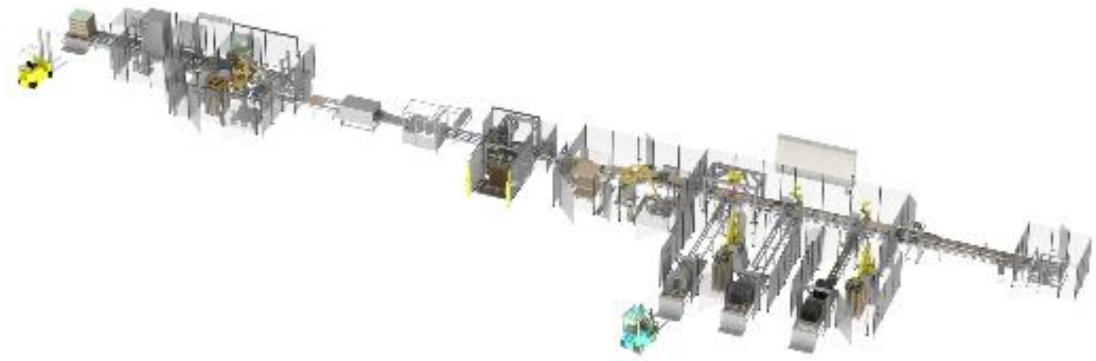


MANUFACTURING SYSTEMS AND SUPPLY CHAIN DYNAMICS

Chapter 11: Reliability Analysis

EPFL, Master MT

Roger Filliger (BFH), Olivier Gallay (UniL)

Course Content

We distinguish:

software reliability

human reliability

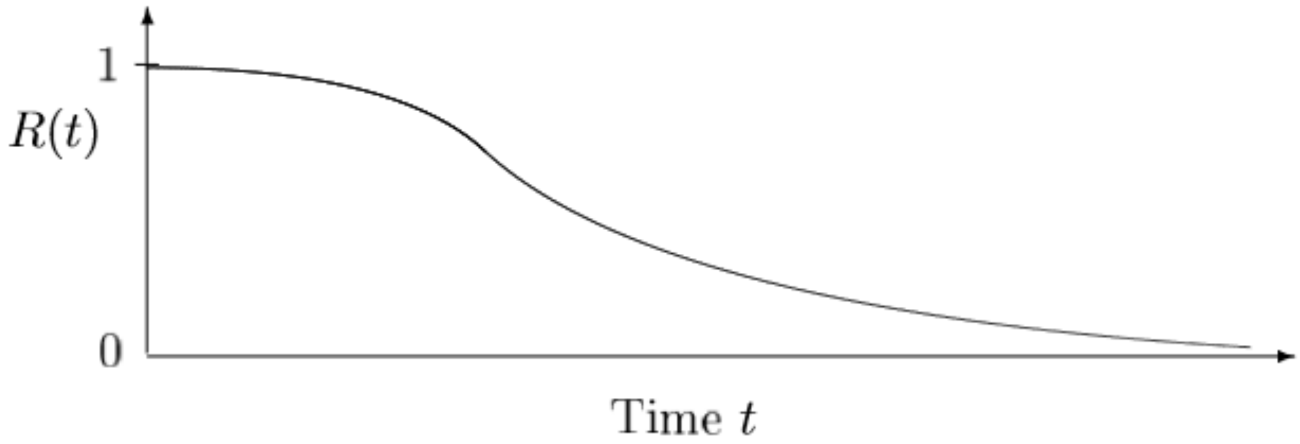
hardware reliability

Reliability of an item (hardware)

Definition 2 *we define the reliability of an item at time t, $R(t)$, as:*

$$R(t) = P(T > t) = 1 - F(t) \qquad (11.3)$$

where T is the (random) time to failure of the item with cumulative distribution function $F(t) := P(T \leq t)$.



Reliability of an item

Example 12 Suppose that the time to failure is exponentially distributed with parameter λ. The resulting reliability function is simply

$$R(t) = e^{-\lambda t}, \quad t > 0$$

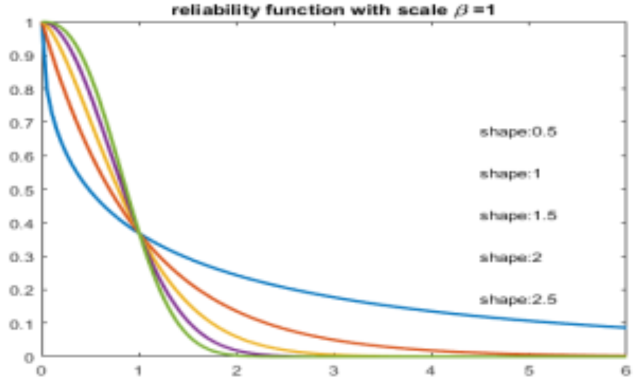
Example 13 If time to failure is uniformly distributed over the interval $[0, \tau]$, the resulting reliability function decreases linearly:

$$R(t) = 1 - \frac{t}{\tau}, \quad 0 \leq t \leq \tau$$

Reliability of an item: Weibull distribution

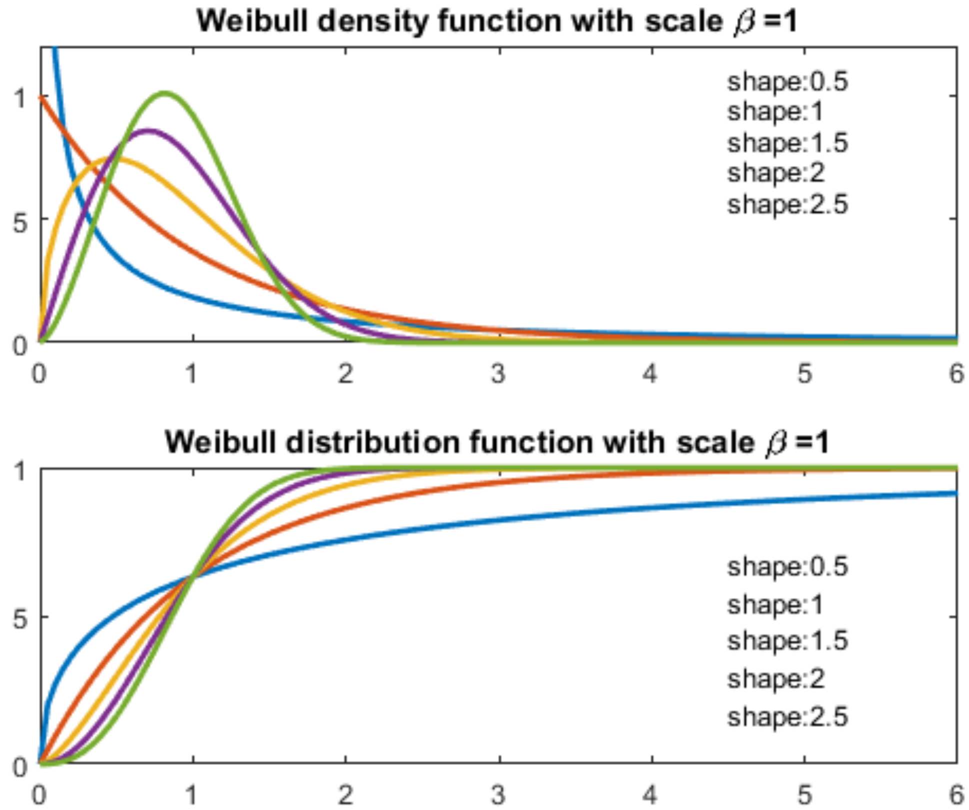
Exercise 44 Waloddi Weibull (1887-1979) found that very often, the cumulative distribution of data on product life can be modeled by a function of the following form:

