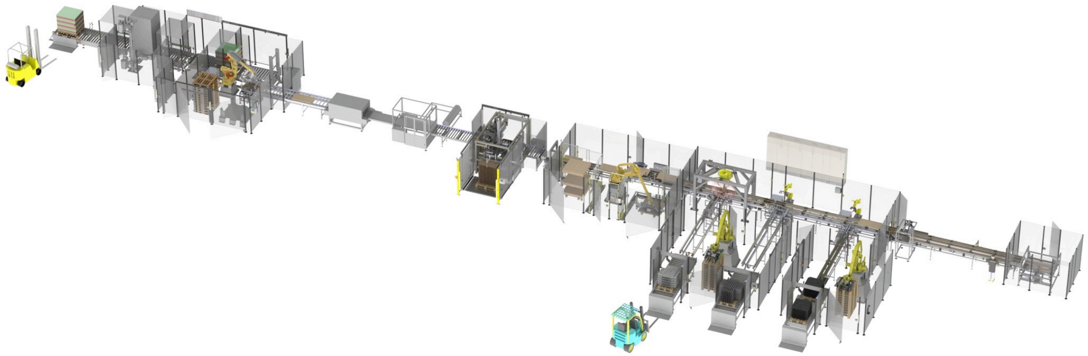


MANUFACTURING SYSTEMS AND SUPPLY CHAIN DYNAMICS

Chapter 2: Inventory Theory

EPFL, Master MT

Roger Filliger (BFH), Olivier Gallay (UniL)

Course Content

1. *Introduction*
2. ***Inventory Theory***
3. *Safety Stock in Manufacturing Systems*
4. *Elements of Queueing Theory*
5. *Productions Flows*
6. *Production Dipole*
7. *Production Lines and Aggregation*
8. *Cooperative Flow Dynamics*
9. *Introduction to Queueing Networks*
10. *Supply Chain Analysis*
11. *Elements of Reliability Analysis*
12. *Maintenance Policies*

Introduction

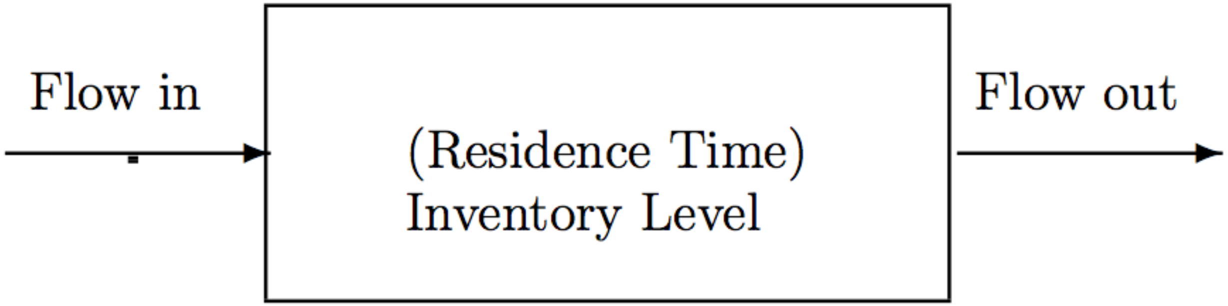
Managing inventories is necessary for any company dealing with physical products (manufacturers, wholesalers, retailers)

In the US, all inventories have in total a value of more than 1 trillion $ (about 4000$ per citizen)

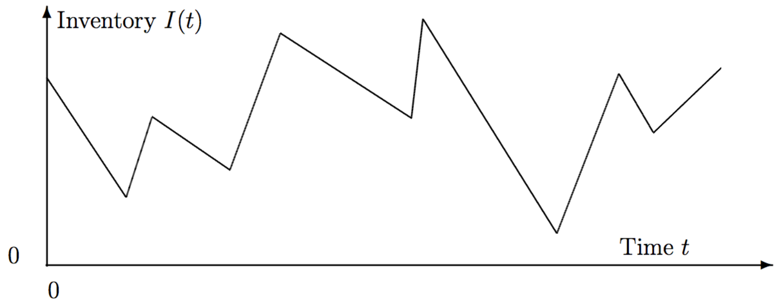
Use **scientific inventory management**:

- Formulate a **mathematical model** for the behavior of the inventory system
- Seek an **optimal inventory policy**

Inventory System



$y(t)$: rate of the input flow at time t
$Y(t)$: cumulative input flow up to time t
$z(t)$: rate of the output flow at time t
$Z(t)$: cumulative output flow up to time t
$I(t)$: **inventory level** at time t



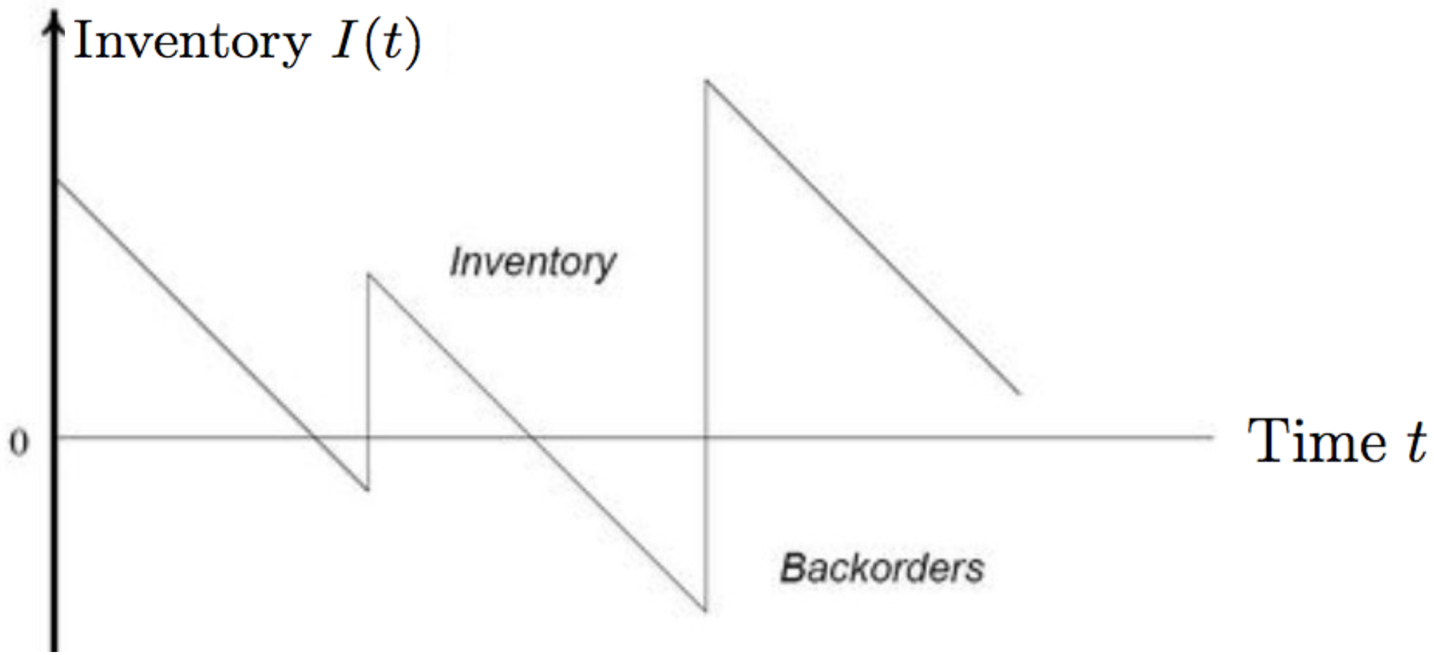
$$I(t) = Y(t) - Z(t) = \int_0^t (y(s) - z(s))ds$$

