



MICRO 372 - Advanced Mechanisms for Extreme Environments

Chapter 4b

Advanced mechanisms design

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Kinematic analysis - definitions

- **Kinematics**: architectural skeleton of a mechanism
- **Internal mobilities**: sum of DOF of individual flexure joints of a mechanisms
- **Kinematic chains**: paths from output to input of a mechanism
- **Kinematic loops**: closed kinematic chains within the mechanism
- **External DOFs**: DOF of the output of a mechanism
- **Internal DOFs**: DOF of intermediate stages in a mechanism
- **Overconstraints**: number of times the output or intermediate stages are constrained several times for a given DOF
- **Isostaticity**: feature of a kinematic with no overconstraint neither internal DOF

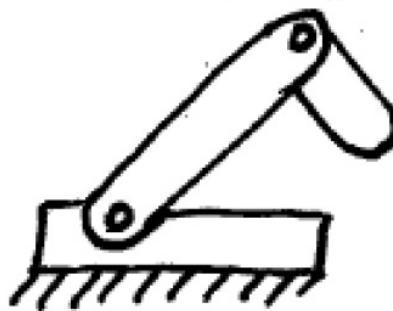
Kinematic analysis – Mobility equation

- Grübler formula using segments and joints numbers:

$$DOF = \sum Mo - 6(k - n + 1)$$

DOF = $\sum Mo - 6(k - n + 1)$

External + internal DOF
Internal mobilities of joints
Number of joints
Number of segments
Fixed base
Number of loops
 $= L$



$$\begin{aligned} k &= 2 \\ n &= 3 \\ Mo &= 2 \end{aligned}$$

Kinematic analysis – Mobility equation

- Grübler formula (with kinematic loops):

$$DOF = \sum Mo - 6L$$

- Modified formulation in 2D (planar analysis):

$$DOF_e + DOF_i = \sum Mo - 3L + OC$$

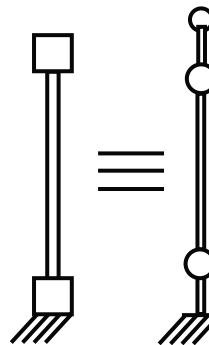
- Modified formulation in 3D (volumetric analysis):

$$DOF_e + DOF_i = \sum Mo - 6L + OC$$

Kinematic analysis – Model of a blade

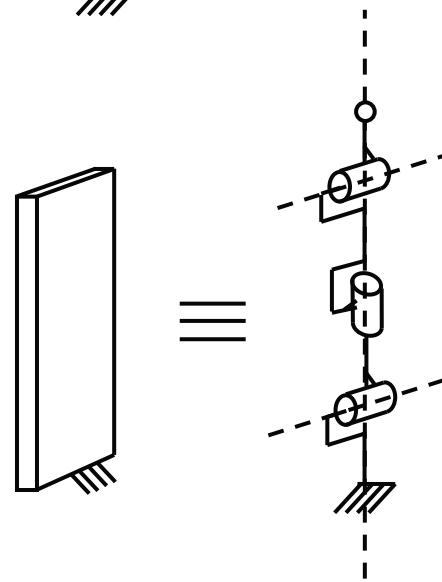
In 2D:

- the kinematic equivalent of a blade is 2 pivots



In 3D:

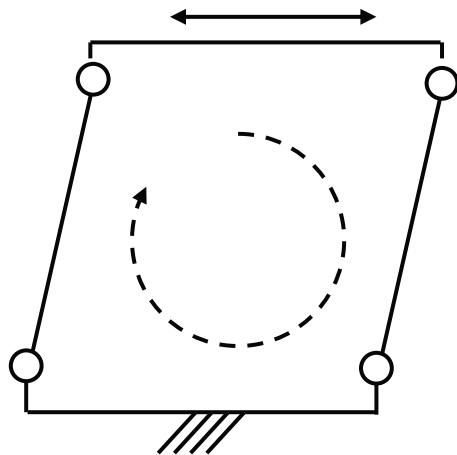
- the kinematic equivalent of a blade is 3 pivots



Kinematic analysis - Parallelogram stage

- $DOF_e = 1$
- $DOF_i = 0$
- $Mo = 4$
- $L = 1$
- $OC = 0$ or $3?$

depends on the analysis type (2D or 3D)



External DOF Internal DOF Internal mobilities of joints Kinematic loop Overconstraints

$$DOF_e + DOF_i = \sum Mo - 3L + OC$$

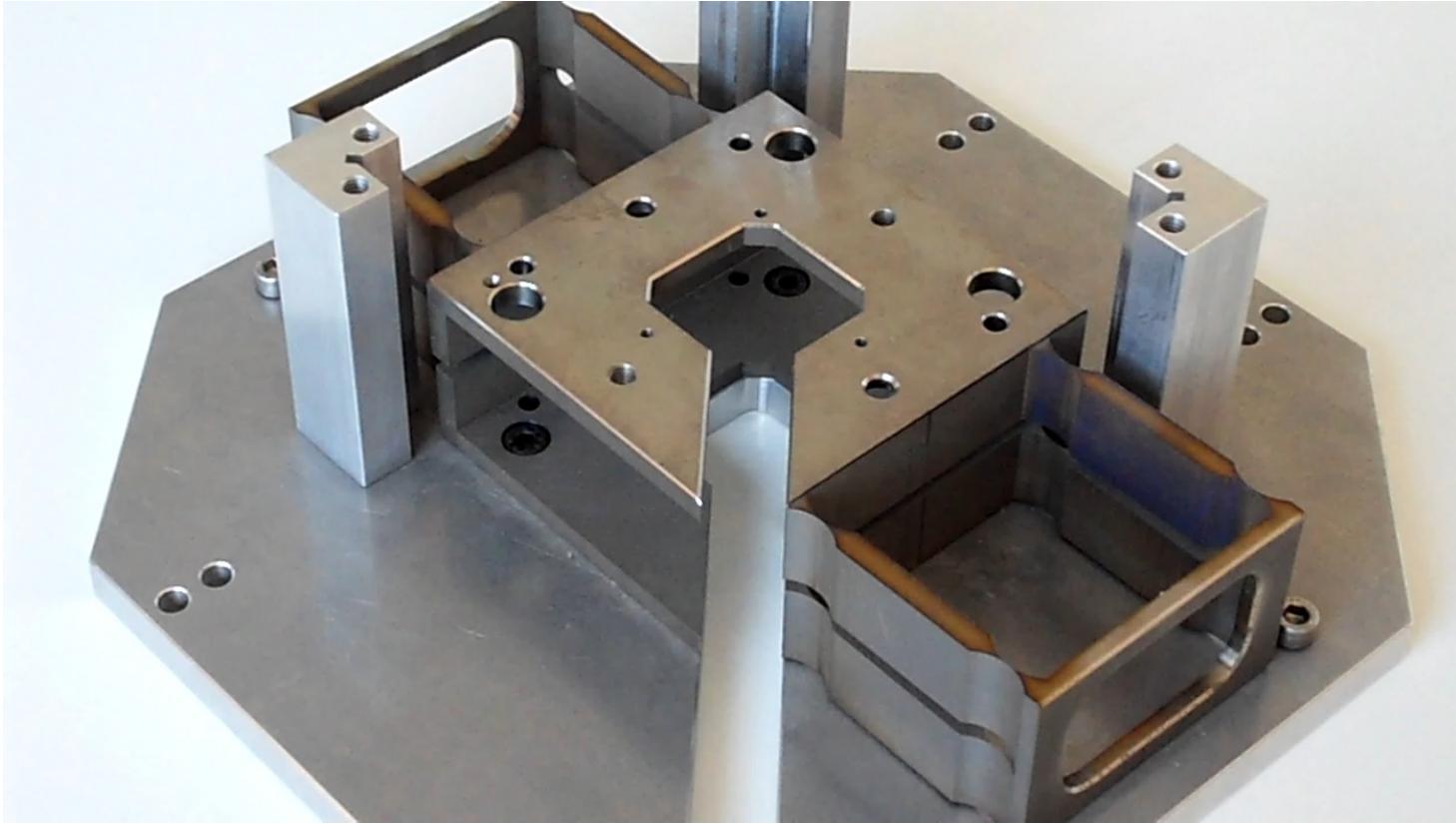
$$1 + 0 = 4 - 3 \cdot 1 + 0$$

$$DOF_e + DOF_i = \sum Mo - 6L + OC$$

$$1 + 0 = 4 - 6 \cdot 1 + 3$$

Each time a planar kinematic loop of blades is closed, 3 out-of-plane overconstraints appear!