$$R(t) = \begin{cases} e^{-((t-\gamma)/\beta)^{\alpha}} & \text{for} \quad t > \gamma \\ 1 & \text{for} \quad t \leq \gamma \end{cases}$$




```
X=0:0.05:6;
scale=0.5:0.5:3;
shape=0.5:0.5:2.5;
figure(1)
for s =shape
Y = wblpdf(X,1,s);
subplot(211), plot(X,Y,'LineWidth',2)
text(4.5,1.2-s/4,['shape:', num2str(s,2)])
axis([0 6 0 1.2])
title('Weibull density function with scale \beta =1')
hold on
end
hold off

for s =shape
Z = wblcdf(X,1,s);
subplot(212), plot(X,Z,'LineWidth',2)
text(4.5,0.8-s/4,['shape:', num2str(s,2)])
title('Weibull distribution function with scale \beta =1')
hold on
end
hold off
```



Reliability of an item

If T follows a Weibull distribution with location parameter γ, scale parameter β and shape parameter α, we have:

$$
\begin{aligned}
E(T) &= \beta\Gamma\left(1+\frac{1}{\beta}\right)+\gamma \\
Var(T) &= \beta^2\left[\Gamma\left(1+\frac{2}{\cancel{\beta}}\right)-\Gamma^2\left(1+\frac{1}{\cancel{\beta}}\right)\right]
\end{aligned}
$$

α α

where $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$.

Use the Matlab command

pd = fitdist(x,'Weibull')

to estimate the Weibull shape and scale parameters fitting the data in x

Reliability of an item

Exercise 45 Suppose that time to failure T follows a truncated normal distribution with density

$$f(x) = \begin{cases} \frac{1}{b\sigma}\phi(\frac{x-\mu}{\sigma}), & x > 0, \\ 0 & \text{otherwise} \end{cases} \tag{11.4}$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

and the normalization

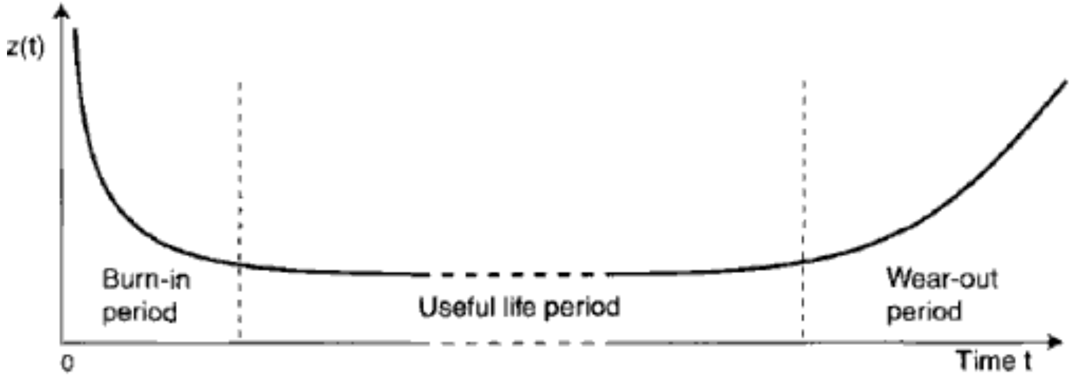
$$b = \frac{1}{\sigma} \int_0^\infty \phi\Big(\frac{x-\mu}{\sigma}\Big) dx$$

For $\mu = 3$ and $\sigma = 1$, draw, using your favorite maths tool, the associated reliability function $R(t)$ for $0 < t < 10$. ⊙

The failure rate z

Definition 3 *The failure rate function $z(t)$ is given by:*

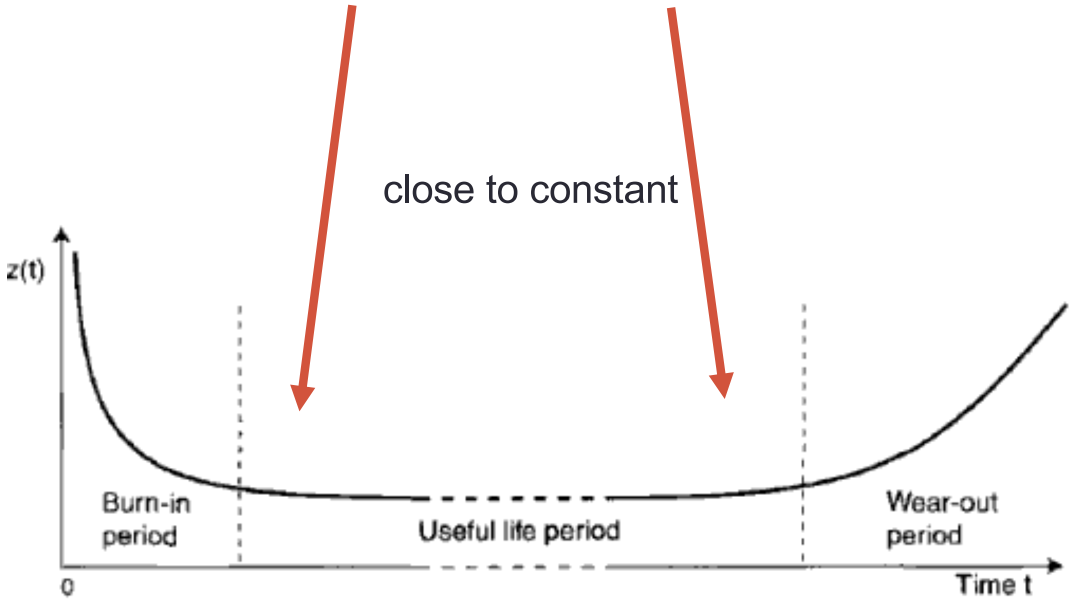
$$z(t) = \lim_{\Delta t \to 0} \frac{P(t < T \leq t + \Delta t \mid T > t)}{\Delta t} = \frac{f(t)}{R(t)} = \frac{f(t)}{1 - F(t)}$$



Typical shape of a failure rate: bathtube

A first (and useful) example

Suppose $T \sim Exp(\lambda)$, show that the associated failure rate is a constant!



z(t), R(t) and F(t) have the same information content!

Exercise 46 Show that $z(t) = -\frac{d}{dt}\ln\big(R(t)\big)$ and, using $R(0) = 1$, that

$$R(t) = e^{-\int_0^t z(u)\,du}$$

and therefore that the failure rate completely defines the density and hence the law of T:

$$f(t) = z(t)e^{-\int_0^t z(u)\,du}$$

⇒ Estimating z(t) is as difficult as estimating f(t) or R(t)

A second example

Exercise 57 Let T_1 and T_2 be two independent times to failure with constant failure rates λ_1 and λ_2. Let $T = T_1 + T_2$

(i) Show that the survivor function of T is:

$$R(t) = P(T > t) = \frac{1}{\lambda_2 - \lambda_1}(\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t}).$$

(ii) Find the corresponding failure rate function $z(t)$, and make a sketch of $z(t)$ as a function of t for selected values of λ_1 and λ_2.