Inventory Theory

Production engineers constantly face **inventory tradeoff** :

Inventory level too small: inventory risks incurring **backorder or lost sales** costs

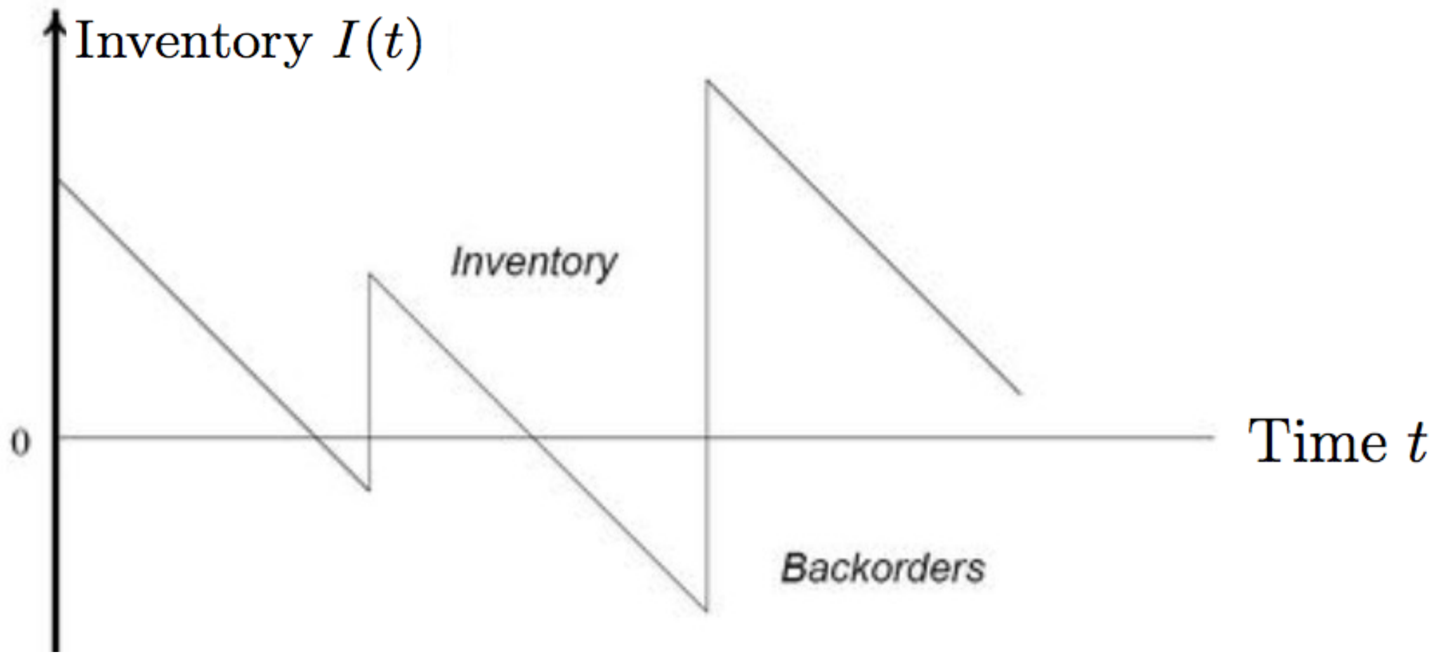
Inventory level too large: inventory increases **holding costs**



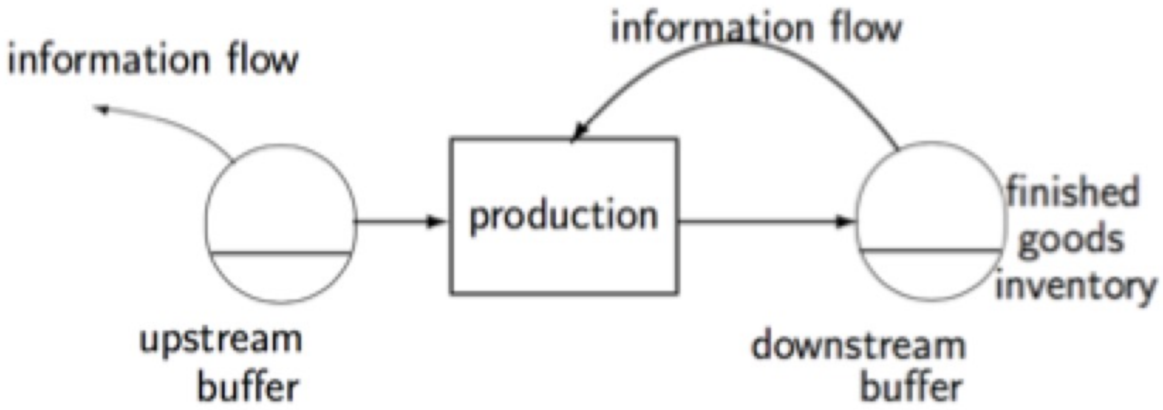
Inventory Theory (continued)

In view of **fluctuating demand** and **production with variability**, is there an **optimal policy for inventory control?**

Example: when your stock goes down to 4, order more!



