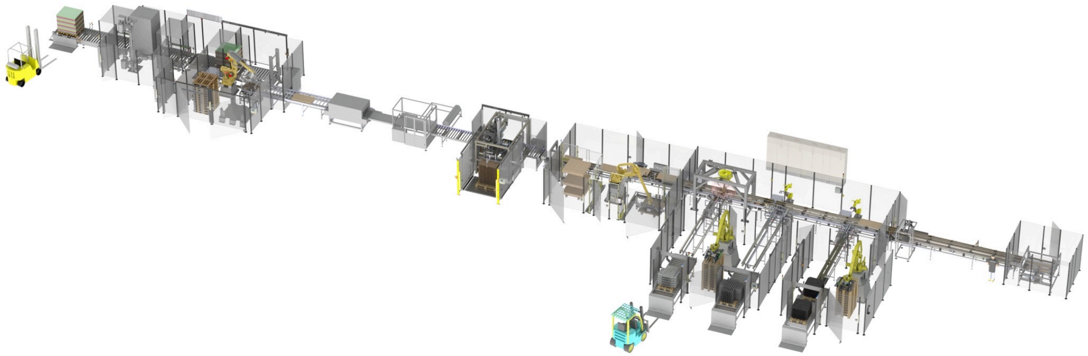

MANUFACTURING SYSTEMS AND SUPPLY CHAIN DYNAMICS

Chapter 9: Introduction to Queueing Networks

EPFL, Master MT

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Course Content

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5. *Productions Flows*
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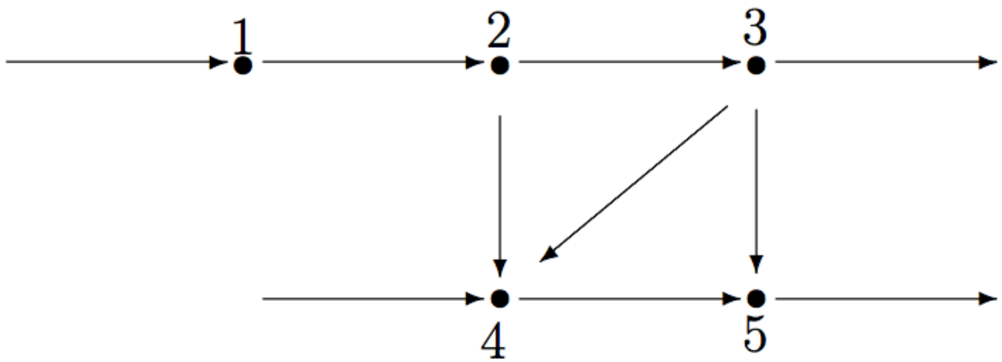
Introduction

Real-world manufacturing systems or supply chains: items might get served in **more than one station arranged in a network structure**

All items may **not require service from** the **same** set of **servers**

Some items may have to **go back** to the **same server** more than once

Example (**open network**):



Introduction

The *k* nodes in the network represent queues

$Q_i(t)$: number of items at node *i* at time *t*

$Q(t) = \sum_i Q_i(t)$: number of items in the queueing network at time *t*

Closed network: no item enters nor leaves the network ($Q(t)$ constant)

Example: service center supporting a fixed set of machines

Open network: input and output flows of items

Closed networks easier to analyze because of conserved quantities

For open networks, when input and output rate are equal, can be approximately modeled as closed networks

Markovian Networks

After service at **node** i, an item moves to **node** j with **fixed conditional probability** P_{ij}

Repeated service at node i : $P_{ii} > 0$

Open network: extra state 0, $P_{00} = 0$, $P_{0j} \geq 0$ (inflow), $P_{i0} \geq 0$ (outflow)

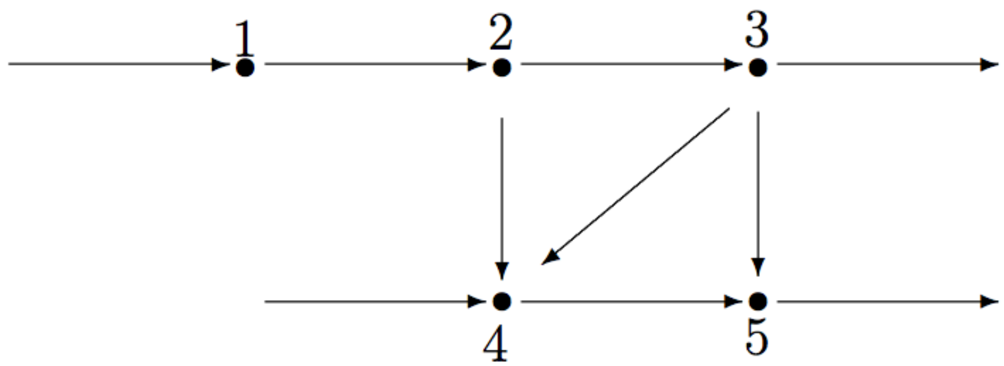
Transition probability matrix (routing matrix):