Kinematic analysis – Overconstrained linear stage



Kinematic analysis - Overconstrained linear stage

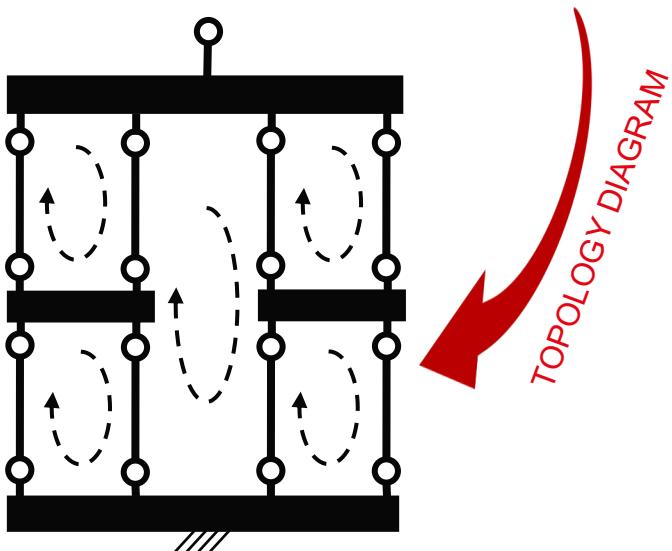
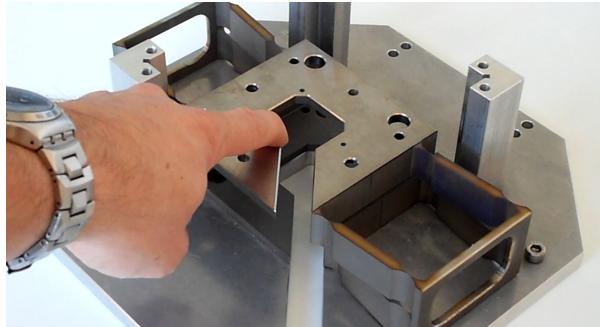


- External Degree of Freedom DOFe: 1
- Internal Degree of Freedom DOFi: 2
- Internal motions of joints Mo: 16
- Kinematic loops L: 5
- Overconstraints: ?

$$DOF_e + DOF_i = \sum Mo - 6L + OC$$

$$1 + 2 = 16 - 6 \cdot 5 + OC$$

$OC = 17$ (12 OC in parallelograms and 5 OC on output)



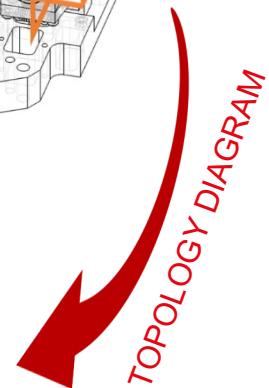
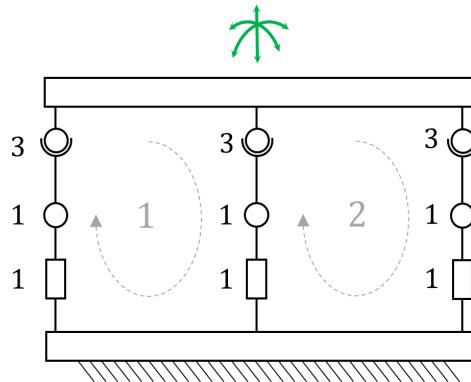
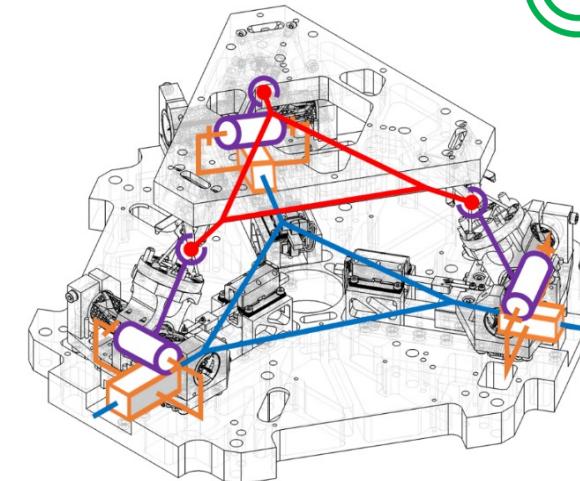
Kinematic analysis – Isostatic PULSAR tripod



- External Degree of Freedom DOFe: 3
- Internal Degree of Freedom DOFi: 0
- Internal motions of joints Mo: 15
- Kinematic loops L: 2
- Overconstraints: 0

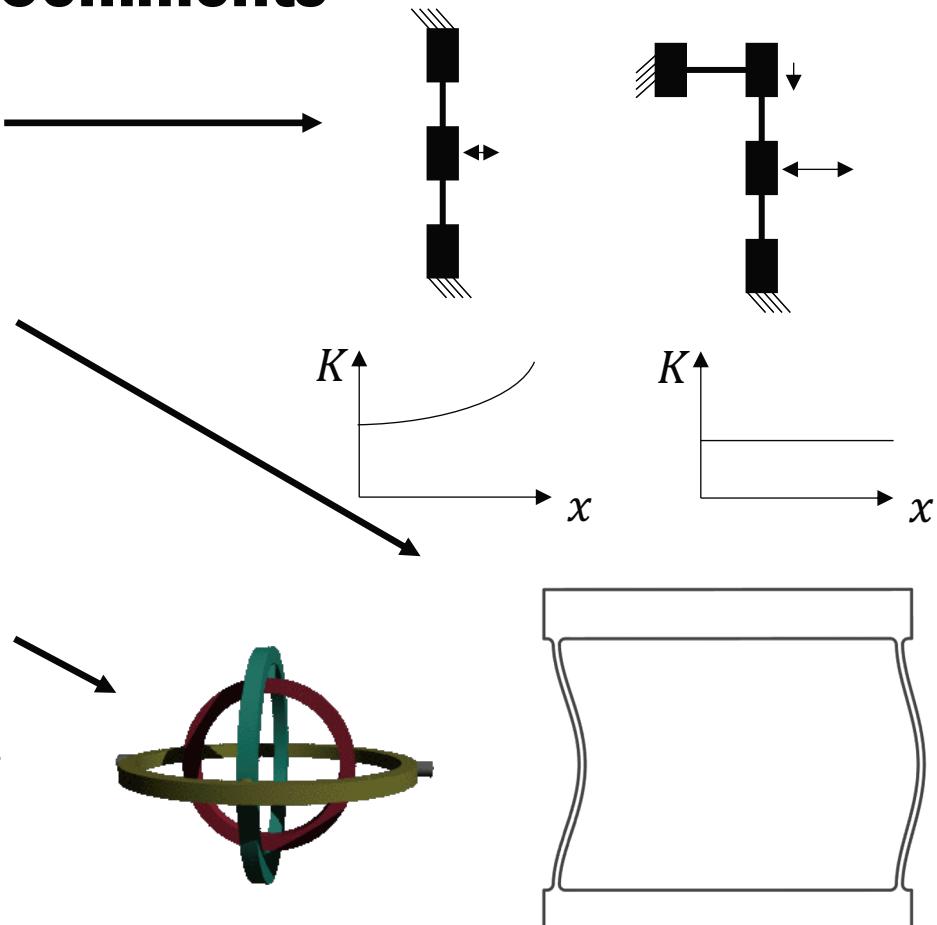
$$DOF_e + DOF_i = \sum Mo - 6L + OC$$

$$3 + 0 = 15 - 6 \cdot 2 + 0$$



Kinematic analysis – Comments

- Internal local DOFs with **limited motion range** versus internal DOF with **large motion range**
- Internal DOFs of the mechanism versus **eigenfrequencies** of the flexures
- Check of the **sub-systems isostaticity**
- Kinematic analysis **does not consider the geometry of the mechanism**, only the **topology**
- Kinematic analysis **does not consider the singularities of the mechanism** (e.g. **gimbal lock**)
- Kinematic analysis **depends on hypothesis**: 2D versus 3D, number of DOF per flexure, etc..
- Interpretation matters!



Kinematic analysis - Conclusions

Internal DOFs:

- Can induce **uncontrolled vibrations**
- **Strokes and stress** on flexures are **not well controlled**
- **Parasitic motion** on output **may be induced** by uncontrolled motion of internal DOFs
- A **nightmare** to **increase the dynamics** of the mechanism!

Overconstraints:

- Large **variations of stiffness** can appear
- Possibly **increases** significantly **the stress level**, even at rest position
- Makes the mechanism **less immune to tolerances variations** (manufacturing and assembly)
- **Can be beneficial** to improve transversal stiffnesses!

Kinematic analysis - Conclusions

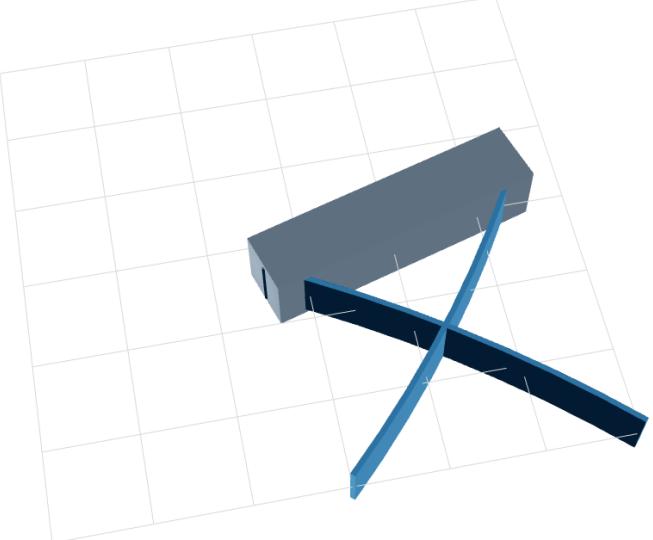
- It is important to **carry out a kinematic analysis** when choosing the architecture of the mechanism, **at an early stage of development**.
- It is preferable to **avoid overconstraints and internal DOFs**.
- We **can accommodate overconstraints**, in particular out-of-plane OC (when a kinematic loop closes). We can also sometimes take advantage of this, for example to stiffen certain transverse stiffnesses or increase robustness.
- **Internal DOFs can be tolerated**, especially if the masses of the moving parts are low, the associated natural frequencies are high, and the system dynamics are low.