Upstream and Downstream Buffers

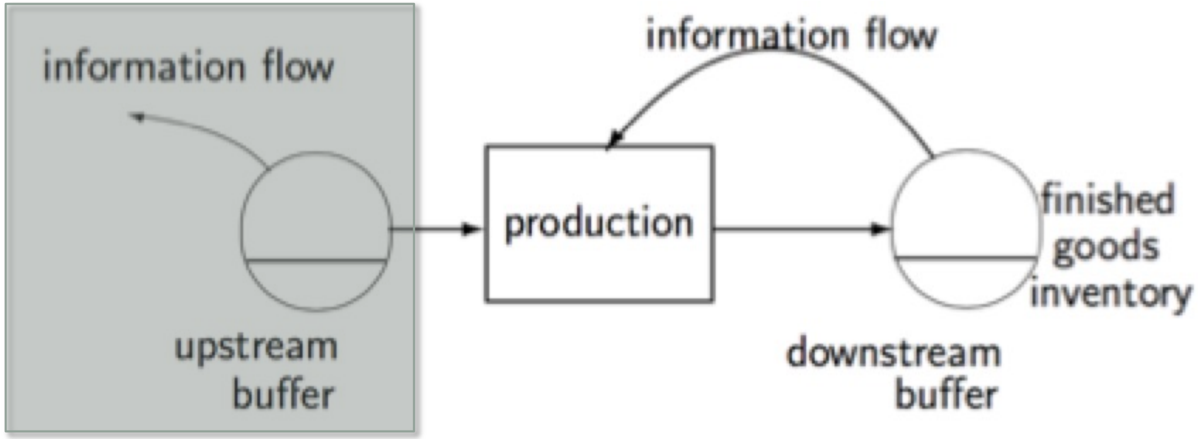


Upstream Buffer: holds material which is going to be processed by the production facility => **raw material**

Downstream Buffer: receives processed goods from the production facility => **finished goods**

Specific upstream and downstream policies

Upstream Inventory Policies

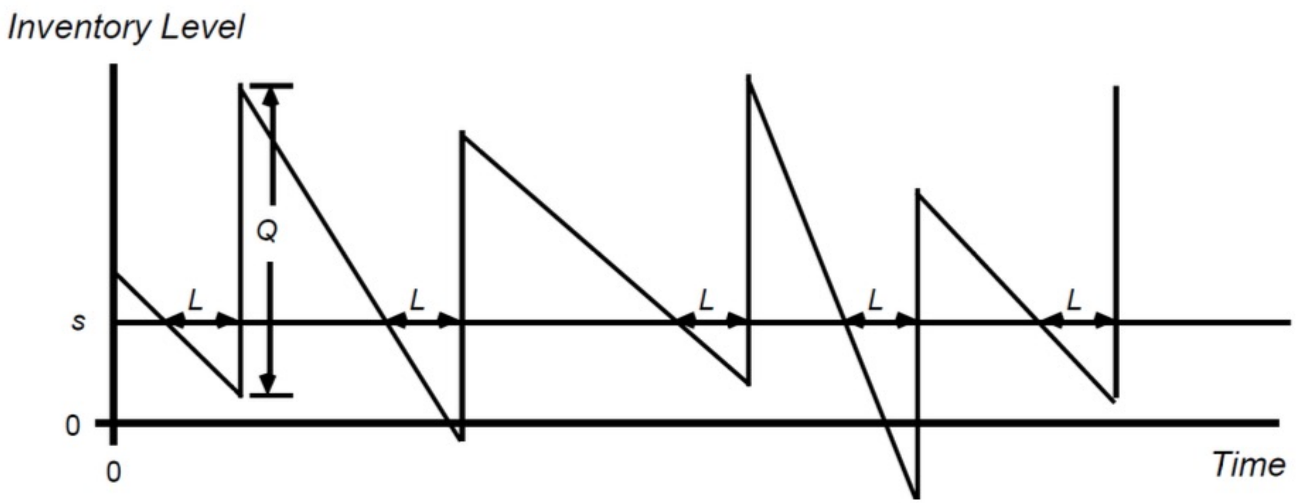


Upstream Inventory Policy: must ensure availability of raw material at the entry of the manufacturing system

Objective: at what time should a production engineer order raw material and how much?

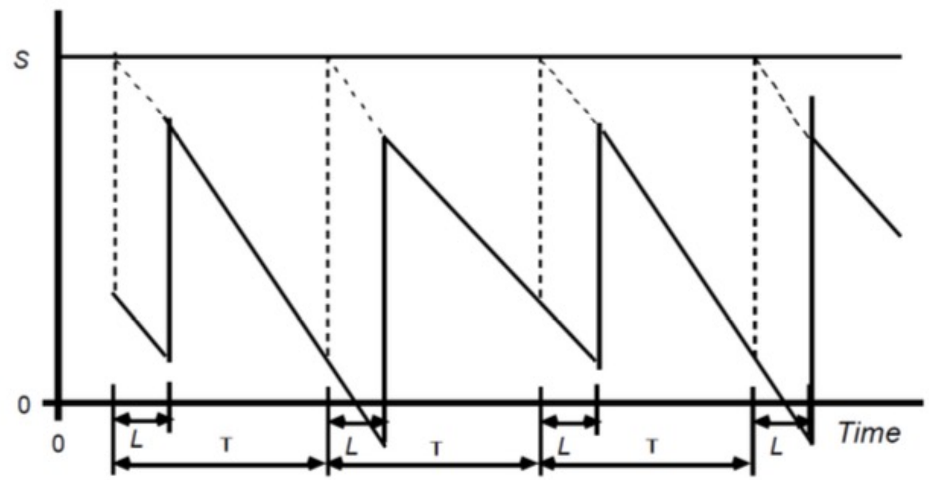
Upstream Buffers: Practical Illustrations

Upstream Inventory Policies: *(s,Q)*-Policy



- The inventory level is **observed at all times**: **continuous review policy.**
- When the level declines to some specified **reorder point *s***, an order is placed for a fixed **lot size *Q***.
- The order arrives to replenish the inventory after a **lead time *L***.
- **Index policy:** threshold together with a time varying index. If the index exceeds the threshold, an appropriate action is triggered (place an order) => efficient and **easily implementable in practice**.

