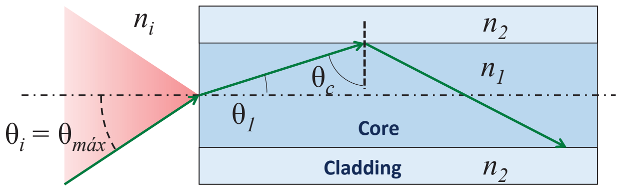
Laser: Theory and Modern Applications

ECOLE POLYTECHNIQUE FEDERALE DE LAUSANNE (EPFL)

Solutions to homework No. 8

1 Problem 1

1.



Critical angle condition:

$$\theta_1 + \theta_c + 90° = 180°$$

$$\theta_1 + \theta_c = 90°$$

$$\theta_1 = 90° - \theta_c$$

I. From air to fiber

$$n_i Sin\theta_i = n_1 Sin\theta_1$$

$$n_i Sin\theta_i = n_1 Sin\theta_1 = n_1 Sin(90° - \theta_c)$$

$$n_i Sin\theta_i = n_1 Cos\theta_c \qquad (1)$$

II. Inside fiber

$$n_1 Sin\theta_c = n_2 Sin 90°$$

$$n_1 Sin\theta_c = n_2$$

$$Sin\theta_c = \frac{n_2}{n_1} \qquad (2)$$

From equation (1):

$$n_i Sin\theta_i = n_1\sqrt{1 - Sin\theta_c^2}$$

Using equation (2):

$$n_i Sin\theta_i = n_1\sqrt{1 - \left(\frac{n_2}{n_1}\right)^2}$$

$$n_i Sin\theta_i = n_1\sqrt{\frac{n_1^2 - n_2^2}{n_1^2}}$$

$$n_i Sin\theta_i = \sqrt{n_1^2 - n_2^2} = NA$$

$$N.A = \sqrt{n_2^2 - n_1^2} = \sqrt{(n_2 - n_1)(n_2 + n_1)} = \sqrt{\Delta \cdot n_1 \cdot 2 \cdot n_1} = \sqrt{2\Delta} \cdot n_1$$

$$(\Delta = \frac{n_2 - n_1}{n_1}, n_2 + n_1 \approx 2n_1)$$

2.

$$N.A = \sqrt{2 \cdot 10^-2} \cdot 1.46 = \sqrt{2} \cdot 0.1 \cdot 1.46 = 1.41 \cdot 0.14 = 0.2$$

3.

$$V = 2\pi \frac{a}{\lambda_0} N.A \leq 2.405$$

$$\Rightarrow \phi = 2a \leq \frac{2.405\lambda_0}{\pi N.A} = 5.4\mu m$$

$$(N.A = 0.22, \lambda_{telecom})$$

$$\Rightarrow \phi \leq \frac{2.405\lambda_{vis}}{\pi N.A} = 1.75\mu m$$

$$(N.A = 0.22, \lambda = 0.5\mu m)$$

If air surronds the solid core:

$$n_1 = 1, N.A = 1.06$$

$$\Rightarrow \phi \leq \frac{2.405\lambda}{\pi 1.06} \approx 0.72\lambda$$

very small core
too small to be produced and too brittle, handling.

4. $8/125$ means $8\mu m$ core and $125\mu m$ cladding diameter

$$V_{fiber} = \pi \cdot \frac{\phi^2}{4} \cdot L = \pi \cdot \frac{\phi'^2}{4} \cdot L'$$

$$\Rightarrow L' = L \cdot \frac{\phi^2}{\phi'^2} = 1000 \cdot 10^3 \cdot (\frac{12.5 \cdot 10^{-5}}{10 \cdot 10^{-2}})^2$$
$$\Rightarrow L' = 10^6 \cdot (1.25)^2 \cdot 10^{-6} \approx 1.45m$$

The preform measures $1.45m$ long $10cm$ in diameter.

