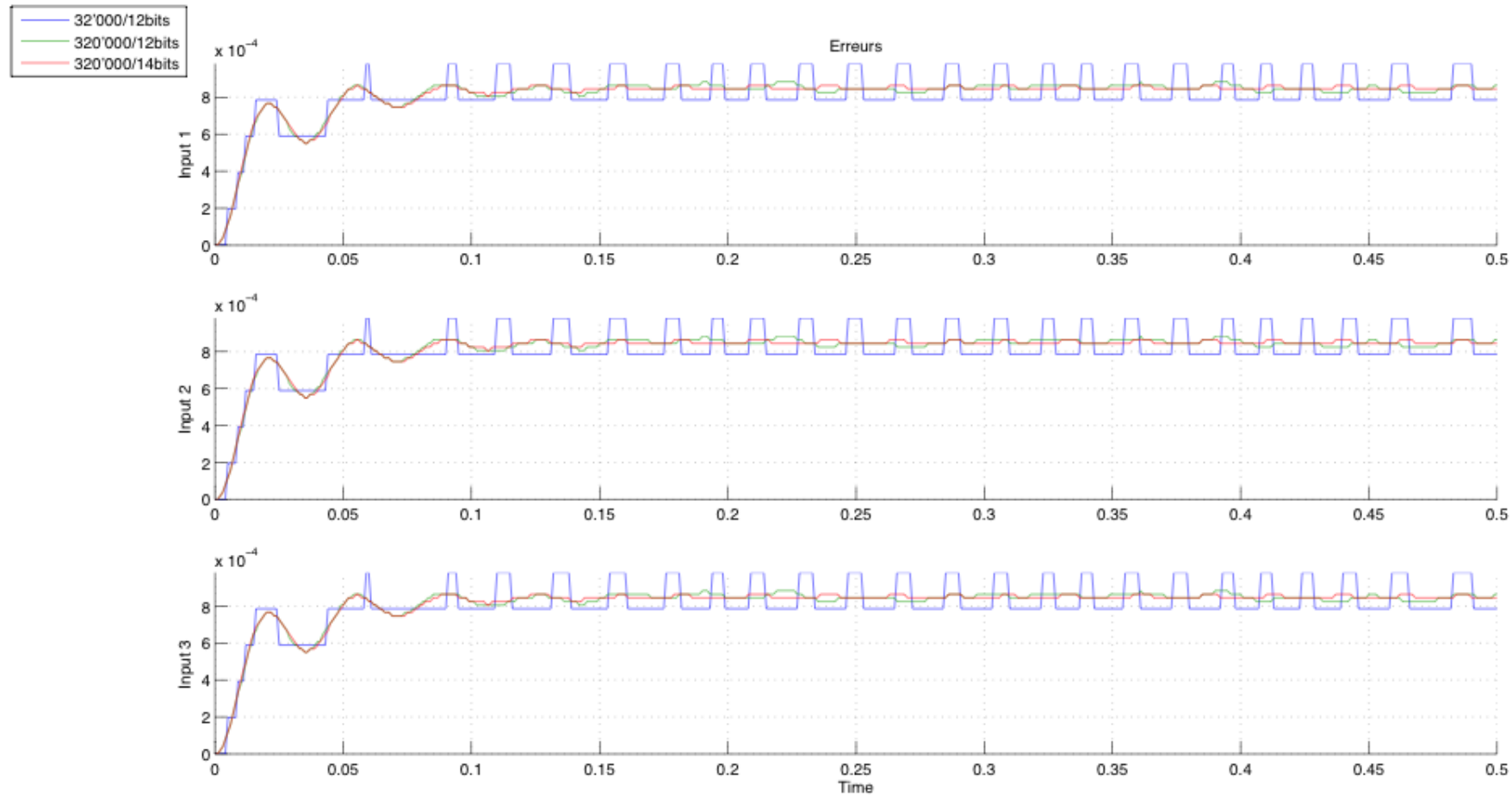


Robot torques

Load $\approx 1\text{kg}$



Errors @Load $\approx 1\text{kg}$



Robot torques

Load = 0kg