Upstream Inventory Policies: *(T, S)*-Policy

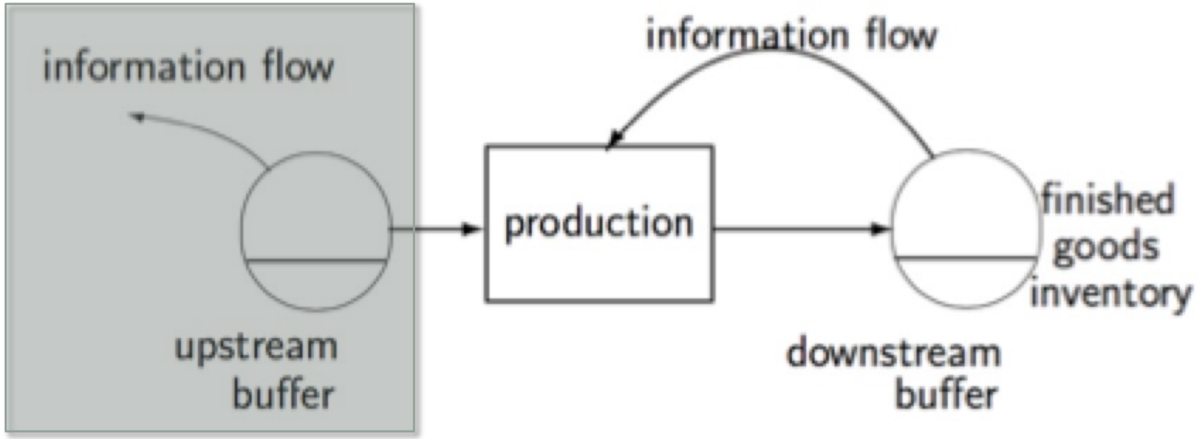


- The inventory level is **observed at time intervals of length *T***: **periodic review policy.**

- If the inventory, upon inspection, is at level y, a quantity $S-y$ is ordered to bring the inventory position back to **order level** S

- The order arrives to replenish the inventory after a **lead time *L***.

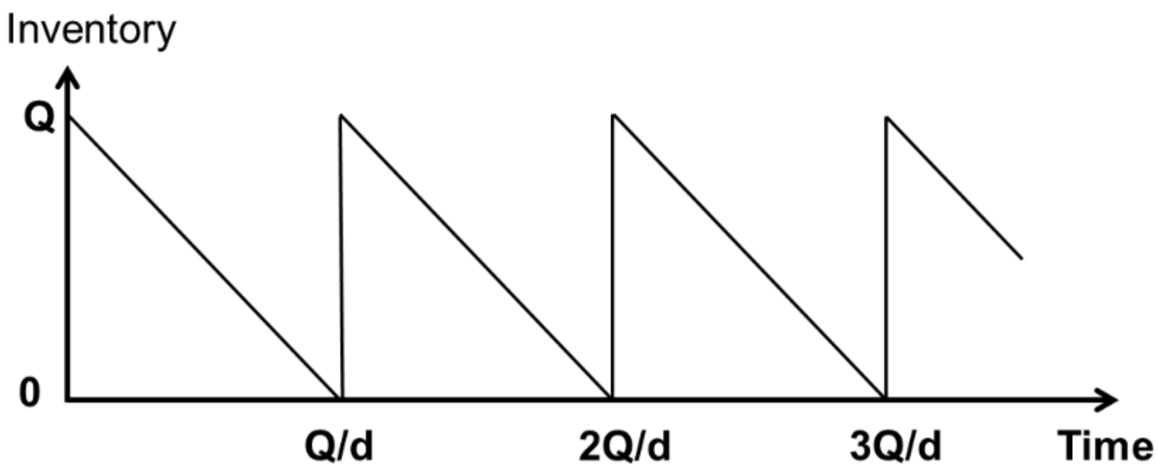
Upstream Inventory Policies



Exercise 7 Explain in your own words why continuous review policies are more effective in reducing variability than periodic review systems.

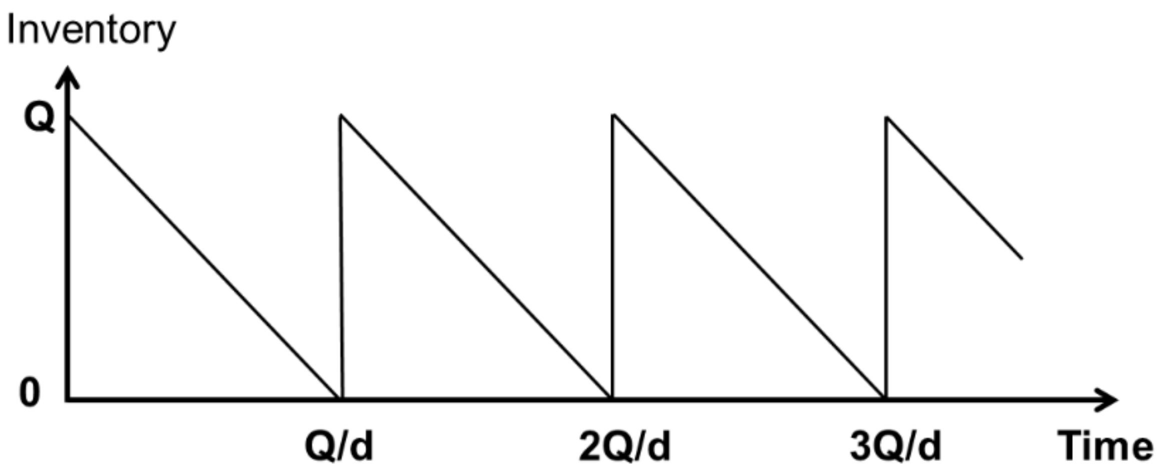
https://www.youtube.com/watch?v=z2CtJLgHX1Q

Deterministic Lot Size Model with no Shortage



- **Economic Order Quantity** (EOQ) inventory model (Harris, 1913)
- **Deterministic**
- Inventory level ranges between 0 and Q (**no shortage**)
- *N.B.* Lead time has no impact if the demand is deterministic and at a constant rate