Kinematic analysis online tool (demo)

whatthedof.com

Flexure Mobility Analysis (v1.8.3) [Load model](#) [Save model](#) [Hide origin](#) [Hide menu](#) [Logout](#)

Planes Nodes Elements **Analyze**



Analysis results

Number of overconstraints	1
Number of DOFs (underconstraints)	1

Visualization

Type [Overconstraints](#) [Underconstraints](#)

Sparsify algorithm (LUQ) [Enable](#) [Disable](#)

Change active underconstraint [-](#) [+](#)

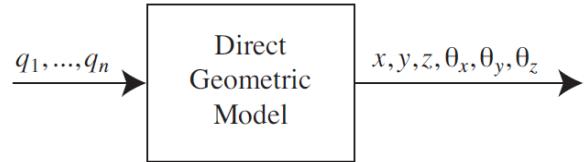
Animate underconstraint [▶](#)

Geometric modeling

- **Direct Geometric Model and Inverse Geometric Model**
- **How to calculate your model?**
 - Analytical implementation
 - Validation through CAD sketches
- **Useful to :**
 - Calculate the motion ranges on actuators, on joints and on output (workspace)
 - Calculate the resolutions on actuator and on output
 - Calculate the parasitic motion
 - Optimize a system
 - Calibrate a system
 - Control a system

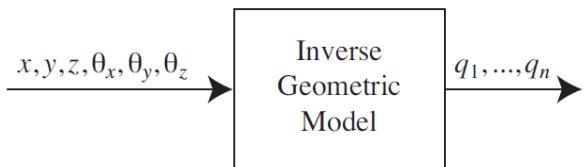
- The Direct Geometric Model (**DGM**) is the mathematical relation that give the output (or tool) positions as a function of the input (or motor) positions:

$$\vec{x}_i = \text{DGM}(\vec{q}_i)$$



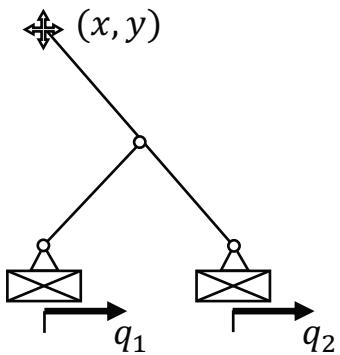
- Inverse Geometric Model (**IGM**) is the mathematical relation that give the input (or motor) positions as a function the output (or tool) positions:

$$\vec{q}_i = \text{IGM}(\vec{x}_i)$$



- x_i are tool coordinates and q_i are motor coordinates

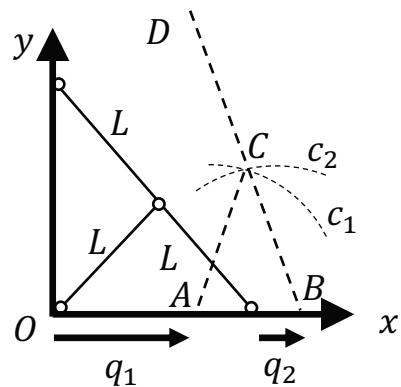
Geometric modeling: DGM of the lambda mechanism



$$[x, y] = DGM([q_1, q_2])$$

$$A: (q_1, 0), B: (\sqrt{2}L + q_2, 0)$$

$$\begin{aligned} c_1: & \text{ (center } A, \text{ radius } L) \\ c_2: & \text{ (center } B, \text{ radius } L) \end{aligned}$$



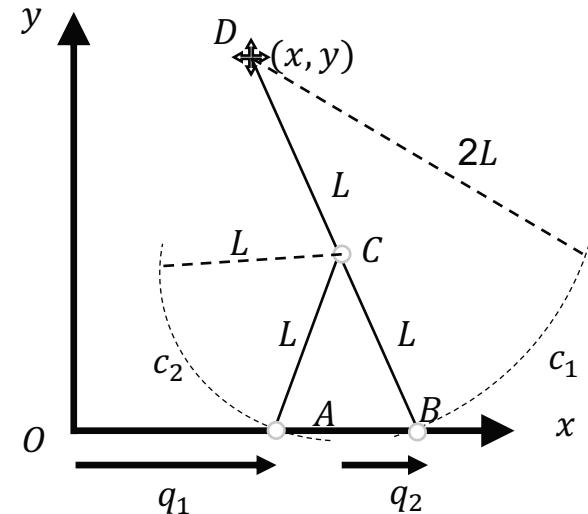
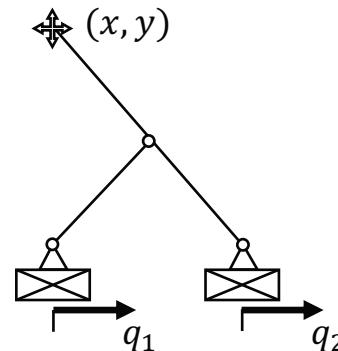
$$C: \text{intersection}(c_1, c_2)$$

$$D: \overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD} = \overrightarrow{OB} + 2\overrightarrow{BC} = 2\overrightarrow{OC} - \overrightarrow{OB}$$

☞ EXERCICE 8

Geometric modeling : IGM of the lambda mechanism (principle only)

- Trace a circle **c1** (center: **D**, radius: **2L**)
- Intersect with line **y = 0** → **B**
- Take the middle point of **BD** → **C**
- Trace a circle **c2** (center: **C**, radius: **L**)
- Intersect with line **y = 0** → **A**
- Or.. use the symmetry of **A** and **B** regards to **C**
- Calculate **q1** and **q2**



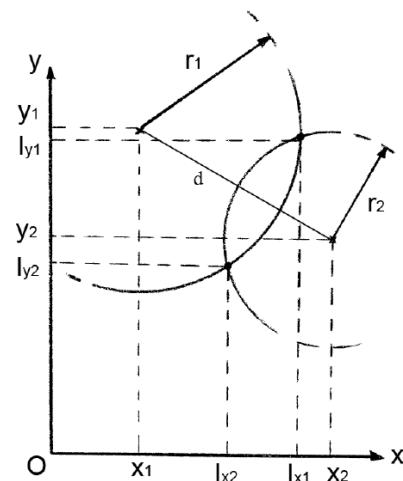
Geometric modeling: DGM and IGM setup

- Define carefully and precisely your **reference axes** and **geometric parameters**
- Draw a 2D (or 3D) **CAD sketch** (it can also be used to animate your CAD model)
- Some **generic models** may exist for known kinematics (e.g., hexapods, delta, ..)
- If it is not the case **do the math**
- No need to **solve analytically the equations**, unless you want for example to implement a fast model on controller
- Use a **CAD sketch to verify your mathematical model**
- **Circle intersections** can be calculated in planar models:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

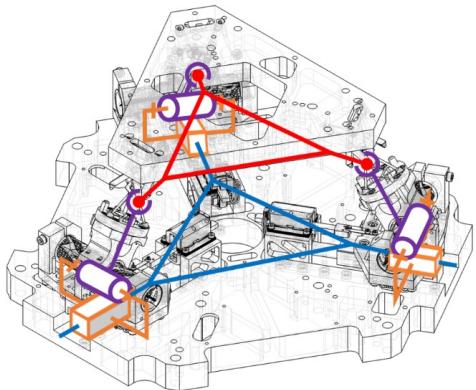
$$Ix_{12} = \frac{x_1 + x_2}{2} + \frac{(x_1 - x_2)(r_1^2 - r_2^2)}{2d^2} \pm \frac{y_2 - y_1}{2d^2} \sqrt{((r_1 + r_2)^2 - d^2)(d^2 - (r_1 - r_2)^2)}$$

$$Iy_{12} = \frac{y_1 + y_2}{2} + \frac{(y_1 - y_2)(r_1^2 - r_2^2)}{2d^2} \pm \frac{x_2 - x_1}{2d^2} \sqrt{((r_1 + r_2)^2 - d^2)(d^2 - (r_1 - r_2)^2)}$$