A last calculatory example

Exercise 58 A component may fail due to two different causes, A and B. It has been shown that the time to failure T_A caused by A is exponentially distributed with parameter λ_A, while the time to failure T_B caused by B has an exponential density with parameter λ_B.

(i) Describe the rationale behind using:

$$f(t) = p \cdot f_A(t) + (1-p) \cdot f_B(t), \quad (\text{for some } p \in [0,1])$$

as the probability density function for the time to failure T of the component.

(ii) Explain the meaning of p in this model.

(iii) Show that a component with probability density $f(t)$ has a DFR function.

Estimate the failure rate z(t):

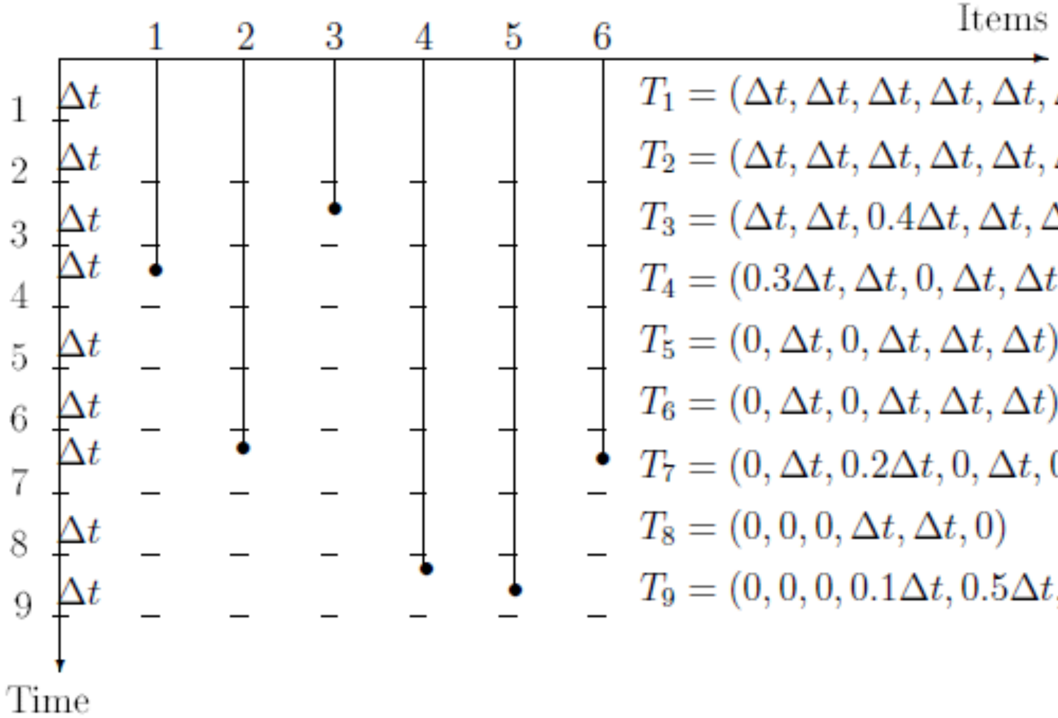
To determine the form of the failure rate function $z(t)$ for a given type of items, the following experiment may be carried out:

Split the time interval $(0, t)$ into disjoint intervals of equal length Δt. Then put n identical items into operation at time $t = 0$. When an item fails, note the time and leave that item out. For each interval i record:

(i) The number of items $n(i)$ that fail in interval i.

(ii) The functioning times for the individual items $(T_{1i}, T_{2i}, ..., T_{ni})$ in interval i. Hence T_{ji} is the time item j has been functioning in time interval i . T_{ji} is therefore equal to 0 if item j has failed before interval i (and equals Δt if the item survives during the whole duration of interval i), where $j = 1, 2, ..., n$.

Thus $\sum_{j=1}^{n} T_{ji}$ is the total functioning time for the items in interval i. Now



$n(1) = 0$ — $T_1 = (\Delta t, \Delta t, \Delta t, \Delta t, \Delta t, \Delta t)$

$n(2) = 0$ — $T_2 = (\Delta t, \Delta t, \Delta t, \Delta t, \Delta t, \Delta t)$

$n(3) = 1$ — $T_3 = (\Delta t, \Delta t, 0.4\Delta t, \Delta t, \Delta t, \Delta t)$

$n(4) = 1$ — $T_4 = (0.3\Delta t, \Delta t, 0, \Delta t, \Delta t, \Delta t)$

$n(5) = 0$ — $T_5 = (0, \Delta t, 0, \Delta t, \Delta t, \Delta t)$

$n(6) = 0$ — $T_6 = (0, \Delta t, 0, \Delta t, \Delta t, \Delta t)$

$n(7) = 2$ — $T_7 = (0, \Delta t, 0.2\Delta t, 0, \Delta t, 0.6\Delta t)$

$n(8) = 0$ — $T_8 = (0, 0, 0, \Delta t, \Delta t, 0)$

$n(9) = 2$ — $T_9 = (0, 0, 0, 0.1\Delta t, 0.5\Delta t, 0)$

Estimate the failure rate z(t):

To determine the form of the failure rate function $z(t)$ for a given type of items, the following experiment may be carried out:

Split the time interval $(0, t)$ into disjoint intervals of equal length Δt. Then put n identical items into operation at time $t = 0$. When an item fails, note the time and leave that item out. For each interval i record:

(i) The number of items $n(i)$ that fail in interval i.

(ii) The functioning times for the individual items $(T_{1i}, T_{2i}, ..., T_{ni})$ in interval i. Hence T_{ji} is the time item j has been functioning in time interval i . T_{ji} is therefore equal to 0 if item j has failed before interval i (and equals Δt if the item survives during the whole duration of interval i), where $j = 1, 2, ..., n$.

Thus $\sum_{j=1}^{n} T_{ji}$ is the total functioning time for the items in interval i. Now

$$z(i) = \frac{n(i)}{\sum_{j=1}^{n} T_{ji}}$$

shows the number of failures per unit functioning time in interval i and is therefore a natural estimate of the failure rate in interval i for the items that are functioning at the start of this interval.

Let $m(i)$ denote the number of items that are functioning at the start of interval i. If Δt is not too big, we have:

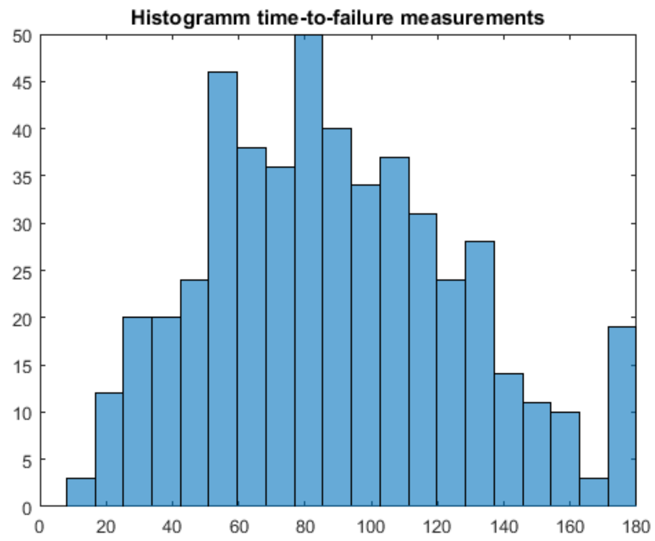
$$z(i) \approx \frac{n(i)}{m(i)\Delta t} \tag{11.7}$$