Deterministic Lot Size Model with no Shortage

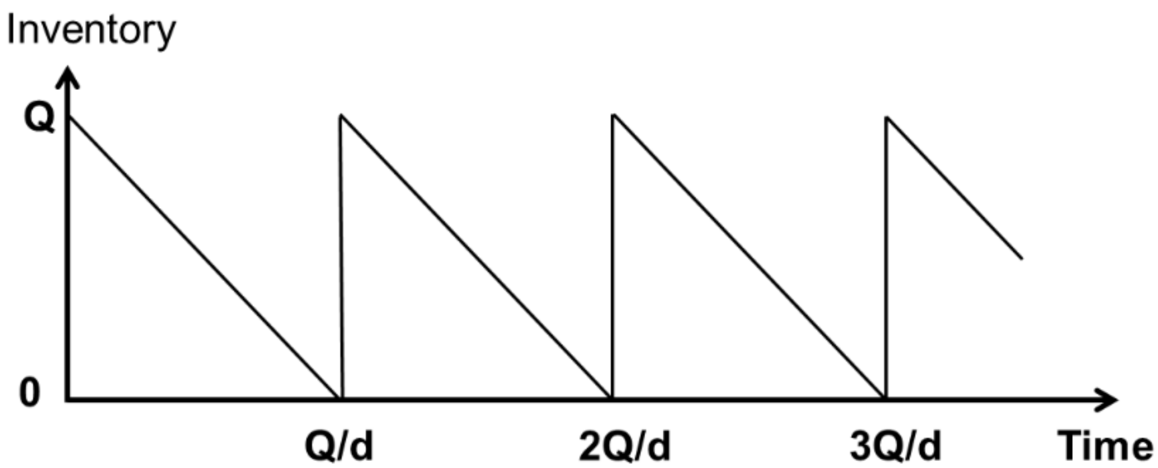


- Periodically, an order of size *Q* (fixed and constant) is placed for the replenishment of the inventory

- The arrival of the order is assumed to occur instantaneously, causing the inventory level to shoot from 0 to *Q* (*N.B.* **no lead time**)

- Between replenishment orders, the inventory decreases at a constant rate *d* (**constant demand**)

Deterministic Lot Size Model with no Shortage

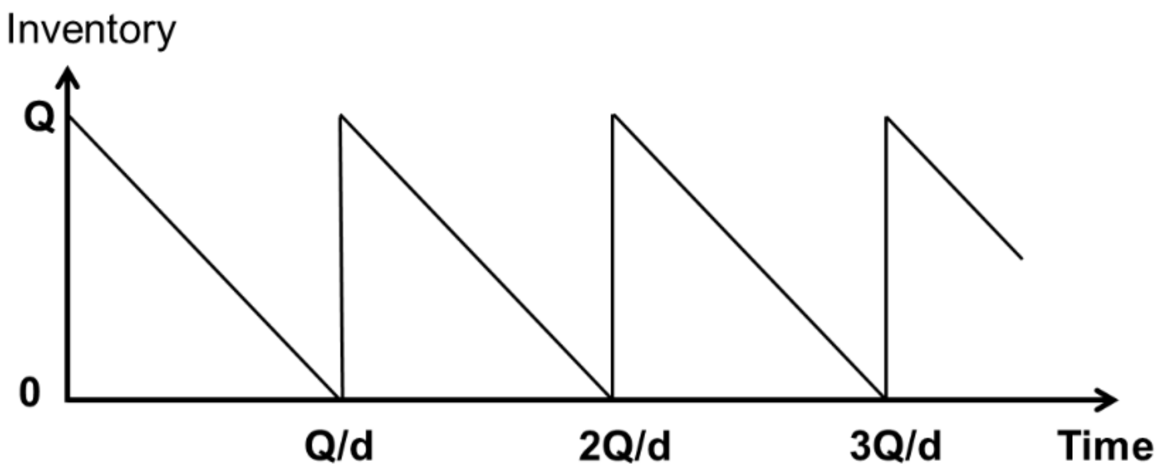


Cost Analysis

- **Set-up costs:** $K\frac{d}{Q}$, Q/d = time between 2 orders
- **Product costs:** cd
- **Holding costs:** $h\frac{Q}{2}$

- **Total costs:** $Y(Q) = K\frac{d}{Q} + cd + h\frac{Q}{2}$

Deterministic Lot Size Model with no Shortage

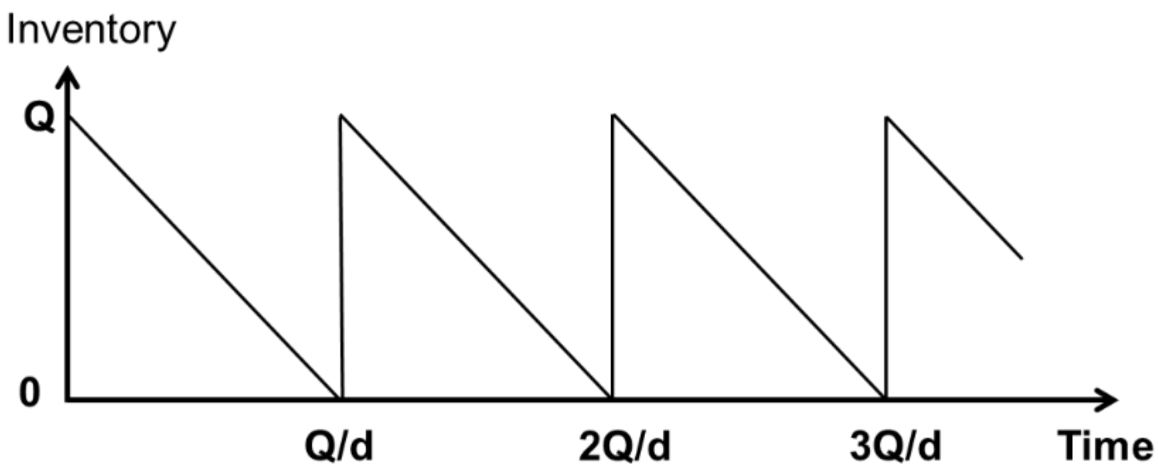


Exercise 8 For the above EOQ-model, show that the optimal order quantity is given by $Q^* = \sqrt{2dK/h}$ and that the resulting inventory costs per time unit are given by:

$$Y^* = \sqrt{dhK/2} + cd + \sqrt{dhK/2},$$

and, therefore that, at the optimum, the holding costs are equal to the setup costs. ⊙

Deterministic Lot Size Model with no Shortage



Optimal order quantity: $Q^* = \sqrt{2dK/h}$

Resulting costs per unit time: $Y^* = \sqrt{dhK/2} + cd + \sqrt{dhK/2}$

There is a **tradeoff between lot size and inventory**: increasing the lot size increases the average amount of inventory on hand, but reduces the frequency of ordering.