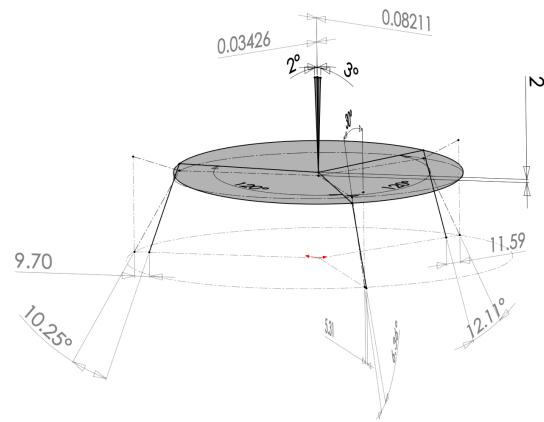
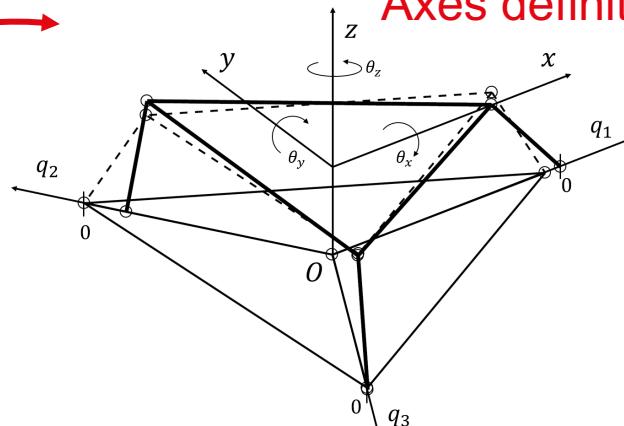


Geometric modeling: example of PULSAR IGM

Kinematic model

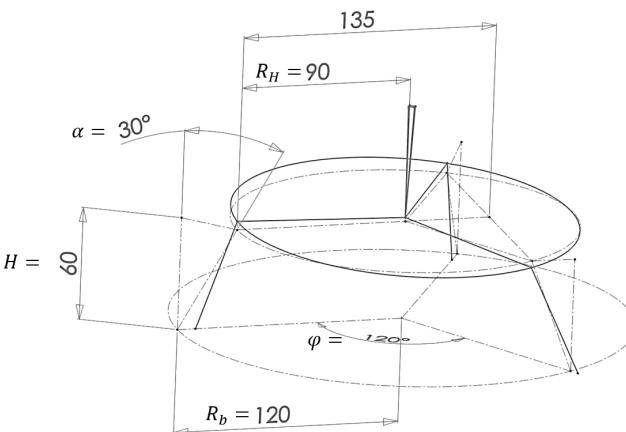


Axes definition



3D sketch

Geometric parameters definition



Geometric modeling: example of PULSAR IGM

The **initial pivot** positions are respectively for each axis:

$$P_{01} = (R_B, 0, 0)^T \quad P_{02} = (R_B \cos(\frac{\pi}{3}), R_B \sin(\frac{\pi}{3}), 0)^T \quad P_{03} = (R_B \cos(\frac{2\pi}{3}), R_B \sin(\frac{2\pi}{3}), 0)^T$$

The **initial gimbal** center positions are: (with $H_v = H \cos \alpha$)

$$P_1 = (R_H, 0, H_v)^T \quad P_2 = (R_H \cos(\frac{\pi}{3}), R_H \sin(\frac{\pi}{3}), H_v)^T \quad P_3 = (R_H \cos(\frac{2\pi}{3}), R_H \sin(\frac{2\pi}{3}), H_v)^T$$

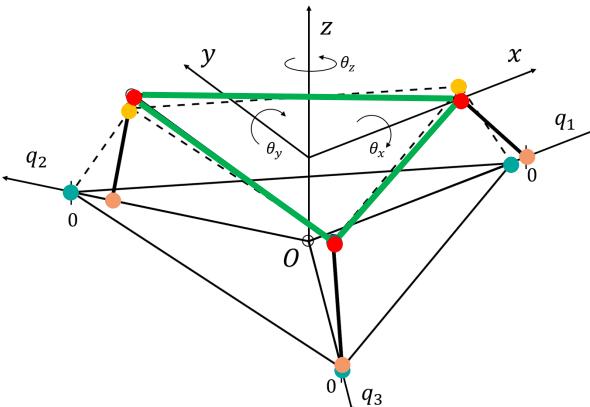
The **triangle** formed by P_1 , P_2 and P_3 is equilateral of length $L = \|P_2 - P_1\|$.

The **gimbal points** $P_{1,new}$, $P_{2,new}$ and $P_{3,new}$ are obtained by applying two **rotational matrices** $R_x(\theta_x)$ and $R_y(\theta_y)$ around the x- and y-axis respectively and a **vertical translation** of z to P_1 , P_2 and P_3 :

$$P_{i,new} = R_y(\theta_y)R_x(\theta_x) \left(P_i + \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \right)$$

$$\text{Where: } R_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_x & -\sin \theta_x \\ 0 & \sin \theta_x & \cos \theta_x \end{bmatrix}$$

$$\text{and } R_y(\theta_y) = \begin{bmatrix} \cos \theta_y & 0 & \sin \theta_y \\ 0 & 1 & 0 \\ -\sin \theta_y & 0 & \cos \theta_y \end{bmatrix}$$



Geometric modelling: example of PULSAR IGM

Then, the **new pivot positions** are for $i = 1, 2, 3$:

$$P_{i,\text{mot}} = \lambda_i \cdot t_i$$

Where λ_i is a **scalar** and t_i the motor axis **direction**:

$$t_1 = (1, 0, 0)$$

$$t_2 = \left(\cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right), 0 \right)$$

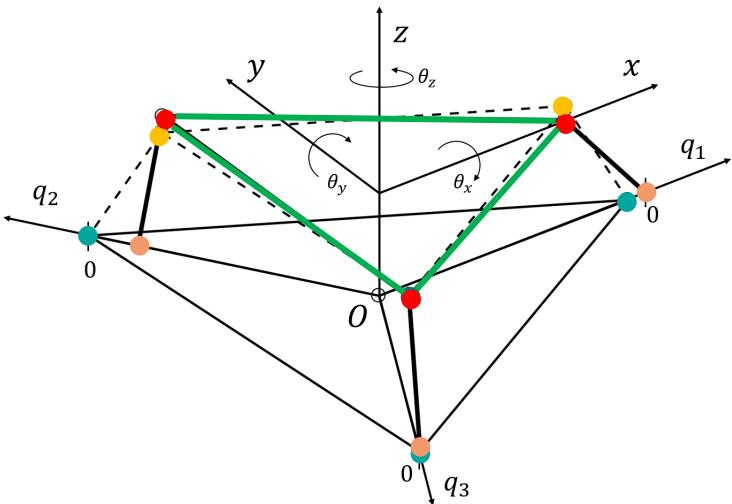
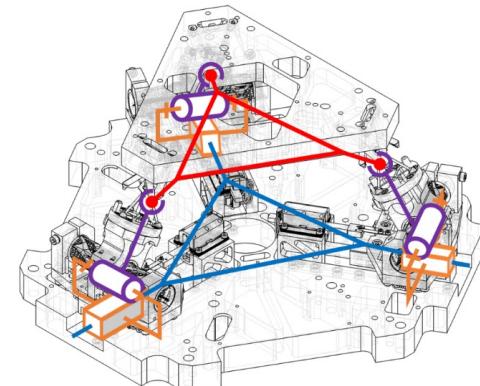
$$t_3 = \left(\cos\left(\frac{2\pi}{3}\right), \sin\left(\frac{2\pi}{3}\right), 0 \right)$$

$$\lambda_i = t_i \cdot P_{i,\text{new}} + \sqrt{(t_i \cdot P_{i,\text{new}})^2 - (\|P_{i,\text{new}}\|^2 - H^2)}$$

Finally, the **motor positions** q_1, q_2 and q_3 are obtained:

$$q_i = \|P_{0i} - P_{i,\text{mot}}\| \cdot \text{sgn}(\lambda_i - \|P_{0i}\|)$$

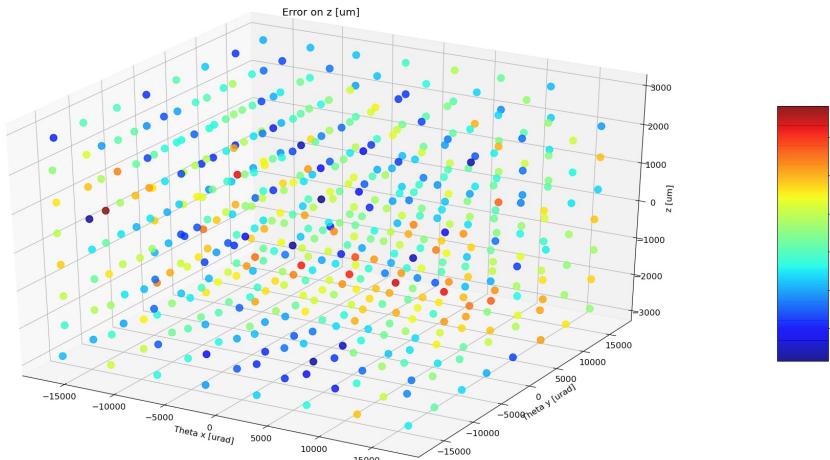
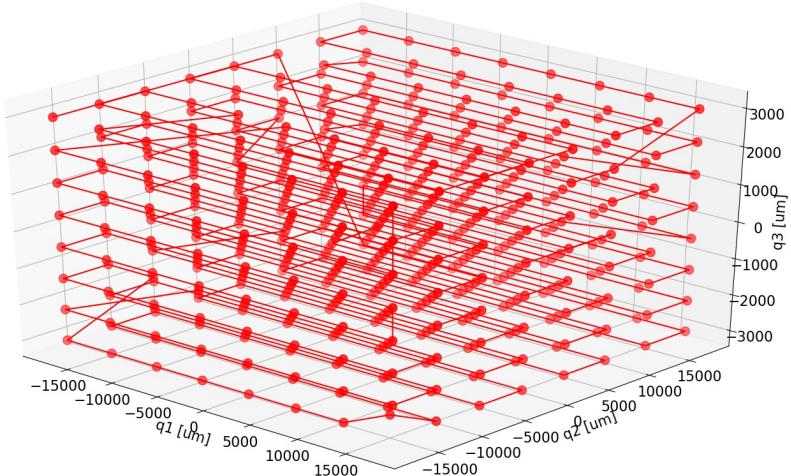
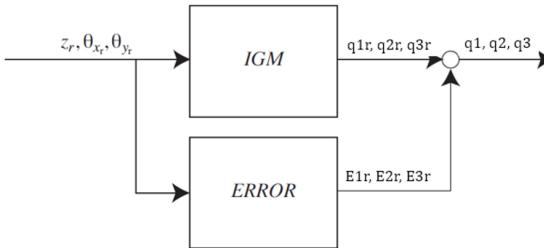
N.B.: the model is not exact!



Geometric modelling: example of PULSAR calibration

- Position measurement is made over the whole workspace
- Error model is a fit at X^{th} order on N axis, including the cross axes terms
- You obtain a model with (many) coefficients

$$q1, q2, q3 = \text{IGM}(z_r, \theta_{x_r}, \theta_{y_r}) + \text{error}(z_r, \theta_{x_r}, \theta_{y_r})$$

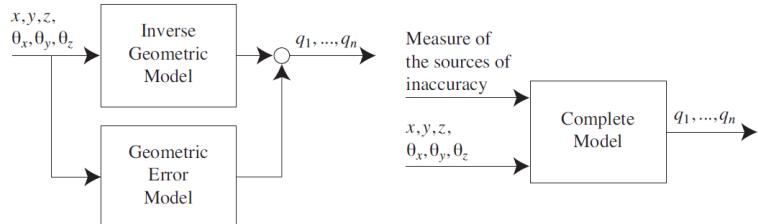


Geometric modelling: example of PULSAR calibration

- Position measurement is performed on the whole working volume
- Error model is fitted at X^{th} order on N axis, including the cross axes terms
 - Standard fit function is used
 - You get a model with many coefficients
- It is also possible to get the **full IGM** instead of just the **error model** through this method

$$\begin{aligned}\Delta\theta_x = & a_1X + a_2Y + a_3Z + a_4X^2 + a_5Y^2 + a_6Z^2 + a_7XY + a_8XZ + a_9YZ + \\ & + a_{10}X^3 + a_{11}Y^3 + a_{12}Z^3 + a_{13}XYZ + a_{14}X^2Y + a_{15}X^2Z + a_{16}Y^2X + \\ & + a_{17}Y^2Z + a_{18}Y^2Z + a_{19}Z^2X + a_{20}Z^2Y + a_{21}\end{aligned}$$

Poly	B1	B2	B3
x	0.000336	-0.13751	0.137711
y	-0.15983	0.078817	0.080533
z	1.735941	1.74542	1.751814
x^*x	-2.88E-08	3.42E-07	3.46E-07
y^*y	5.36E-07	1.01E-07	1.00E-07
z^*z	6.17E-05	5.57E-05	5.63E-05
x^*y	0	-4.21E-07	4.37E-07
y^*z	-1.09E-05	4.79E-06	4.92E-06
x^*z	0	-8.23E-06	8.32E-06
x^*x^*x	0	-9.38E-13	1.16E-12
y^*y^*y	0	0	0
z^*z^*z	3.50E-09	2.77E-09	2.89E-09
x^*x^*y	0	2.04E-12	2.45E-12
x^*x^*z	0	4.47E-11	5.08E-11
y^*y^*x	0	-1.23E-12	1.87E-12
y^*y^*z	9.03E-11	1.66E-11	1.89E-11
z^*z^*x	0	-6.14E-10	6.43E-10
z^*z^*y	-8.35E-10	3.58E-10	3.93E-10
x^*y^*z	0	-5.12E-11	5.86E-11
Offset	-124.361	-99.3174	220.4672



Flexure optimization

- **Flexure model: FEM or analytical**
 - **FEM model**
 - Faster to set up, longer to solve than analytical
 - **Example:** COMAM flexure lattices
 - **Analytical model**
 - Must be validated on a FEM nominal case (no need for high precision)
 - Run on millions of parameter sets
 - **Example:** Vibration isolation platform gimbal
- **Model inputs**
 - Geometric parameters, material
 - Discrete or continuous values depending on the optimization method

- **Optimization methods:**
 - Brute force
 - Monte-Carlo
 - Latin square
 - Genetic algorithm
 - Gradient based
 - ...

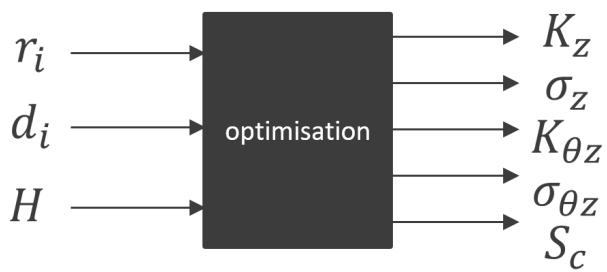
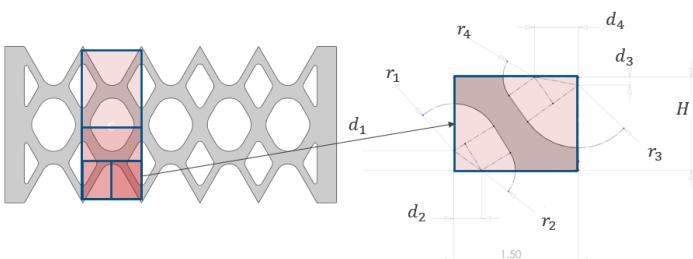
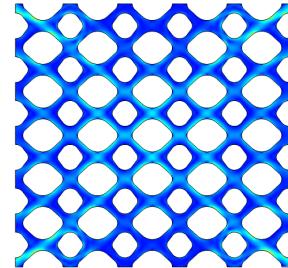
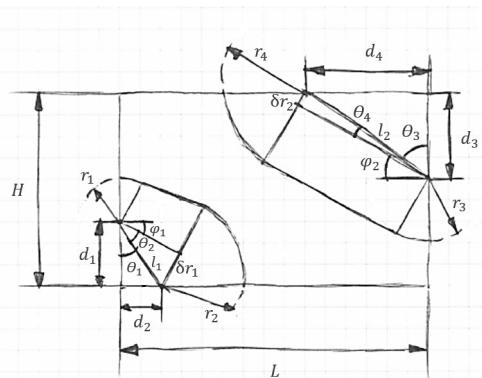
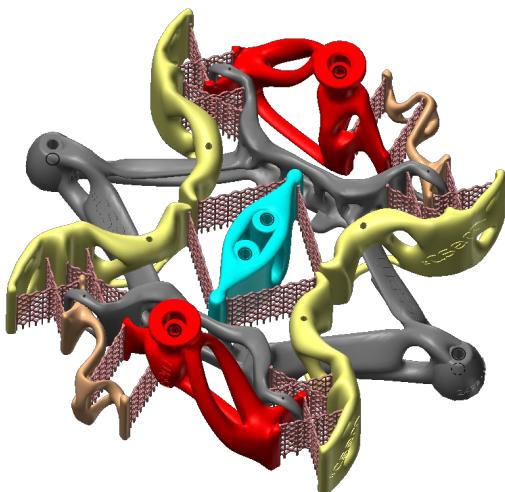
- **Input filters**
 - Based on impossible geometries
 - Based on overhang angle for AM
 - ...

- **Model outputs:**
 - Mass
 - Max stress
 - Stiffnesses
 - Eigenmodes
 - Buckling loads
 - ...

- **Output filters**
 - Limit stress
 - Required stiffnesses
 - ...