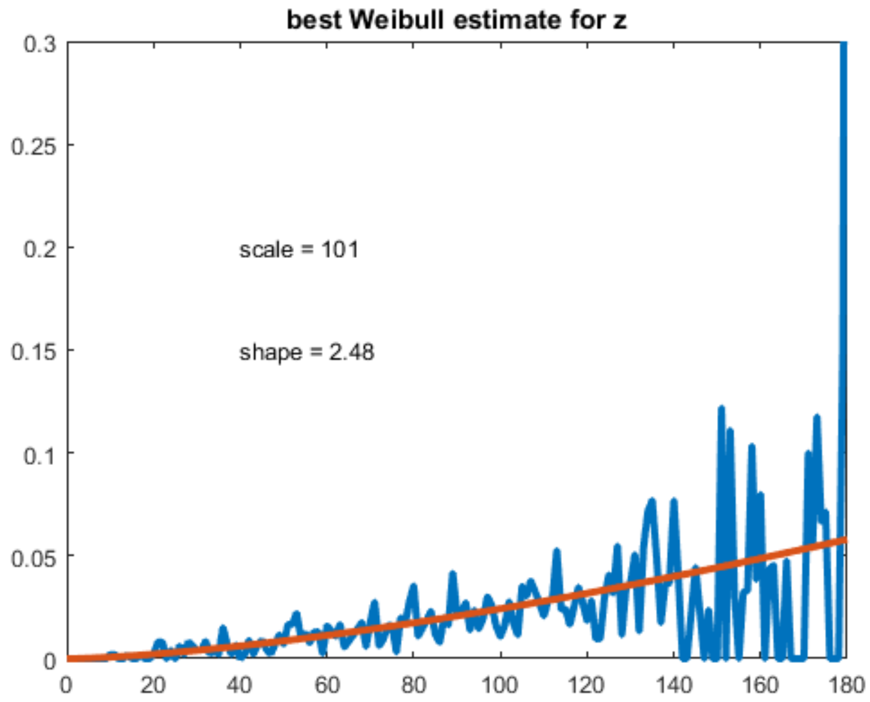
and hence

$$z(i)\Delta t \approx \frac{n(i)}{m(i)} = \frac{\text{The number of items that fail in interval } i}{\text{the number of items that are functioning at the start of interval}}$$

Estimate the failure rate z(t):

Exercise 49 Download from our Moodle platform the time-to-failure data "TimeToFailure" from 500 identical workpieces and estimate the failure rate function $z(t)$. Using "fitdist" on moodle, fit a Weibull distribution through $z(i)$ and give your best estimate for the scale and shape parameter. ⊙





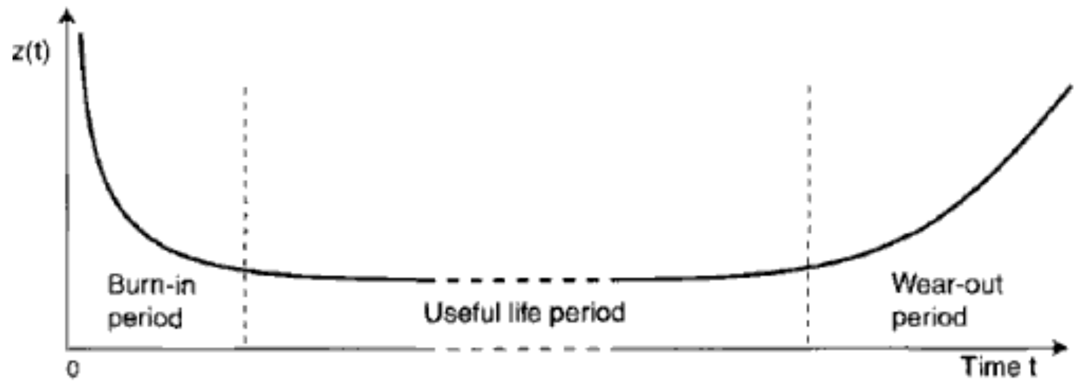
Estimate the failure rate z(t):

```
%Exercise 49
TLimit=180;
deltaT=1;
load('TimeToFailure.mat');%read Data
data=TimeToFailure;
Index=find(data>TLimit);%collect data surviving TLimit
data(Index)=TLimit;%define upper bound as =TLimit
figure(1)
histogram(data,20)
title('Histogramm time-to-failure measurements')
T=0:deltaT:TLimit;
epsilon=10^(-9);
nFailure=zeros(size(T));
m=zeros(size(T));
z=zeros(size(T));
for i=1:length(T)
    nFailure(i)=sum(and(data>=((i-1)*deltaT),data<(i*deltaT)));
    m(i)=sum(data>=(i-1)*deltaT);
    if m(i)==0
        z(i)=nFailure(i)/(m(i)*deltaT+epsilon);
    else
        z(i)=nFailure(i)/(m(i)*deltaT);
    end
end
figure(2)
pd=fitdist(data','Weibull');
y = pdf('Weibull',T,pd.A,pd.B)./(1-cdf('Weibull',T,pd.A,pd.B));
plot(T,z,T,y,'LineWidth',3)
title('best Weibull estimate for z')
axis([0 180 0 0.3])
text(40,0.20,['scale = ', num2str(pd.A,3)])
text(40,0.15,['shape = ', num2str(pd.B,3)])
```

Increasing failure rate: aging
decreasing failure rate: hardening/debugging

A life distribution T is said IFR (increasing failure rate) if the associated **failure rate function** $z(t)$ **is increasing**.

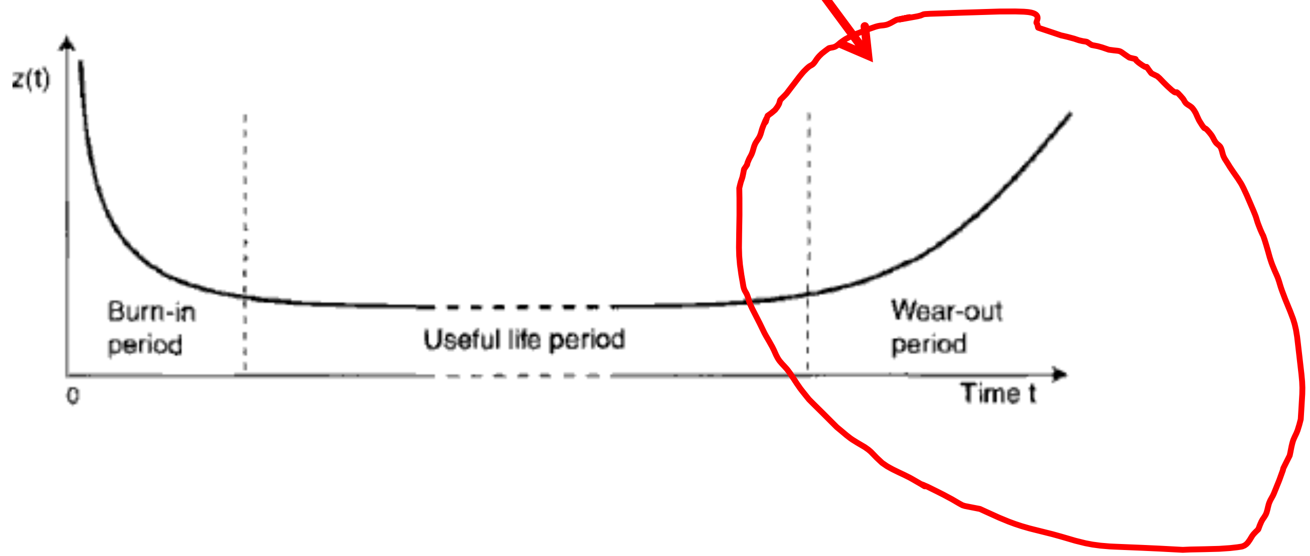
A life distribution T is said DFR (decreasing failure rate) if the associated **failure rate function** $z(t)$ **is decreasing**.