Deterministic Lot Size Model with no Shortage

Exercise 9 A second insight that follows from the above EOQ-model is that **holding and setup costs are fairly insensitive to lot size.**

To see this, show that for $c = 0$ (because lot size does not affect the sensitivity of setup costs):

$$\frac{Y}{Y*} = \frac{1}{2}\left(\frac{Q}{Q*} + \frac{Q^*}{Q}\right)$$

To understand the above formula, suppose that $Q = 2Q^*$, which implies that we use a lot size twice as large as the optimal one. Then, the ratio of the resulting holding plus setup costs to the optimum is equal to 1.25. That is, a 100% error in the lot size results in a 25% error in terms of costs. Notice that, if $Q = Q^*/2$, we also get an error of 25% percent in terms of costs. ⊙

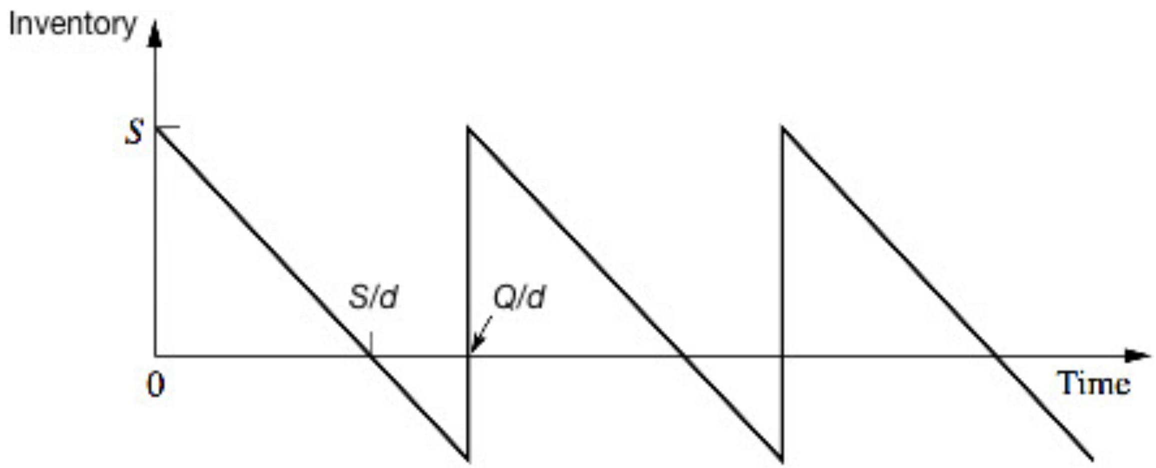
Deterministic Lot Size Model with no Shortage

Exercise: Using AnyLogic, simulate the dynamics of the deterministic lot size model with no shortage. Then, add uncertainty to the demand and/or the lead-time, and observe the resulting behavior.

Other Inventory Models

Deterministic Lot Size Model with Planned Shortages

- When customers are willing to accept a reasonable delay in filling their orders

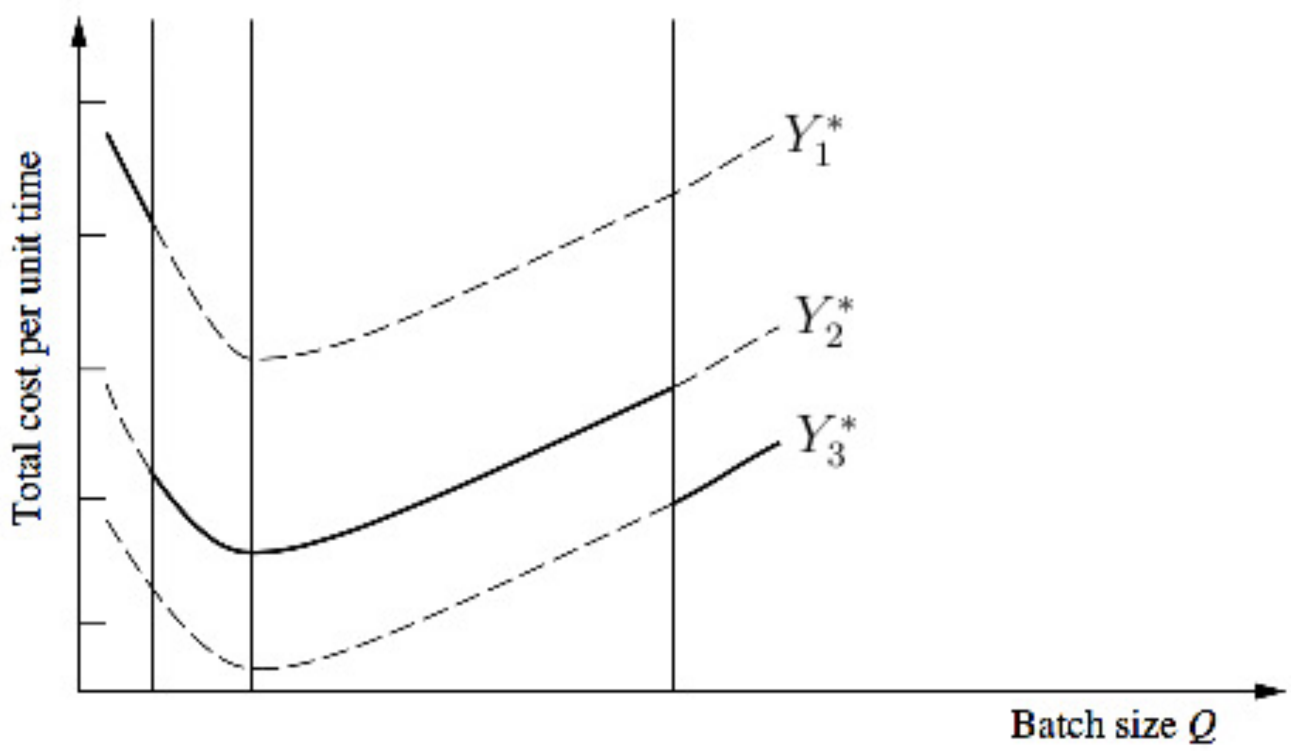


$$Q^* = \sqrt{\frac{2dK}{h}}\sqrt{\frac{p+h}{p}} \qquad S^* = \sqrt{\frac{2dK}{h}}\sqrt{\frac{p}{p+h}}$$

