Laser Theory and Modern Applications

Homework 2: Absorption cross-section, spontaneous emission

Fall 2024

Questions

1. **Absorption cross section:**

   Absorption cross section In class it was shown that the absorption cross section within the electron oscillator model (which has a damping term β) takes the following form:

   $$\sigma(\omega)=\frac{e^2}{2mc\epsilon_0}\frac{\beta}{\left(\omega_0-\omega\right)^2+\beta^2}$$

   Here ω denotes the angular frequency. Show that the absorption cross section on resonance for a lifetime (radiatively) broadened transition is given by (for $f=1$):

   $$\sigma\left(v_0\right)=\frac{\lambda^2}{4\pi}$$

   And also show that there would be a factor 3 different between pure classical model and quantum corrected model.

   To derive this equation, use for the damping β in the Lorentz model the expression for the radiative damping (from the Larmor formula, as shown in class, see last question on this sheet). Note that the relation between spontaneous emission lifetime and the introduced damping term is $\beta=A_{12}$.

2. **Absorption of a YAG Laser crystal:**

   Absorption of a YAG Laser crystal Assume that a Nd:YAG crystal is placed inside a beam of monochromatic resonant light. The length of the crystal is $L=10$ cm. The Nd:YAG has its maximim absorption at a wavelength of 1064 nm and has a refractive index of $n=1.5$. The Nd ions inside the crystal decay - when excited by resonant light - relax to the ground state with a spontaneous emission time of $5\cdot10^{-4}$ s. Analysing the absorption shows that it has a Lorenzian lineshape which is inhomogeneously broadenened and has a spectral width of $\Delta\omega/2\pi=3\cdot10^{12}$ Hz.

- Based on this information, calculate first the absorption cross section of the Nd:YAG Laser crystal on resonance. What is the corresponding fraction of light of a monochromatic resonant light beam that passes through the crystal, if all ions are in their ground state and the density is $N = 10^{19}$ ions /cm^3 ?

Hint: The relation between the absorption coefficient and the cross section is $a(\omega) = \sigma \cdot N$, where N is the density (in units of inverse Volume) of ions in the medium.

Note: Nd:YAG is an abbreviation for Neodium doped Yttrium aluminium garnet (YAG, Y3A15O12). This material is used to make solid state lasers.

3. **Absorption of Sodium:**

Absorption of Sodium Estimate the absorption for 589 nm radiation (D1 transition) in Sodium (Na) atomic vapor containing $2.7 \cdot 10^{18}$ atoms /m^3 at 200 Degrees Celcius[1]. How much light will pass through a vapor cell of 1 cm length? Use the formula for the Doppler broadened linewidth to arrive at the correct expression. The spontaneous emission rate of Sodium D1 transition is $A_{12} \approx 2\pi \times 10\text{MHz}$. Note that Sodium lamps are widely used in street lamps.

4. **Balmer Series:**

The swiss school teacher Balmer derived a simple relation that describes the emission wavelength λ of atomic Hydrogen. It is given by:

$$\lambda = B \frac{n^2}{n^2 - 4}$$

where n is an integer. Derive this relation based on the de Broglie assumption and the assumption that stable orbits fulfill the resonance condition. The de Broglie relation states that also electrons are waves and have a wavelength given via the relation $p = h/\lambda$, where h is the Planck constant and p is the momentum of the electron. The de Broglie condition for the orbits implies that the condition $2\pi r_n = n\lambda$ needs to be satisfied. Here λ is the de Broglie wavelength and n is an integer number. Derive[2] the energy Balmer series relation (for transitions from higher order n to the level $n = 2$) and show that, notably $B = (4\pi\epsilon_0)^2 \cdot \frac{16\pi\hbar^3 c}{me^4}$.

5. **The Kramers Kronig relation:**

The refractive index and the absorption are the real and imaginary part of the same holomorphic function. Real and imaginary part of holomorphic functions are related via the Kramers Kronig relation. In practice it is thus sufficient to measure only one quantity and derive the other from it. Show that the refractive index and the absorption satisfy the following relation:

$$n_R(\omega) - 1 = \frac{c}{2\omega}\frac{\omega_0 - \omega}{\beta}a(\omega)$$

From the given absorption spectrum of glass BK7, compute and plot the index of refraction of glass BK7. Use $\lambda_0 = 100$ nm and $\beta = 200$MHz.

6. **Doppler broadening:**

1. A Helium-Neon gas laser is composed of a mixture of Helium and Neon atoms. Find the most probable velocity for Helium atoms and Neon atoms at room temperature (T = 25°C). Plot the velocity distribution of both atom species on the same graph at room temperature. Molar mass of Neon: 20.18 g. Molar mass of Helium: 4.00 g.

Hint: use the Maxwell Bolzmann velocity distribution in $3d$

2. Derive the one dimensional Maxwell distribution :

$$f(v_x)\,dv_x = \left(\frac{m}{2\pi kT}\right)^{1/2} e^{-\frac{mv_x^2}{2kT}}\,dv_x$$

Hint: integrate the cartesian three dimensional form of the distribution over v_y and v_z and use :

$$\int_{-\infty}^{\infty} e^{-ax^2}dx = \frac{\sqrt{\pi}}{\sqrt{a}}$$

3. Find the full width at half maximum (FWHM) of the Doppler broadened absorption probability function S:

$$S(\nu) = \left(\frac{m_x c^2}{2\pi kT\nu_0^2}\right)^{1/2} e^{-m_x c^2(\nu-\nu_0)^2/2kT\nu_0^2}$$

Compute the absorption bandwidth (@ FWHM) of the a Helium Neon gas at T=400K Consider the emission wavelength of 632.8 nm for Neon.

[01] Sodium D Line Data, Daniel A. Steck, Los Alamos National Laboratory, 14 October 2003, {http://steck.us/alkalidata}

[2] Hint: Equate the centripetal force $\left(F = \frac{mv^2}{r}\right)$ to the coulomb force $\left(F = \frac{e^2}{4\pi\epsilon_0 r^2}\right)$. See Milonni "Lasers" chapter