Increasing failure rate: aging

A life distribution T is said IFR (increasing failure rate) if the associated **failure rate function** $z(t)$ **is increasing**.

A life distribution T is said DFR (decreasing failure rate) if the associated **failure rate function** $z(t)$ **is decreasing**.



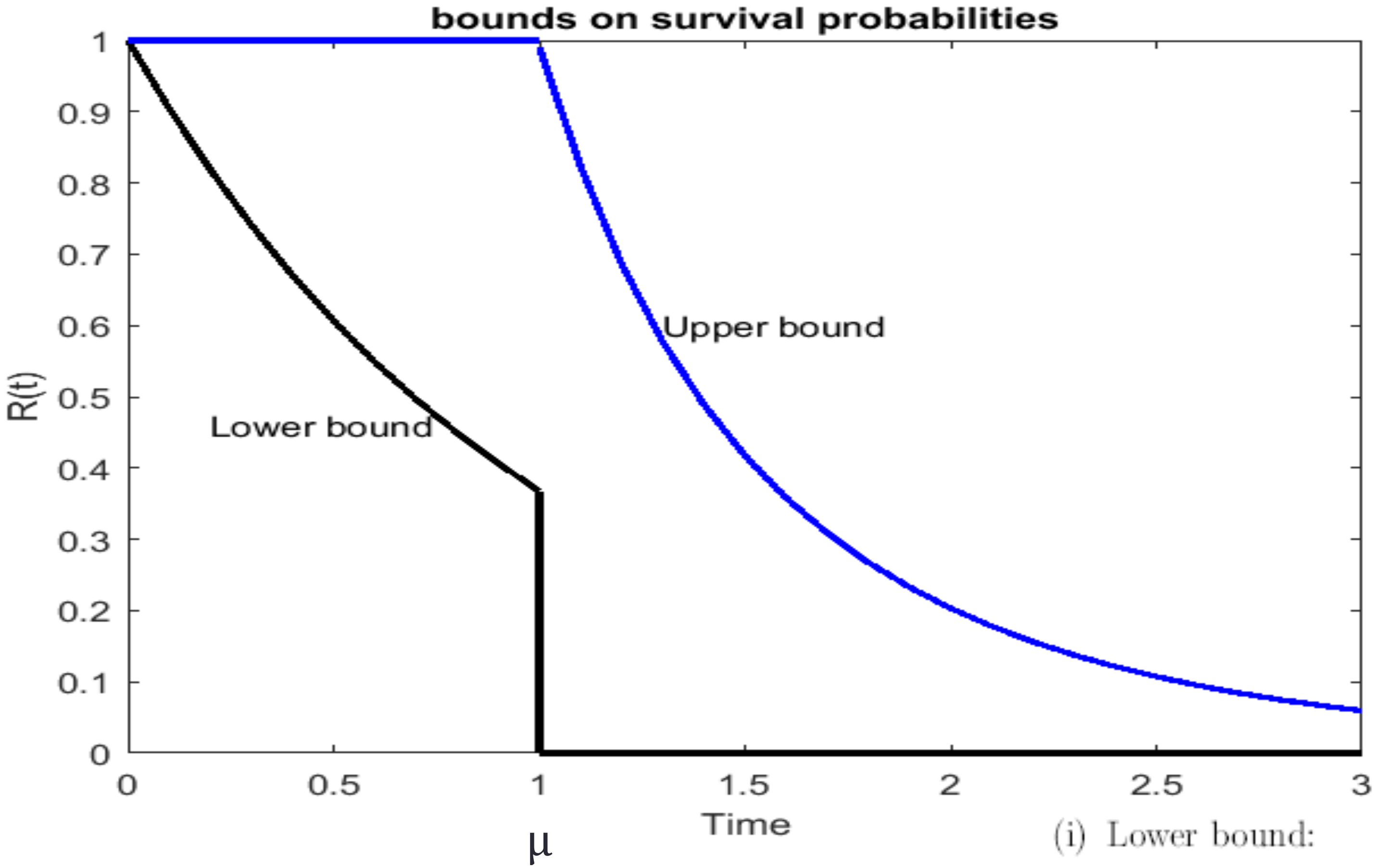
Increasing failure rate: aging

to a Weibull distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$: $F(t) = 1 - e^{-(t/\beta)^\alpha}$ for $t > 0$. We have:

$$z(t) = \frac{d}{dt}\left(t/\beta\right)^\alpha = \frac{\alpha}{\beta}\left(t/\beta\right)^{\alpha-1}$$

Since $\frac{d}{dx}\left(x^{\alpha-1}\right) = (\alpha-1)x^{\alpha-2}$, the Weibull distribution is IFR for $\alpha > 1$ and DFR for $\alpha < 1$. For $\alpha = 1$, the distribution is reduced to an exponential distribution with parameter $1/\lambda$, and hence is IFR as well as DFR.

Bounds for IFR with E(T)=μ=1



(i) Lower bound:

$$R(t) \geq \begin{cases} e^{-t\mu} & \text{if } t < \mu \\ 0 & \text{if } t \geq \mu \end{cases}$$

(ii) Upper bound:

$$R(t) \leq \begin{cases} 1 & \text{if } t \leq \mu \\ e^{-t\omega} & \text{if } t > \mu \end{cases}$$

where ω depends on t and satisfies $1 - \omega\mu = e^{-\omega t}$.

Bounds for IFR with E(T)=μ

Exercise 50 A company monitors gearboxes on vehicles by attaching wireless sensors to each gearbox to take vibration readings. The vibration signals are analyzed by a digital signal processing toolbox and indicate, as a result, the moment for necessary maintenance:

Gearbox	Operating hours before maintenance
1	8550
2	3215
3	9460
4	6650
5	7056
6	7604

Give a founded estimate for the reliability of a gearbox after 5000 operative hours. ⊙

(i) Lower bound:

$$R(t) \geq \begin{cases} e^{-t\mu} & \text{if } t < \mu \\ 0 & \text{if } t \geq \mu \end{cases}$$

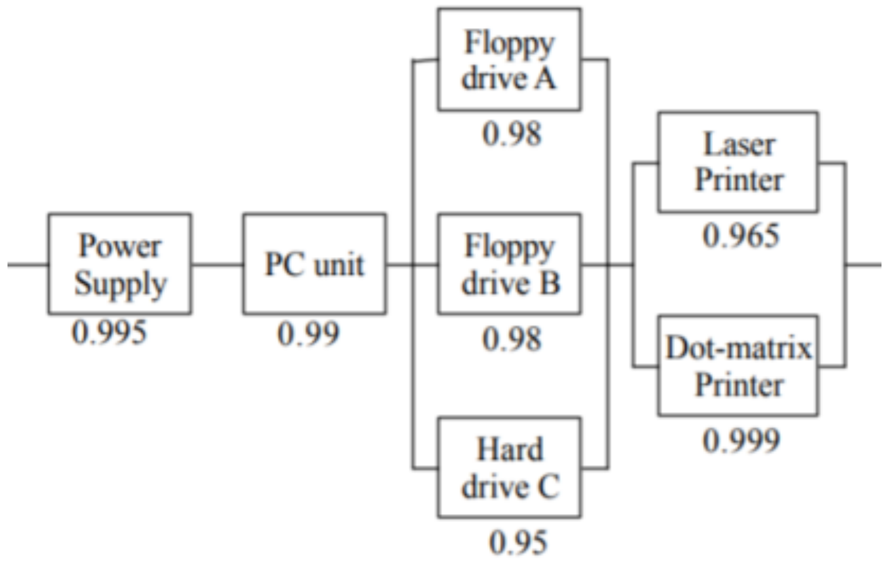
(ii) Upper bound:

$$R(t) \leq \begin{cases} 1 & \text{if } t \leq \mu \\ e^{-t\omega} & \text{if } t > \mu \end{cases}$$

where ω depends on t and satisfies $1 - \omega\mu = e^{-\omega t}$.