5.

$$\eta = 1 - e^{-F \cdot \frac{S_{core}}{S_{tot}} \cdot \alpha \cdot L}$$

Effective absorption coefficient:

$$\alpha_{eff} = F \cdot \frac{S_{core}}{S_{tot}} \cdot \alpha$$
$$\eta = 1 - e^{-\alpha_{eff} \cdot L}$$
$$\frac{dB}{m} = \frac{10 log_{10}(\frac{Power_{in}}{Power_{out}})}{length[m]}$$
$$\Rightarrow X[\frac{dB}{m}] \cdot \overbrace{l[m]}^{1meter} = 10 log_{10} \frac{P_{in}}{P_{out}}$$

light propagates in core

$$\Rightarrow P_{out} = P_{in} \cdot e^{-\alpha \cdot L}$$
$$\Rightarrow X[\frac{dB}{m}] = 10 log_{10} e^{\alpha[\frac{1}{m}]}$$
$$\Rightarrow \frac{X[\frac{dB}{m}]}{10} = log_{10} e^{\alpha[\frac{1}{m}]}$$
$$\Rightarrow 10^{\frac{X[\frac{dB}{m}]}{10}} = e^{\alpha[\frac{1}{m}]}$$
$$\Rightarrow \alpha = ln(10^{\frac{X[\frac{dB}{m}]}{10}})$$
$$(X = 55\frac{dB}{m}) \Rightarrow \alpha = 12.6[\frac{1}{m}]$$
$$\alpha_{eff} = F \cdot \frac{S_{core}}{S_{tot}} \cdot \alpha$$
$$\Rightarrow \alpha_{eff} = 0.8 \cdot (\frac{12\mu m}{400\mu m})^2 \cdot 12.6$$
$$\Rightarrow \alpha_{eff} = 0.9 \cdot 10^{-2}[\frac{1}{m}]$$
$$\eta(L = 1m) = 1 - e^{-\alpha_{eff} \cdot 1} = 0.009$$
$$\eta(L = 5m) = 1 - e^{-\alpha_{eff} \cdot 5} = 0.044$$
$$\eta(L = 10m) = 1 - e^{-\alpha_{eff} \cdot 10} = 0.0861$$
$$\eta(L = 20m) = 1 - e^{-\alpha_{eff} \cdot 20} = 0.1647$$
$$\eta(L = 50m) = 1 - e^{-\alpha_{eff} \cdot 50} = 0.3624$$

2 Problem 2

1. The propagation constant of a fiber is

$$\beta_{\ell m} = (n_1^2 k_0^2 - (\ell + 2m)^2 \frac{\pi^2}{4a^2})^{\frac{1}{2}} \tag{1}$$

The V-parameter of a fiber is given by

$$V = 2\pi \frac{a}{\lambda_0} NA$$

and the number of modes supported by the fiber with core radius a, numerical aperture NA at wavelength λ_0 is

$$\text{for M} \gg 1, \quad M \approx \frac{4}{\pi^2} V^2 = \frac{4}{\pi^2} 4\pi^2 \frac{a^2}{\lambda_0^2} NA^2 = 16 \frac{a^2}{\lambda_0^2} NA^2$$

$$NA \approx n_1 \sqrt{2\Delta} \quad \Rightarrow \quad M = 16 \frac{a^2}{\lambda_0^2} n_1^2 2\Delta, \quad \text{with} \lambda_0 = \frac{2\pi}{k_0}$$

$$M = \frac{4a^2}{\pi^2} \cdot 2 \cdot k_0^2 \cdot n_1^2 \cdot \Delta \tag{2}$$

if we now insert eq. 2 into eq. 1

$$\beta_{\ell m} = (n_1^2 k_0^2 - n_1^2 k_0^2 (\ell + 2m)^2 \frac{2\Delta}{M})^{\frac{1}{2}} \qquad = n_1 k_0 (1 - 2 \cdot (\ell + 2m)^2 \frac{\Delta}{M})^{\frac{1}{2}}$$

since Δ is small $\sqrt{1+\delta} \approx 1+\frac{\delta}{2}$, $|\delta| \ll 1$

$$\beta_{\ell m} = n_1 k_0 (1 - \frac{(\ell+2m)^2}{M} \cdot \Delta)$$

2. $v_{\ell m} = \frac{d\omega}{d\beta_{\ell m}}$, let's express $\beta_{\ell m}$ as a function of ω

$$n_1 k_0 = n_1 \cdot \frac{2\pi}{\lambda_0} = \frac{\omega n_1}{c} \quad \text{and} \quad M = \frac{4a^2}{\pi^2} \cdot 2\Delta \cdot \frac{\omega^2 n_1^2}{c^2}$$