$$\mathbf{P} = \begin{bmatrix} 0 & P_{01} & \dots & P_{0k} \\ P_{10} & P_{11} & \dots & P_{1k} \\ \vdots & \vdots & \vdots & \vdots \\ P_{k0} & P_{k1} & \dots & P_{kk} \end{bmatrix}$$

Rows of the matrix sum to 1

Markovian Networks

Example 9:



$$\mathbf{P} = \begin{bmatrix} 0 & P_{01} & 0 & 0 & P_{04} & 0 \\ 0 & 0 & P_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & P_{23} & P_{24} & 0 \\ P_{30} & 0 & 0 & 0 & P_{34} & P_{35} \\ 0 & 0 & 0 & 0 & 0 & P_{45} \\ P_{50} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Markovian Networks: Transition Matrix

Relative throughput: rate of items passing through each node

$$\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_k)$$

Under stable conditions, item arrival rates λ_i at node i must reach **input-output parity** (**balance equations**)

$$\boldsymbol{\lambda}\mathbf{P} = \boldsymbol{\lambda}$$

Similar to the situation where computing the limiting distribution of a Markov chain (Chapter 4)

No normalizing constraints => solution only indicates relative throughput in the network

Markovian Networks: Transition Matrix

Mean Throughput Time Exclusive of Waiting:

$1/\mu_j$: **mean time** item spends at node *j* and

ν_{ij} : **expected number of visits** at node *j* for item starting at node *i*

Mean throughput time (exclusive of waiting) for item starting at node *i* :

$$\sum_{j=1}^{k} \frac{\nu_{ij}}{\mu_j}$$

Computation of ν_{ij} : convert 0 to an absorbing state:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ P_{10} & P_{11} & \dots & P_{1k} \\ \vdots & \vdots & \vdots & \vdots \\ P_{k0} & P_{k1} & \dots & P_{kk} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{R} & \mathbf{Q} \end{bmatrix}$$

Markovian Networks: Transition Matrix

We have:

$$(\mathbf{I}_k - \mathbf{Q})^{-1} = \begin{bmatrix} \nu_{11} & \nu_{12} & \dots & \nu_{1k} \\ \nu_{21} & \nu_{22} & \dots & \nu_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ \nu_{k1} & \nu_{k2} & \dots & \nu_{kk} \end{bmatrix}$$

α_i : probability that initial state is node *i*

Total throughput time of an item (exclusive of waiting) :

$$\sum_{i=1}^{k} \alpha_i \sum_{j=1}^{k} \frac{\nu_{ij}}{\mu_j}$$

Markovian Networks: Transition Matrix

Exercise 29 We are given the open queueing system represented in Figure 9.2 with 3 states 1, 2, 3.

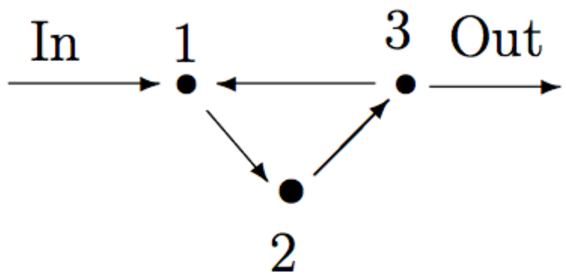


Figure 9.2: A simple open network.