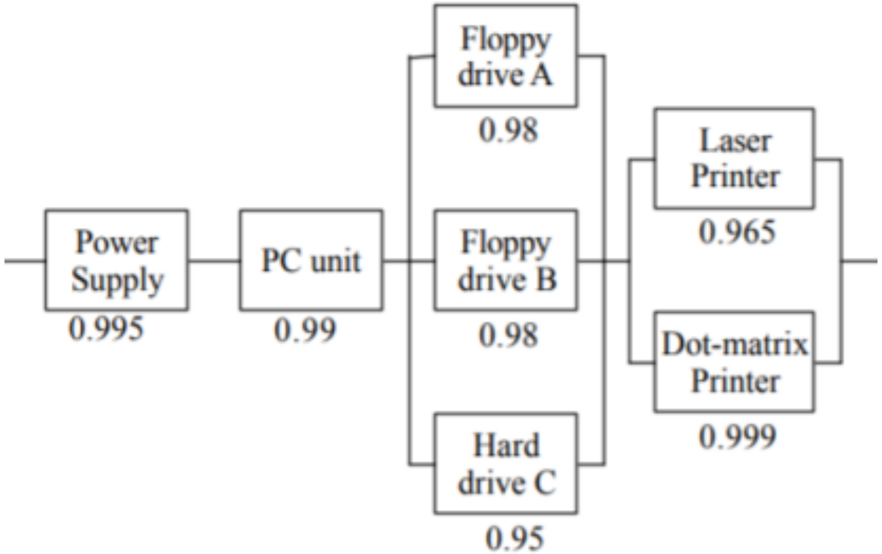
Reliability of a system composed of subsystems

R(t)=?



Reliability of a system

R(t)=?

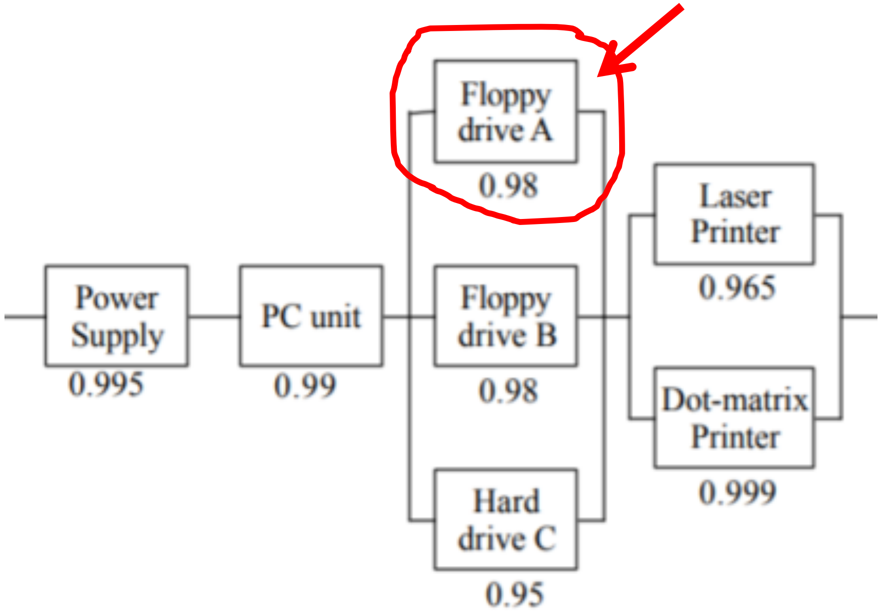


Consider a system consisting of n components, and suppose that each component is either functioning or has failed. In this paragraph[2], we ask how the system structure influences overall system reliability.

Reliability of a system

For component i, we define the indicator variable x_i by:

$$x_i = \begin{cases} 1, & \text{if the i'th component is functioning} \\ 0, & \text{if the i'th component has failed} \end{cases}$$



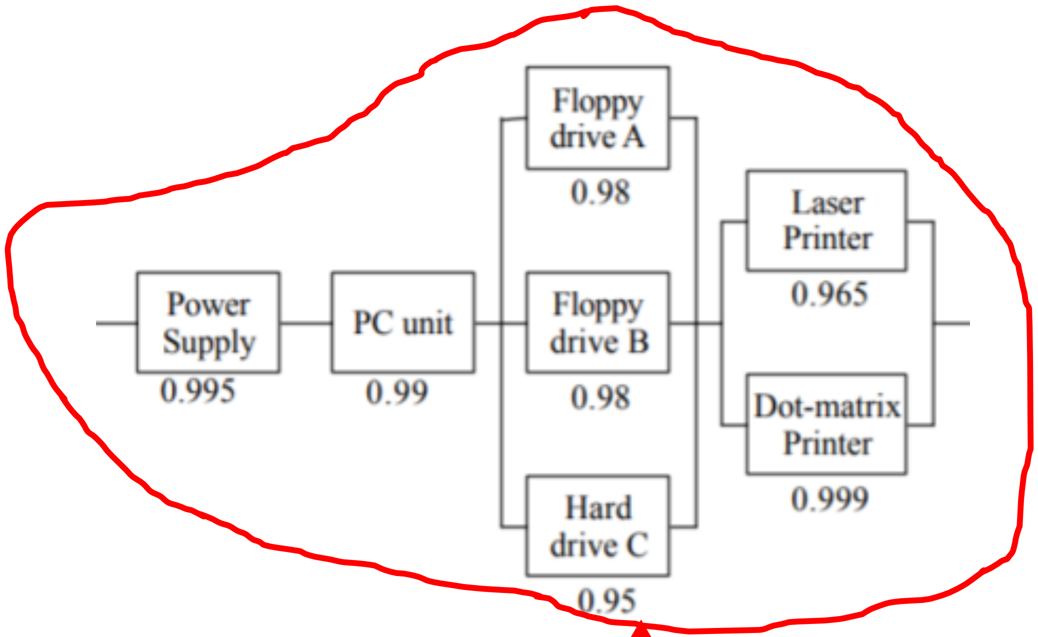
The vector $\mathbf{x} = (x_1, ..., x_n)$ is called the state vector of the system.

$$\phi(\mathbf{x}) = \begin{cases} 1, & \text{if the system is functioning when the state vector is } \mathbf{x} \\ 0, & \text{if the system has failed when the state vector is } \mathbf{x} \end{cases}$$

Reliability of a system

For component i, we define the indicator variable x_i by:

$$x_i = \begin{cases} 1, & \text{if the i'th component is functioning} \\ 0, & \text{if the i'th component has failed} \end{cases}$$



The vector $\mathbf{x} = (x_1, ..., x_n)$ is called the state vector of the system.