Flexure optimization – Example 1

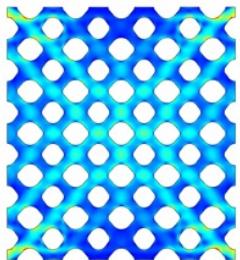
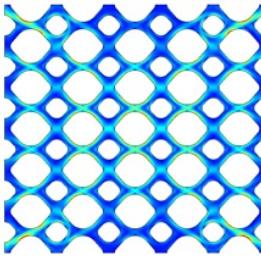
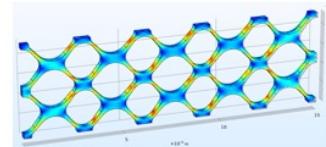
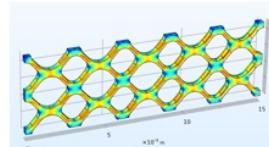
- Example: lattice flexures
- Optimized through FEM
- Monte-Carlo parameters sets generation
- Unit cell of mesh geometric parameters as inputs



Flexure optimization – Example 1

- Outputs:
 - Rotation and vertical stiffnesses and stress (K_z , Σ_z , K_{rz} , Σ_{rz})
 - Surface constancy (CS)
- Stiffness K_z is transformed into compliance C_z , so to be minimized

Symbol	K_z	$C_z = 1/K_z$	Σ_z	K_{rz}	Σ_{rz}	CS
Min/max	↗	↘	↘	↘	↘	↘
Weight		0.6	0.8	0.9	0.5	0.4

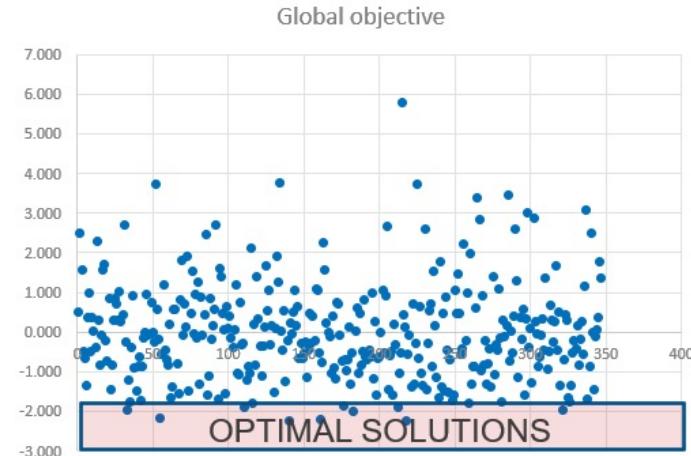
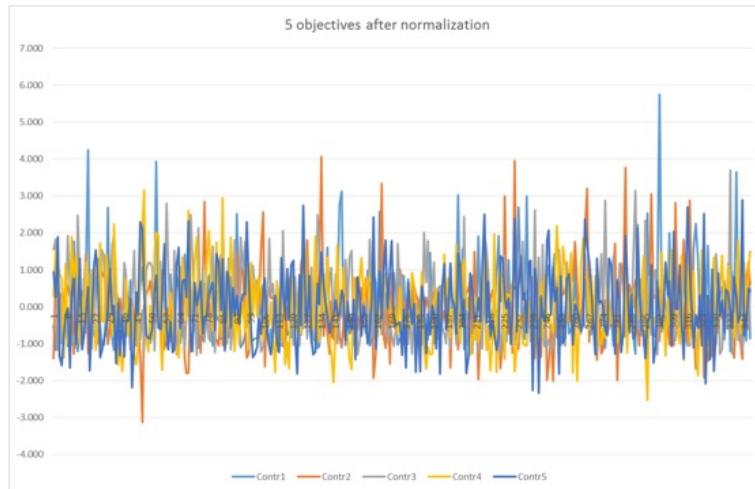
K_z max (sol 291)Sigma_z min (sol 45)K_{rz} min (sol 302)Sigma_{rz} min (sol 296)

CS min (sol 242)

Flexure optimization – Example 1

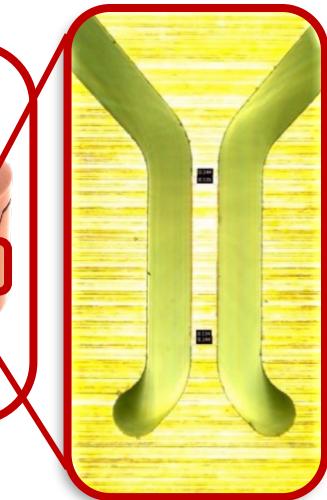
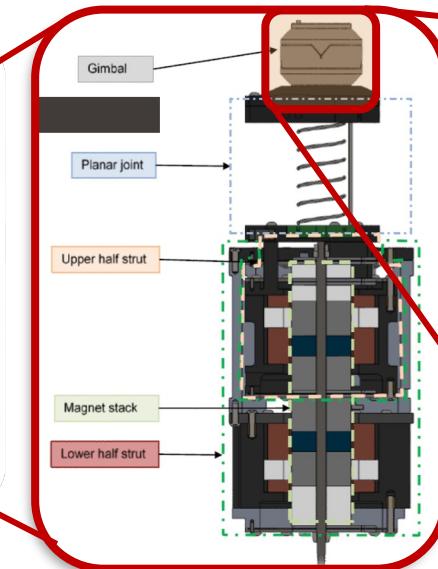
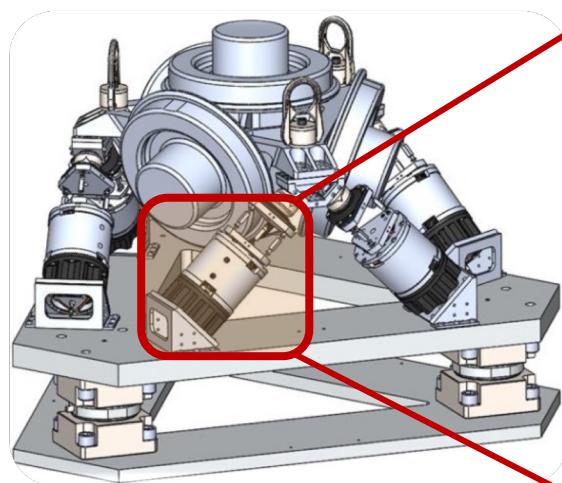
- **Objective function:** summation of normalized and weighted contributions
- **Weights** are chosen based on the relative importance and criticality of the achievement of each objective

$$G_{obj} = w_{C_z} \frac{C_z / \bar{C}_z - 1}{STD(C_z / \bar{C}_z - 1)} + w_{\sigma_z} \frac{\sigma_z / \bar{\sigma}_z - 1}{STD(\sigma_z / \bar{\sigma}_z - 1)} + w_{K_{\theta_z}} \frac{K_{\theta_z} / \bar{K}_{\theta_z} - 1}{STD(K_{\theta_z} / \bar{K}_{\theta_z} - 1)} + w_{\sigma_{\theta_z}} \frac{\sigma_{\theta_z} / \bar{\sigma}_{\theta_z} - 1}{STD(\sigma_{\theta_z} / \bar{\sigma}_{\theta_z} - 1)} + w_{CS} \frac{CS / \bar{CS} - 1}{STD(CS / \bar{CS} - 1)}$$



Flexures optimization - Example 2

- **Vibration isolation and suppression system** for precision payloads in space
- **Hexapod** including six electromagnetic damping struts
- **Gimbal flexure optimization** made analytically



Flexures optimization - Example 2

- Baseline and specifications



Input parameters

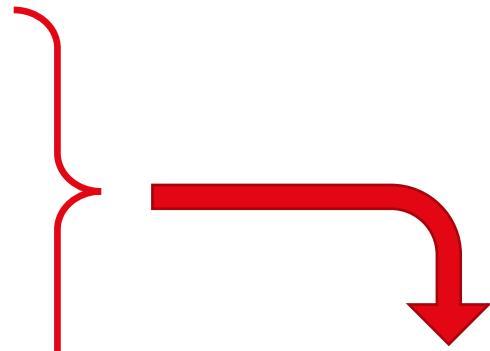
Material	17-4 PH	
Young's Modulus	E	196 GPa
Pivot blade Outer Diameter	D	20 mm
Blade length	L	12 mm
Blade thickness	h	0.26 mm
Blade width	b	9 mm
Pivot internal diameter	di	2 mm
Joint overall length (flat-flat)	l	mm
Max angle stroke	0.018 rad	

Analysis stiffness values (results)

K_lateral_x	35782	N/m
K_lateral_y	35782	N/m
K_axial_z	3.8E+07	N/m
K_bending_rx	0.430	Nm/rad
K_bending_ry	0.430	Nm/rad
K_torsion_rz	1.15	Nm/rad