Show that:

$$(\nu_{ij}) = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

and calculate the expected total throughput time of a customer as a function of μ_j if s/he is starting respectively in state 1, 2 or 3. $\odot$

Queues in Series

Service facilities located **in series** (open network), and items passing through them **sequentially**

Example: **assembly lines**

Each node: ***M/M/s*** queue

Items from outside the network always start at the first facility

No blocking between successive service stations (=> waiting rooms of **infinite capacity**)

Queues in Series: 2 Servers

Two *M/M/1* queues in series (waiting rooms of infinite)

Arrivals at first queue: Poisson process with rate λ

Service times exponentially distributed with mean $1/\mu_j$

{Q1(t), Q2(t)} : **vector of Markov chains** with states *(n1, n2)* (*n1, n2* non-negative integers)

Arrival to second queue, as $t \to \infty$: Poisson process with rate λ

The two queues are **independent** *(for $t \to \infty$) with stationary distribution:*

$$p_{n_1,n_2} = (1-\rho_1)\rho_1^{n_1}(1-\rho_2)\rho_2^{n_2}$$

Finite time => more complex, no factorization

Queues in Series: 2 Servers

Exercise 30 For a two-stage serial queue, we note (n, m) the state where there are n customers at service 1 and m customers at service 2. Establish the balance equations:

$$\begin{array}{rcll} \text{rate that process leaves} & = & \text{rate that it enters} & \\ \lambda p_{0,0} & = & \mu_2 p_{0,1} & n = m = 0 \\ (\lambda + \mu_1) p_{n,0} & = & \mu_2 p_{n,1} + \lambda p_{n-1,0} & m = 0, n > 0 \\ (\lambda + \mu_2) p_{0,m} & = & \mu_2 p_{0,m+1} + \mu_1 p_{1,m-1} & n = 0, m > 0 \\ (\lambda + \mu_1 + \mu_2) p_{n,m} & = & \mu_2 p_{n,m+1} + \mu_1 p_{n+1,m-1} + \lambda p_{n-1,m} & n, m > 0 \end{array}$$

and verify that $p_{n,m} = (1 - \rho_1)\rho_1^n (1 - \rho_2)\rho_2^m = p_n p_m$ solves the above system.

Establish further that the average number L of customers in the system is given by:

$$L = \sum_{n,m} (n + m) p_{n,m} = \frac{\rho_1}{1 - \rho_1} + \frac{\rho_2}{1 - \rho_2}$$

and that from this, we see that the average time a customer spends in the system is equal to:

$$W = \frac{L}{\lambda} = \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda}$$

⊙

Queues in Series: *k* Servers

Generalization**: *k M/M/1* queues** in series (waiting rooms of infinite)

Items arrive from outside the system to server *i*, *i* = *1,…,k* in accordance with independent Poisson processes with rate r_i

Once an item is served by server *i* : it joins server *j* with probability P_{ij}

$1 - \sum_{j=1}^{k} P_{i,j}$: probability an item leaves the system

Total arrival rate at server *j* : $\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i P_{i,j}$

Stationary distribution of the **number of items at node *j*** :

$$P(n \text{ customers at server } j) = \left(\frac{\lambda_j}{\mu_j}\right)^n \left(1 - \frac{\lambda_j}{\mu_j}\right)$$

Queues in Series: *k* Servers

Mean number of items in the system:

$$L = \sum_{j=1}^{k} \text{average number of customers at server } j = \sum_{j=1}^{k} \frac{\lambda_j}{\mu_j - \lambda_j}$$

Average time an item spends in the system (Little's law):

$$W = \frac{\sum_{j=1}^{k} \frac{\lambda_j}{\mu_j - \lambda_j}}{\sum_{j=1}^{k} r_j}$$

Queues in Series: *k* Servers

Example 10 We consider here a queueing system composed of two servers, where manufacturing jobs from outside the system arrive at server 1 following a Poisson process with rate 4 per minute and at server 2 following a Poisson process with rate 5 per minute. The service rates for servers 1 and 2 are respectively 8 and 10 manufacturing jobs per minute. A job, upon completion of service at server 1, is equally likely to go to server 2 or to leave the system (i.e., $P_{1,1} = 0$, $P_{1,2} = \frac{1}{2}$); whereas a departure from server 2 will go 25 percent of the time to server 1 and will depart the system otherwise (*i.e.*, $P_{2,1} = 0.25$, $P_{2,2} = 0$).

Determine L (mean number of jobs in the system) and W (mean time a job spends in the system).