Structure function

$$\phi(\mathbf{x}) = \begin{cases} 1, & \text{if the system is functioning when the state vector is } \mathbf{x} \\ 0, & \text{if the system has failed when the state vector is } \mathbf{x} \end{cases}$$

Series structure and parallel structure

Example 15 The series structure: A series system is functioning if and only if all components are functioning. Hence, its structure function is given by

$$\phi(\mathbf{x}) = \min(x_1, ..., x_n) = \prod_{i=1}^{n} x_i$$

$\diamond$

Example 16 The parallel structure: A parallel system is functioning if and only if at least one of its components is functioning. Hence, its structure function is given by

$$\phi(\mathbf{x}) = \max(x_1, ..., x_n) = 1 - \prod_{i=1}^{n}(1 - x_i)$$

Structure function

$$\phi(\mathbf{x}) = \begin{cases} 1, & \text{if the system is functioning when the state vector is } \mathbf{x} \\ 0, & \text{if the system has failed when the state vector is } \mathbf{x} \end{cases}$$

The k-out-of-n structure

Example 17 The k-out-of-n structure: The series and parallel systems are both special cases of a k-out-of-n system. Such a system is functioning if and only if at least k of the n components are functioning. As $\sum_{i=1}^{n} x_i$ equals the number of functioning components, the structure function of a k-out-of-n system is given by

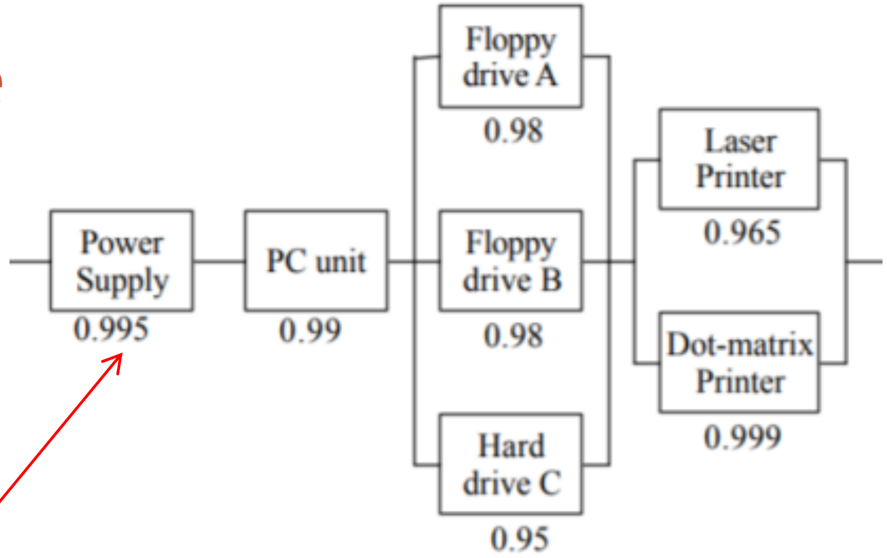
$$\phi(\mathbf{x}) = \begin{cases} 1, & \text{if } \sum_{i=1}^{n} x_i \geq k \\ 0, & \text{if } \sum_{i=1}^{n} x_i < k \end{cases}$$

◊

Structure function

$$\phi(\mathbf{x}) = \begin{cases} 1, & \text{if the system is functioning when the state vector is } \mathbf{x} \\ 0, & \text{if the system has failed when the state vector is } \mathbf{x} \end{cases}$$

Reliability of a structure



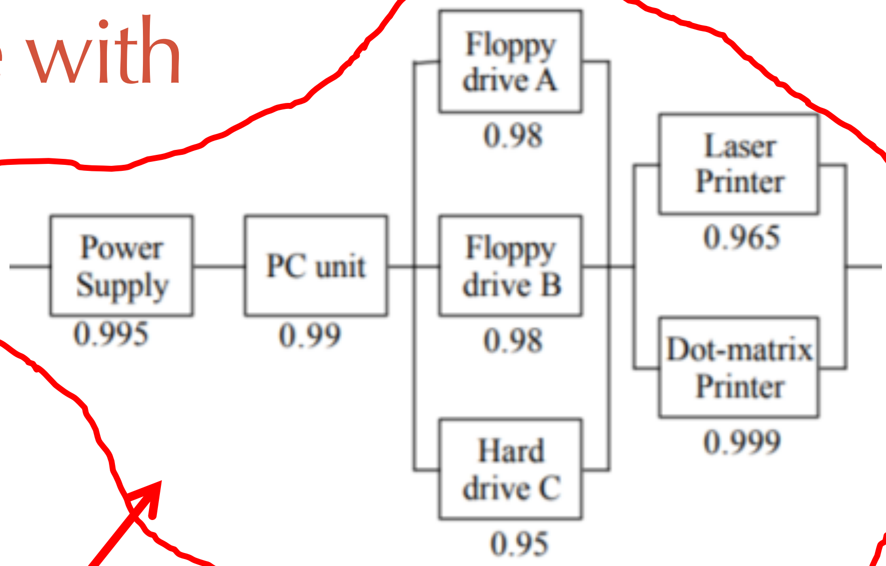
Let us suppose that the state of the i'th component is a Bernoulli random variable X_i, such that

$$P(X_i = 1) = p_i = 1 - P(X_i = 0)$$

The value p_i is the reliability of the i'th component. We define the reliability R of the system by

$$R = P(\phi(\mathbf{X}) = 1), \quad \text{where} \quad \mathbf{X} = (X_1, ..., X_n)$$

Reliability of a structure with independent components



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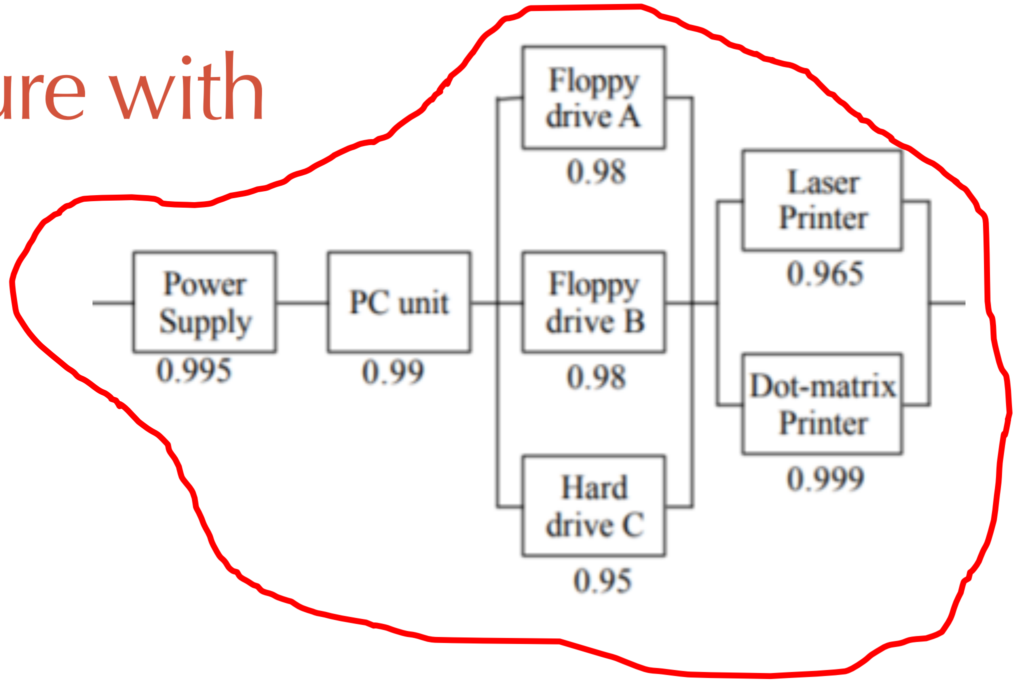
The value p_i is the reliability of the i'th component. We define the reliability R of the system by