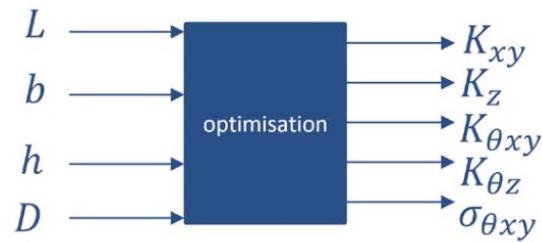
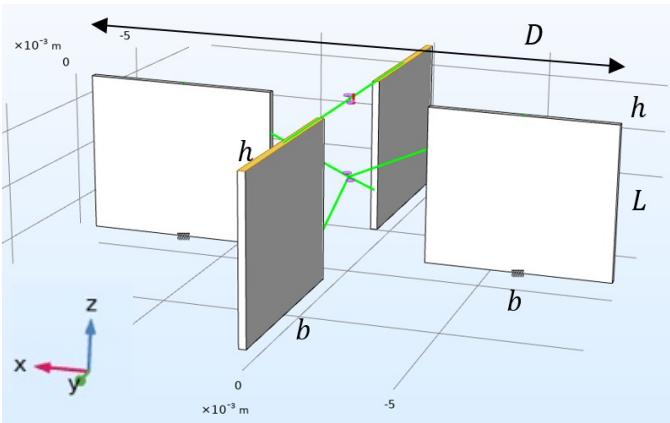
Target stiffness values (specs)

K_lateral_x	1E+06	N/m
K_lateral_y	1E+06	N/m
K_axial_z	12000000	N/m
K_bending_rx	1.15	Nm/rad
K_bending_ry	1.15	Nm/rad
K_torsion_rz	600	Nm/rad



Flexure optimization – example 2 (optimization procedure)

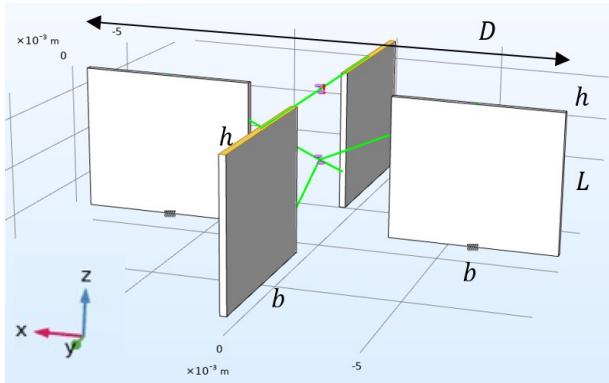
- Parameter definition
- Analytical formulation of the stiffnesses
- Validation of the stiffnesses formulae
- Generation of sets of parameters
 - Define parameters boundaries
 - Monte Carlo set up (uniform distribution on each param.)
- Outputs assessment
 - Calculate K_{xy} , K_z , K_{rxy} , K_{rz} for each set of parameters
 - Calculate the functional von Mises stress for each set of parameters
- Optimization process
 - Define the weights for each individual objective
 - Define a global objective function including the weights and normalizations
 - Find the optimum of the global objective function
- Verification of the optimal solution (reanalysis)



Flexure optimization – example 2

Analytical modeling and model validation through FEM:

- Lateral stiffness
$$K_{xy} = \frac{2Eb^3h^3}{L^3(h^2 + b^2)}$$
- Axial stiffness
$$K_z = \frac{Ebh}{L}$$
- Bending stiffness
$$K_{\theta_{xy}} = \frac{Ebh^3(D - b)^2}{L(2h^2 + 6(D - b)^2)}$$
- Torsional stiffness
$$K_{\theta_z} = \frac{Ebh^3(D - b)^2}{4L^3} + \frac{Gbh^3}{3L}$$
- Bending stress
$$\sigma_{\theta_{xy}} = \frac{\theta_{xy}Eh}{2L}$$



Flexure optimization – example 2

- What is the value of the **bending stiffness** $K_{\theta_{xy}} = ?$

$$\frac{1}{K_{\theta_{xy}}} = \frac{1}{2K_{trac\theta}} + \frac{1}{2K_{pb}}$$

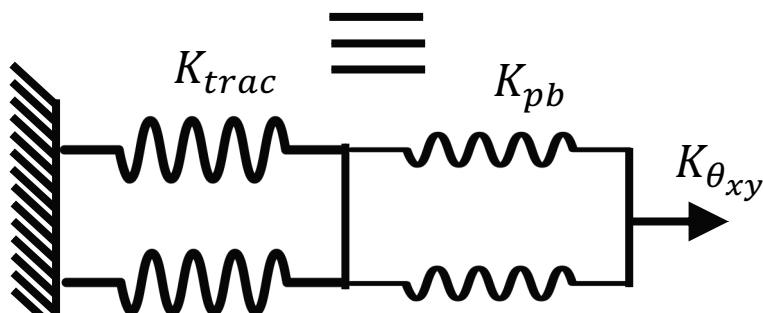
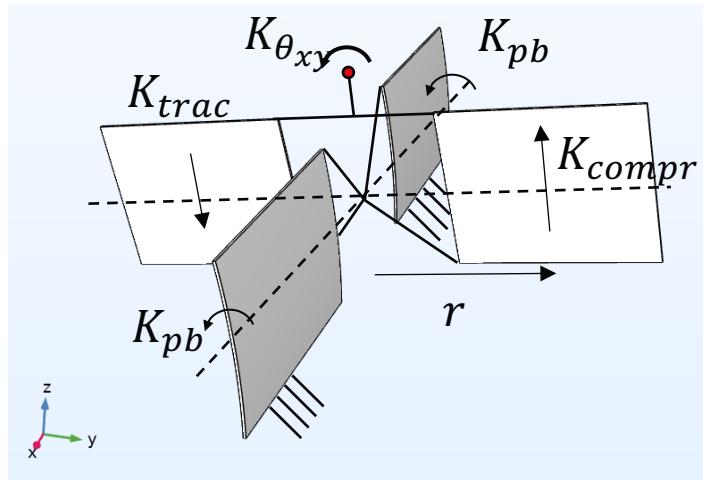
$$K_{trac\theta} = \left(\frac{C}{\theta} = \frac{F \cdot r}{z/r} = \right) K_{trac} \cdot r^2$$

$$r = \frac{D - b}{2}$$

$$K_z = \frac{Ebh}{L} \quad K_{pb} = \frac{Ebh^3}{12L}$$



$$K_{\theta_{xy}} = \frac{Ebh^3(D - b)^2}{L(2h^2 + 6(D - b)^2)}$$



$$K_{compr} = K_{trac}$$

$$K_{pb}$$

Flexure optimization – example 2

- This correspond to a **design** with:
 - Very short leaf springs
 - Thin leaf springs
 - Large diameter
 - Medium inner diameter
- All **stiffness requirements** are fulfilled
- Optimal solution **parameters**:

Property	Symbol	Value	Unit
Height	L	1	mm
Width	b	12.5	mm
Thickness	h	140	um
Diametre	D	41	mm



MIN

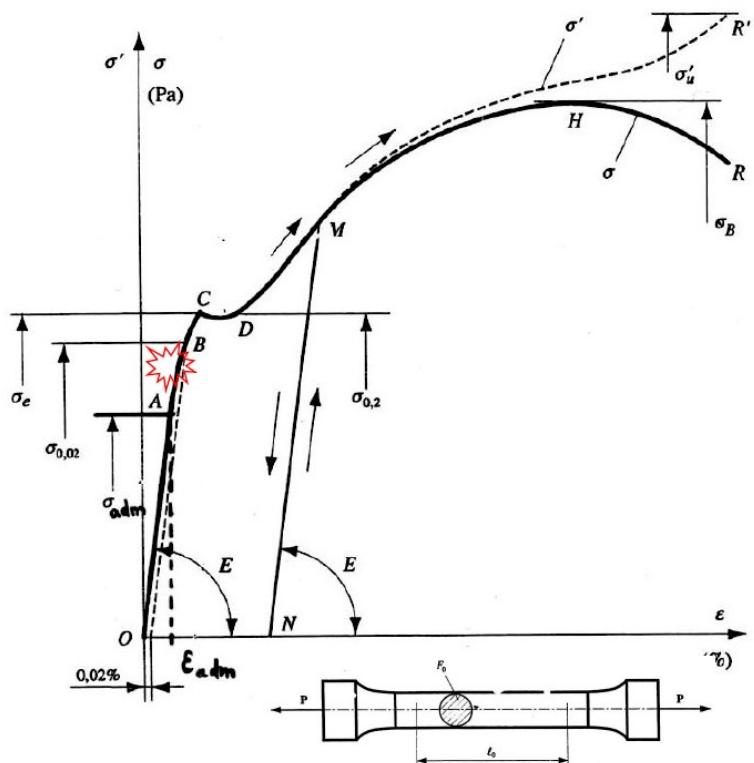
MAX

MIN

Axis	Spec	Stiffness FEM	Unit	Ratio spec
x	1.00	5.35	MN/m	5.35
y	1.00	5.35	MN/m	5.35
z	12.00	63.92	MN/m	5.33
rx	1.15	0.96	Nm/rad	0.83
ry	1.15	0.96	Nm/rad	0.83
rz	600	627	Nm/rad	1.05

Fatigue and material aspects

- The stroke of flexure is proportional to : $\frac{\sigma_{adm}}{E}$



Good materials for flexures:

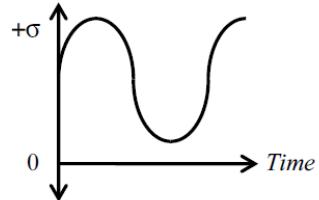
- Spring steels, carbon steels
 - e.g. X220CrVMo13-4 (K190)
 - X3NiCoMoTi18-9-5 (W720)
- Spring stainless-steels
 - e.g. X10CrNi18-8
 - N700 ESU (17-4 PH)
- Titanium alloys
 - e.g. Ti Al6 V4
- Copper-beryllium alloys
 - e.g. XHMS
- Bronze alloys
 - e.g. Cu Ni 15 Sn 8
- Aluminum alloys
 - e.g. AlCuSiMn (Avional)
 - AlZnMgCu1.5 (Perunal)

Fatigue and material aspects – tests

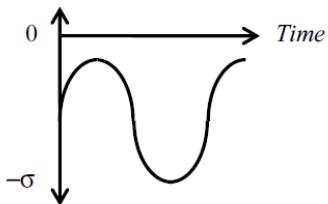
$$R = \sigma_{\min} / \sigma_{\max}$$

(R-1 is most common type for flexures)

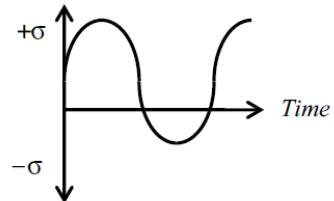
$0 < R < 1$
Tension-Tension



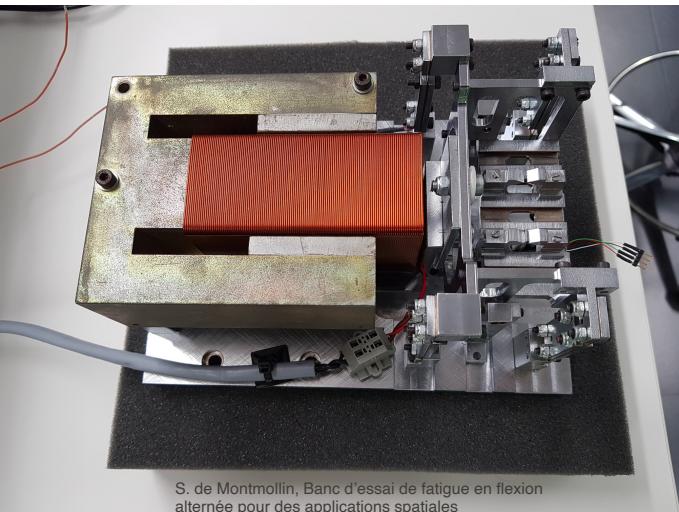
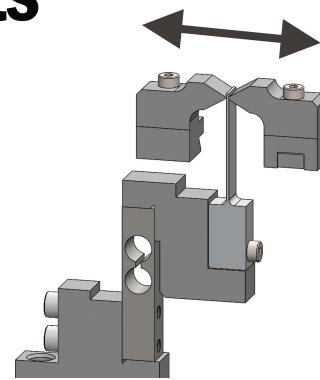
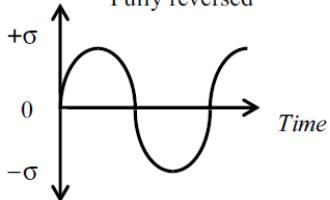
$R > 1$
Compression-Compression



$R < 0$
Tension-Compression



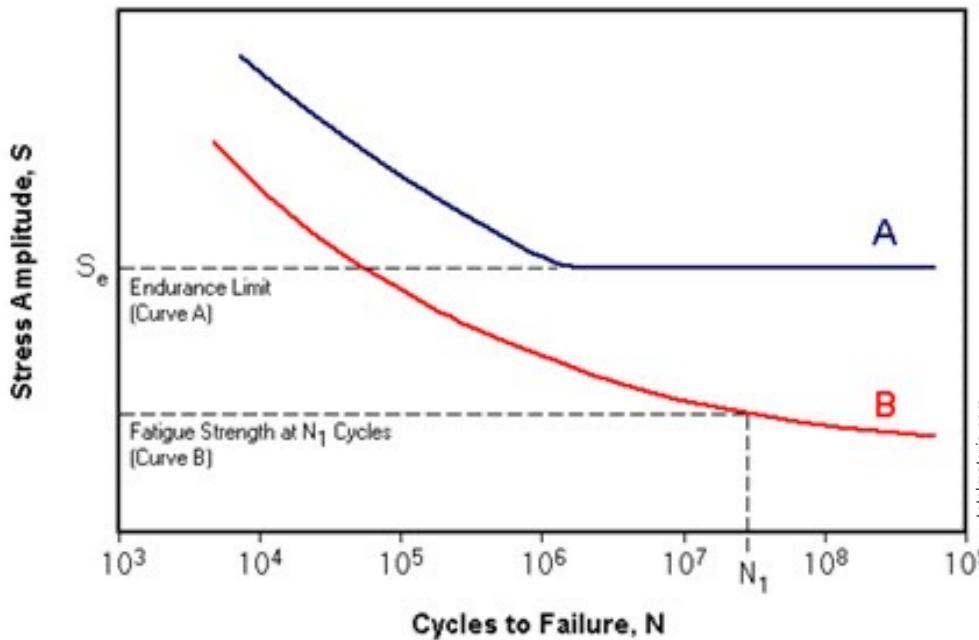
$R = -1$
Tension-Compression
Fully reversed



Fatigue and material aspects

- Examples of Wohler curves (S/N curve)
 - A has infinite lifetime (with fatigue plateau)
 - B has no infinite lifetime (without fatigue plateau)

EXERCICE 7



Fatigue and material aspects - standards

- Failure probability and lifetime duration

ISO 3800 Threaded fasteners – Axial load fatigue testing

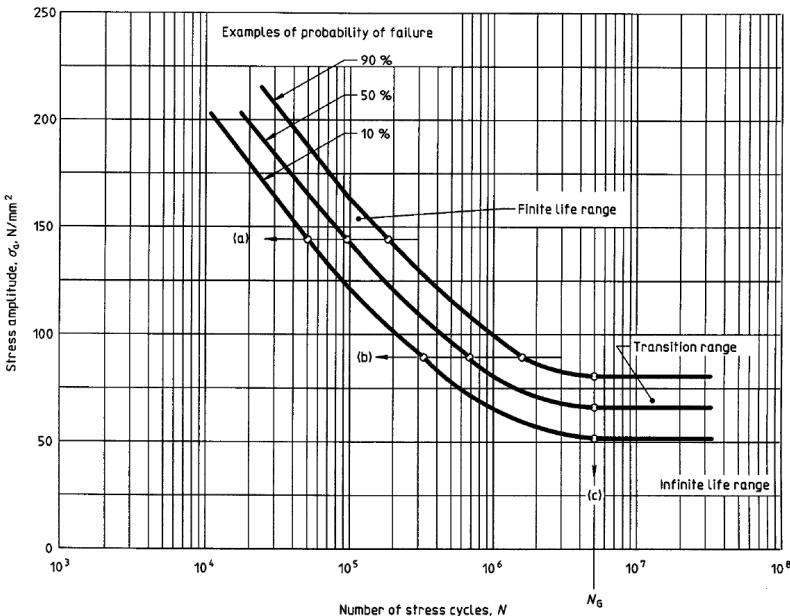
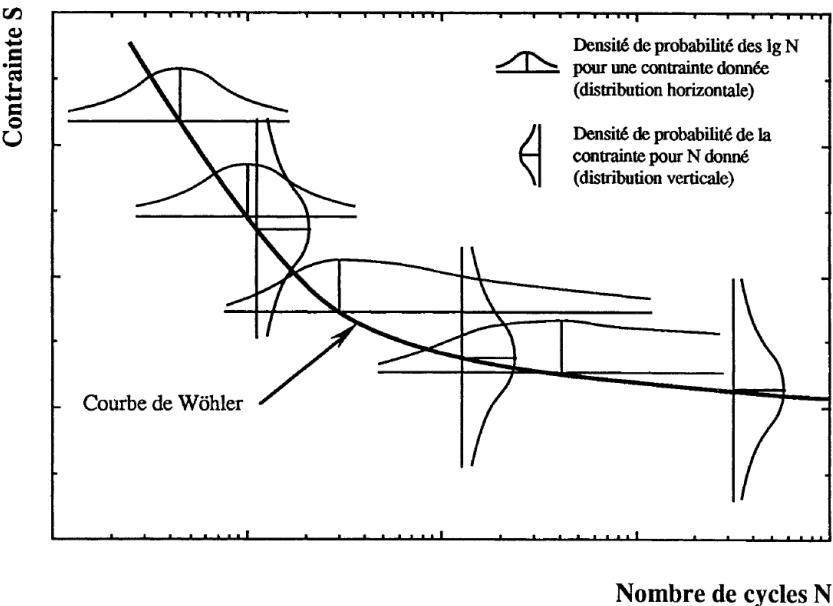


Figure 10 — Wöhler curve (S/N curve) $\sigma_a = f(N)$

AFNOR – A 03-405 Produits métalliques - Essais de fatigue - Traitement statistique des données.



Week 6 exercises and homework

- Exercise
 - on MOODLE : EXO_6_Kinematic_analysis.pdf