Other Inventory Models

Deterministic Lot Size Model with Quantity Discount

- When the unit cost of an item depends on the quantity in the batch



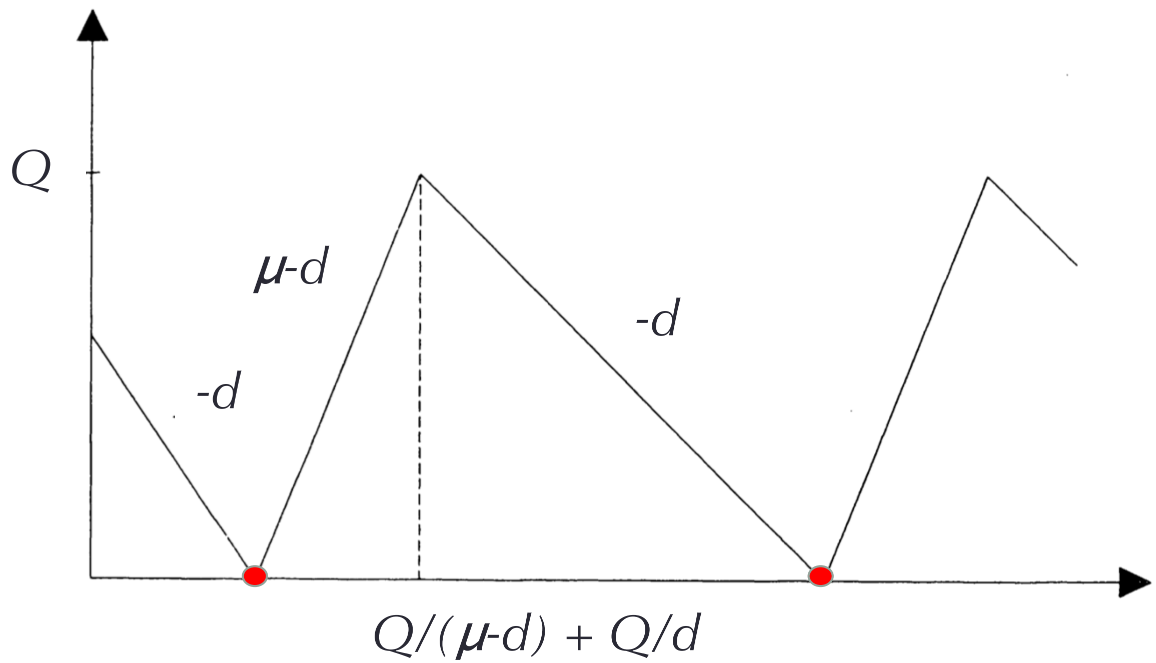
Optimal Stocking in Batch Production

Stock control with **replenishment through production**.

Continuous withdrawals from the finished part inventory (rate d).

d: demand rate, μ: production rate, $\mu > d$

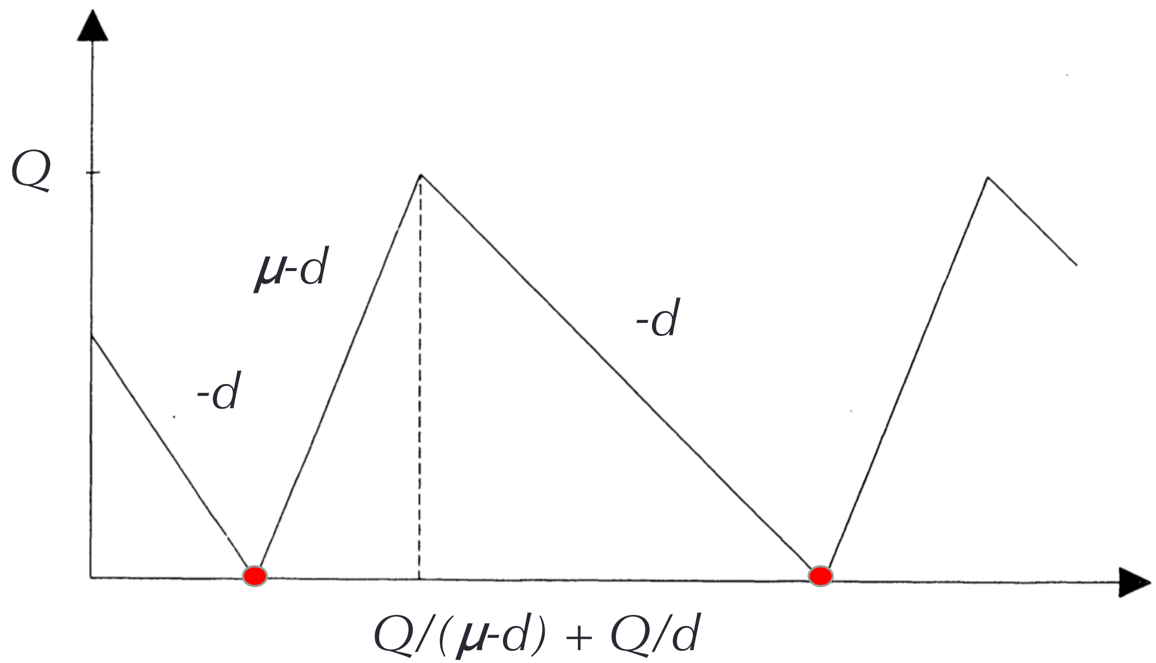
Q: net lot size (minus continuous withdrawal during production)



Optimal Stocking in Batch Production

Total costs per unit time: $$Y(Q) = \frac{K + cQ + h\frac{Q}{2} \cdot Q\left(\frac{1}{d} + \frac{1}{\mu - d}\right)}{Q\left(\frac{1}{d} + \frac{1}{\mu - d}\right)}$$

Optimal lot size: $$Q^* = \sqrt{\frac{2K}{h} \cdot \frac{1}{\frac{1}{d} + \frac{1}{\mu - d}}}$$



News Vendor Problem

In the EOQ inventory model, demand was assumed to be deterministic

Here, **demand is random**

We assume **single stock replenishment**

Use case: a person who purchases newspapers at the beginning of the day, sells a random amount of them to clients, and then must **discard any leftovers**

Objective: determine quantity of newspapers for the stock at the beginning of the day (upstream buffer)

News Vendor Problem

Trade-off between:

- Ordering too much (waste)
- Ordering too little (excess demand is lost)

Other examples (**perishable products**):

- Restaurant
- Fashion
- High-Tech
- Christmas Trees

News Vendor Problem

Information to consider:

- **anticipated demand**
- **costs of buying too many** newspapers **or too few** of them

To develop a formal model, we make the following **assumptions**:

(1) Products are **separable**.
(2) Planning is done for a **single period**
(3) **Demand** X is random with a **known probability density function** $g(x)$
(4) **Deliveries** of the order quantity Q are **made in advance of demand**.
(5) **Costs** of overage c_o or shortage c_s are **linear**

News Vendor Problem

Expected inventory costs in function of ordered quantity Q:

$$Y(Q) = c_o \int_0^Q (Q - x) g(x)\, dx + c_s \int_Q^\infty (x - Q) g(x)\, dx$$

The value $Q*$ that minimizes $Y(Q)$ satisfies:

$$G(Q^*) = \frac{c_s}{c_o + c_s}$$

Conclusion: in an environment of **uncertain demand**, the appropriate order quantity depends on **both** the distribution of demand and the relative costs of ordering too much or too little.