$$R = P(\phi(\mathbf{X}) = 1), \quad \text{where} \quad \mathbf{X} = (X_1, ..., X_n)$$

$$R = R(\mathbf{p}), \quad \textit{where} \quad \mathbf{p} = (p_1, ..., p_n).$$

Reliability of a structure with independent components



We can link this reliability to time t by using the models discussed in the former paragraph by setting

$p_i(t) = P(\text{component } i \text{ is functioning at time } t) = P(\text{lifetime of } i > t) = 1 - F_i(t)$

We then have

$$R(t) = 1 - F(t) = R(1 - F_1(t), 1 - F_2(t), ..., 1 - F_n(t))$$

Reliability of a structure with independent components

Example 18 The series structure: The reliability function of the series system of n independent components is given by

$$R(\mathbf{p}) = P(X_i = 1 \text{ for all } i = 1, ..., n) = \prod_{i=1}^{n} p_i$$

Reliability of a structure with independent components

Example 18 The series structure: The reliability function of the series system of n independent components is given by

$$R(\mathbf{p}) = P(X_i = 1 \text{ for all } i = 1, \ldots, n) = \prod_{i=1}^{n} p_i$$

Exercise 51 A module of a satellite monitoring system has 500 components in series. The reliability of each module component is 0.999.

(1) Find the reliability of the module.

(2) If the number of the module is reduced to 200, what is the reliability?

Based on this simple example, we note that for series systems, reliability increases if the number of components decrease! ⊙

Reliability of a structure with independent components

Example 19 The parallel structure: The reliability function of the parallel system of n independent components is given by

$$R(\mathbf{p}) = P(X_i = 1 \text{ for some } i = 1, ..., n) = 1 - \prod_{i=1}^{n}(1 - p_i)$$

Exercise 52 A system has three parallel components, A, B, C with respective reliabilities 0.95, 0.92, 0.9.

(1) Find the reliability of the system.

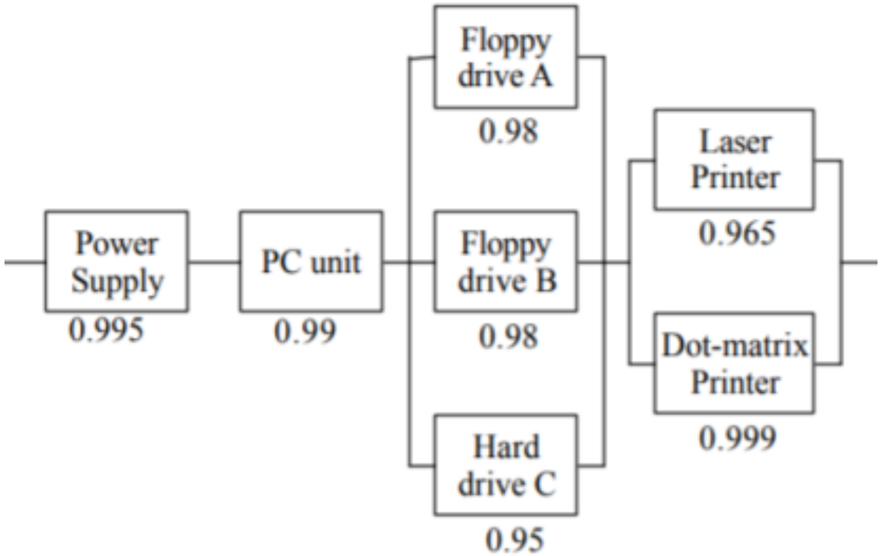
(2) Find the reliability of the system if C is out of order.

Based on this simple example, we note that for parallel systems, reliability decreases if the number of components decreases! ⊙

Reliability of a structure with independent components

Proposition: If $R(\mathbf{p})$ is the reliability function of a system of independent components, then $R(\mathbf{p})$ is an increasing function of $\mathbf{p}$ (for a proof: see [20], page 581).

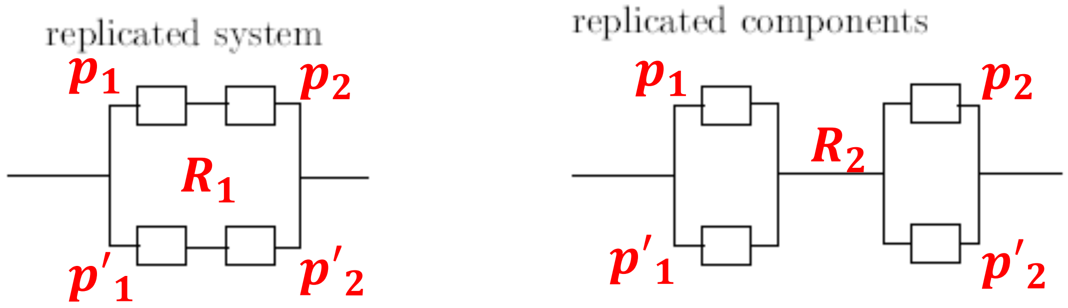
Reliability of a structure with independent components



Exercise 53 Find the reliability of the system shown in figure (11.6) ⊙

Roule 0 for the design of backup structures

Example

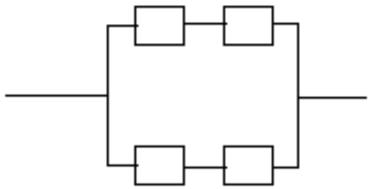


?: $R_1 < R_2$

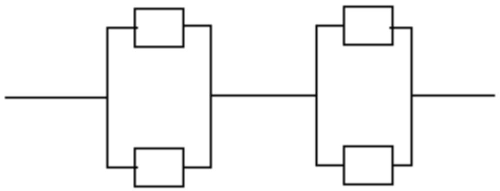
?: $R_1 > R_2$

Roule 0 for the design of backup structures

replicated system



replicated components



$$\begin{aligned} P(\text{the system functions}) &= P(\text{at least one of the two systems function}) \\ &= 1 - P(\text{neither of the system function}) \\ &= 1 - \Big([1 - R(\mathbf{p})][1 - R(\mathbf{p}')]\Big) \end{aligned}$$

$$P(\text{the system functions}) = R\Big(1 - (1 - \mathbf{p})(1 - \mathbf{p}')\Big)$$

General

Proposition: For any system reliability function R and component reliability vectors $\mathbf{p}$ and $\mathbf{p}'$:

$$R\Big(1 - [1 - \mathbf{p}]\,[1 - \mathbf{p}']\Big) \geq 1 - \Big[1 - R(\mathbf{p})\Big]\Big[1 - R(\mathbf{p}')\Big]$$

Replication at the component level is more effective: $R_1 < R_2$