$$\begin{aligned}
\beta_{\ell m} &\approx \frac{\omega n_1}{c}\left(1 - (\ell+2m)^2 \cdot \Delta \frac{\pi^2}{4a^2}\frac{1}{2\Delta}\frac{c^2}{\omega^2 n_1^2}\right) \\
&= \frac{\omega n_1}{c} - (\ell+2m)^2 \cdot \Delta \frac{\pi^2}{4a^2}\frac{1}{2\Delta}\frac{c}{\omega n_1} \\
\frac{d\beta_{\ell m}}{d\omega} &= \frac{n_1}{c} + (\ell+2m)^2 \cdot \Delta \frac{\pi^2}{4a^2}\frac{1}{2\Delta}\frac{c^2}{\omega^2 n_1^2} \cdot \frac{n_1}{c} \\
&= \frac{n_1}{c} \cdot \left(1 + \frac{(\ell+2m)^2}{M} \cdot \Delta\right) \\
\Rightarrow v_{\ell m} &= \frac{c}{n_1} \cdot \left(1 + \frac{(\ell+2m)^2}{M} \cdot \Delta\right)^{-1}
\end{aligned}$$

$$\begin{aligned}
v_{\ell m} &= \frac{c}{n_1} \cdot \left(1 + \frac{(\ell+2m)^2}{M} \cdot \Delta\right)^{-1} \\
&\overset{M \gg \Delta}{=} \frac{c}{n_1} \cdot \left(1 - \frac{(\ell+2m)^2}{M} \cdot \Delta\right)
\end{aligned}$$

$$\begin{aligned}
v_{max} &\overset{\ell+2m=2}{=} \frac{c}{n_1} \cdot \left(1 - \frac{4}{M} \cdot \Delta\right) \approx \frac{c}{n_1} \\
v_{min} &\overset{\ell+2m=\sqrt{M}}{=} \frac{c}{n_1} \cdot (1-\Delta)
\end{aligned}$$

3. Pulse broadening

$$\begin{aligned}
\Delta\tau &= \frac{1}{2}\left(\frac{L}{v_{min}} - \frac{L}{v_{max}}\right) \\
&= \frac{L}{2}\left(\frac{n_1}{c(1-\Delta)} - \frac{n_1}{c}\right) \\
&= \frac{L n_1}{2c}\left(\frac{1}{1-\Delta} - 1\right) \\
&= \frac{L n_1}{2c}\left(\frac{1-(1-\Delta)}{1-\Delta}\right) \\
&= \frac{L n_1}{2c} \cdot \frac{\Delta}{1-\Delta}
\end{aligned}$$

$$\Rightarrow \Delta\tau = \frac{100 \cdot 10^3 \cdot 1.46}{2 \cdot 3 \cdot 10^8} = 24.6 \cdot 10^{-7} = 2.46 \mu s$$

4. Pulse broadening for graded index fiber

$$\begin{aligned}
\Delta\tau &= \frac{L}{2}\left(\frac{n_1}{c}\cdot\frac{1}{1-\frac{\Delta^2}{2}} - \frac{n_1}{c}\right) \\
&= \frac{Ln_1}{2c}\left(\frac{1}{1-\frac{\Delta^2}{2}} - 1\right) \\
&= \frac{Ln_1}{2c}\left(\frac{2-(2-\Delta^2)}{2-\Delta^2}\right) \\
&= \frac{Ln_1}{2c}\cdot\frac{\Delta^2}{2-\Delta^2} \\
&= 12ns
\end{aligned}$$

5. The gradient index fiber delivers the least temporal pulse broadening.

3 Problem 3

$$\begin{aligned}
E_y &= A \cdot e^{-p(|x|-d)} \cdot e^{-j\beta z} \qquad |x| \geq d \\
E_y &= B \cdot \cos{(hx)} \cdot e^{-j\beta z} \qquad |x| \leq d \\
\frac{\partial^2}{\partial x^2} E_y &= p^2 \cdot E_y(x,z) \qquad |x| \geq d
\end{aligned}$$

into wave equation:

$$p^2 \cdot E_y(x,z) + (k_0^2 n_1^2 - \beta^2)E_y(x,z) = 0 \Rightarrow p^2 = \beta^2 - k_0^2 n_1^2$$

$$\frac{\partial^2}{\partial x^2} E_y = -h^2 \cdot E_y(x,z) \qquad |x| \leq d$$

into wave equation:

$$-h^2 \cdot E_y(x,z) + (k_0^2 n_2^2 - \beta^2)E_y(x,z) = 0 \Rightarrow \beta^2 = k_0^2 n_2^2 - h^2$$