Queues in Series: *k* Servers

Solution: The total arrival rates to servers 1 and 2 – call them λ_1 and λ_2 respectively– can be obtained by solving the system:

$$\begin{aligned}\lambda_1 &= 4 + \frac{1}{4}\lambda_2 \\ \lambda_2 &= 5 + \frac{1}{2}\lambda_1\end{aligned}$$

implying that $\lambda_1 = 6$ and $\lambda_2 = 8$. Hence:

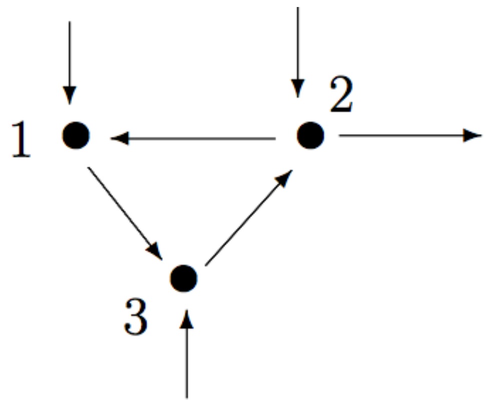
$$P(n \text{ customers at server } 1, m \text{ customers at server } 2) = \left(\frac{3}{4}\right)^n \frac{1}{4} \left(\frac{4}{5}\right)^m \frac{1}{5}$$

and $L = \frac{6}{8-6} + \frac{8}{10-8} = 7$ jobs, $W = \frac{L}{9} = \frac{7}{9}$ minutes. $\Diamond$

Queues in Series: *k* Servers

Exercise 31 We consider a queueing system composed of three servers where manufacturing jobs from outside the system arrive at servers 1, 2 and 3 following a Poisson process with rate 1 job per time unit (see Figure 9.3). The service rates for servers 1, 2 and 3 are respectively 10, 15 and 12 jobs per time unit.

Determine L (mean number of customers in the system) and W (mean time a customer spends in the system). ⊙



Queues in Series: *k* Servers

Exercise 32 We consider a queueing system composed of three servers, following the topology displayed in Figure 9.4. The service rates for servers 1, 2 and 3 are respectively 10, 15 and 12 jobs per time unit.

Determine L (mean number of customers in the system) and W (mean time a customer spends in the system). We assume that each queue is $M/M/1$ with

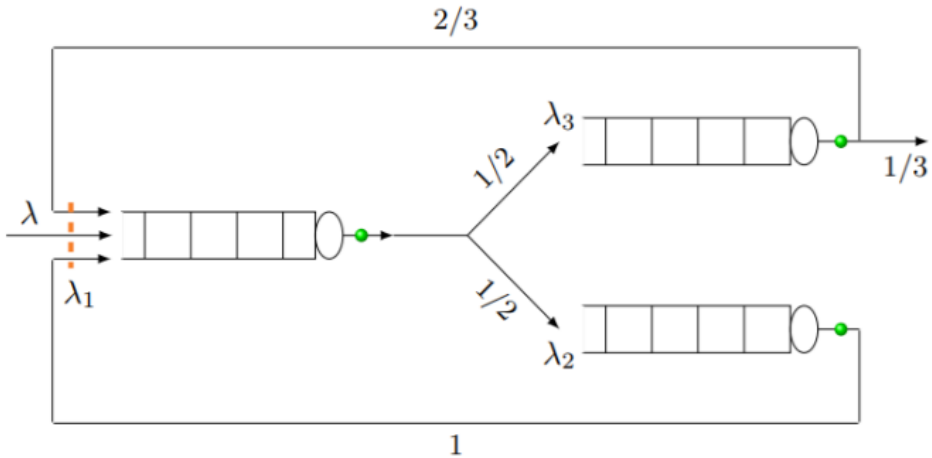


Figure 9.4: Open network with bifurcation and loop.

infinite storage capacity. Moreover, we suppose that arrival and departure rates are exponentially distributed.

Queues in Series: *k* Servers

Show that:

- For the steady-state distribution, we find:

$$\rho_1 = \frac{\lambda_1}{\mu_1} = \frac{6\lambda}{\mu_1}, \quad \rho_2 = \frac{\lambda_2}{\mu_2} = \frac{3\lambda}{\mu_2}, \quad \rho_3 = \frac{\lambda_3}{\mu_3} = \frac{3\lambda}{\mu_3},$$

  and therefore the stationary distribution of having i jobs at server 1, j jobs at server 2 and, and k jobs at server 3, is given by:

$$\pi_{ijk} = (1-\rho_1)\rho_1^i(1-\rho_2)\rho_2^j(1-\rho_k)\rho_3^k$$

- The average number of jobs in all queues is equal to:

$$L = \sum_{i=1}^{3} \frac{\rho_i}{1-\rho_i}$$

⊙