News Vendor Problem

Since $G(Q*)$ represents the probability that demand is less or equal than $Q*$, $Q*$ should be chosen such that the probability of having enough stock to meet demand is $\frac{c_s}{c_o + c_s}$

Since $G(x)$ increases with x, we see:

- **Increasing c_s (shortage costs) will increase $Q*$**

=> *When shortage costs are high, meaning that the profits when selling one unit of product is large, we want to have a limited probability of shortage and order more products for the initial stock.*

- **Increasing c_o (overage costs) will decrease $Q*$**

=> *When overage costs are high, meaning that the buying price of the products is large, we want to limit the probability of overage and order less products for the initial stock.*

News Vendor Problem

Exercise 10 In the context of the news vendor problem formulated above, suppose that the demand follows a normal distribution of the form $X \sim \mathcal{N}(\mu, \sigma^2)$. In this situation, show that the optimal order quantity Q^* is given by:

$$Q^* = \mu + z\sigma$$

where z is such that $\Phi(z) = \frac{c_s}{c_o + c_s}$ with $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ is the cumulative distribution function of the standard normal distribution $\mathcal{N}(0, 1)$.

Conclude that, if the demand X is normally distributed, then when the variability of the demand increases, it will increase the order quantity if $\frac{c_s}{c_o + c_s} > 0.5$ and decrease it if $\frac{c_s}{c_o + c_s} < 0.5$. $\odot$

News Vendor Problem

Exercise 11 Suppose that a set of lights costs 1\$ to make and distribute and sells for 2\$. Any sets not sold by Christmas will be discounted to 0.5\$. In terms of the above modelling notation, this means that the overage costs per unit are the lost amount per excess set of lights: $c_o = \$(1-0.50) = 0.5\$$. The shortage costs per unit are the lost profit from a sale, or $c_s = \$(2-1) = 1\$$. Suppose further that demand has been forecast to be 10’000 units with a standard deviation of 1’000 units, and that the normal distribution is a reasonable representation for the demand. How much units should the firm produce? (Answer: 10’430 units) ⊙

News Vendor Problem: Other Models

With **initial stock**

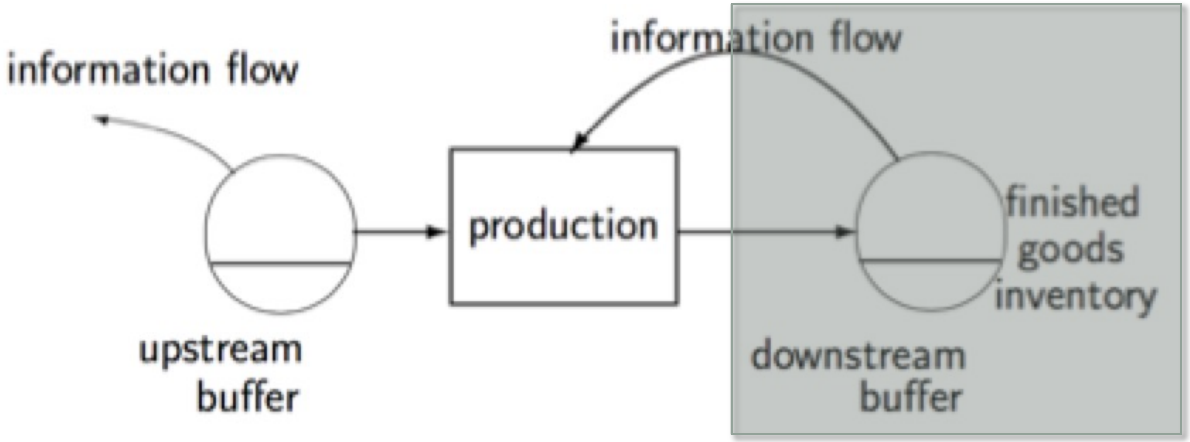
With **non-linear** holding and/or shortage **costs**

With **set-up costs**

Determine **optimal sales period length**

Overbooking

Downstream Inventory Policies



Downstream Inventory Policy: must ensure for the market (or for the next service element in a production or supply chain) availability of finished goods

Objective: how many finished goods should the production engineer keep in the downstream buffer?

Downstream Buffers: Practical Illustrations

Downstream Inventory Policies

Make-to-Stock:

- **Items are available when a customer arrives**
- **Large demand volumes** and **cheap raw material and products**.

Make-to-Order:

- **Production starts after order arrives**.
- **Low demand volumes** and **expensive raw materials or products**

Conflicting objectives:

- Make-to-Stock: need **large inventories** to prevent from stock-outs but **small inventories** to keep costs low!
- Make-to-Order: need **high production capacities** to allow early and reliable delivery promises, but **low production capacities** to keep costs low!

Downstream Inventory Policies: Single-Part Production-Inventory Model

Formulated as a **fluid model**

d: demand rate
x(t): content of the downstream buffer at time *t*
M: maximum shortage
K: maximum inventory level

Then, $-M \leq x(t) \leq K$

The machine produces finished goods at rate $u(t) \in [0, U]$, with U being the maximum production capacity

Objective: determine **at which rate *u(t)* the machine must produce at any time *t*?**

Downstream Inventory Policies: Single-Part Production-Inventory Model

x(t) satisfies the following dynamical equation:

$$\dot{x}(t) = \begin{cases} \max\{u(t)-d,0\} & \text{if } \; x(t) = -M \\ u(t)-d & \text{if } \; -M < x(t) < K \\ \min\{u(t)-d,0\} & \text{if } \; x(t) = K \end{cases}$$

Scheduling problem: determine the **optimal control** *u(t)* which minimizes a given cost function

Running cost associated to positive inventory (modelling handling costs) and shortage (modelling unsatisfied demand): $g\left(x(t)\right) = h^{+}x^{+}(t) + h^{-}x^{-}(t)$

u(t) must minimize the long-term average expected costs:

$$J = \lim_{T\to\infty} \frac{1}{T} E \int_0^T g(x(t))dt.$$

complex problem!