Solution for exercise 9 – Part I:
Design and force balancing of a flexure-based parallel stage

Context:

A flexure-base parallel stage is subjected to fluctuating linear accelerations along the x-axis due to motion of its base. To be insensitive to this specific perturbation, we wish to force balance the mechanism.

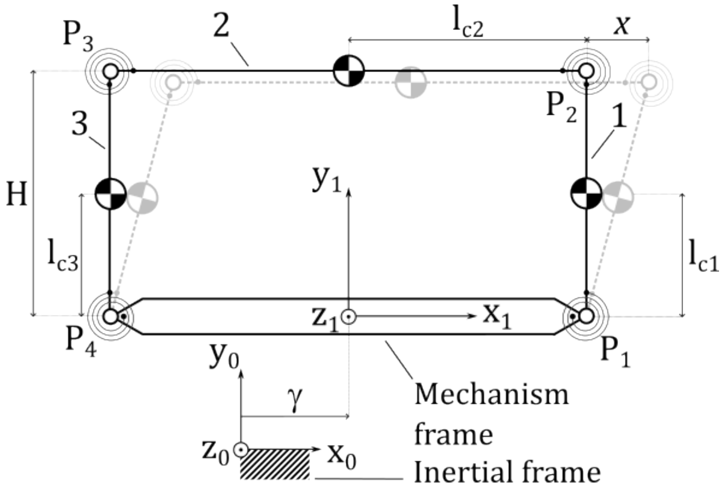

Fig. 1: Flexure-based parallel stage Pseudo-Rigid-Body-Model before its force balancing.

Figure 1 shows the Pseudo-Rigid-Body-Model (PRBM) of the studied flexure-based parallel stage before its force balancing. The mechanism is shown in its rest position (black lines) and in its deformed position (transparent black lines). The mechanism has one degree-of-freedom and is composed of three moving rigid bodies: solids **1**, **2** and **3** articulated in P_1, P_2, P_3 and P_4.

Each solid $i = 1 \dots 3$, has:

- a mass m_i,
- a center of mass (COM) c_i (represented by the symbol ◕) located by a distance l_{ci},
- an inertia tensor $\bar{\bar{J}}_i$ written at c_i.

Each pivot $P_{i=1\dots4}$ has an angular stiffness k_i that represents the intrinsic stiffness of the future flexure-based implementation.

First, the parallel stage mechanism is actuated by a voice coil whose permanent magnet is fixed to the mechanism's base and whose coil is fixed to solid **2**. The voice coil applies a force F at c_2 along the x-axis. The parameter x is used to locate the displacement of solid **2** relative to its base.

Last, the linear displacement that corresponds to the motion of the mechanism's base is represented by the parameter γ.

Part I: Force balancing of the flexure-based parallel stage

In order to force balance the mechanism, we propose to redistribute its mass by lowering the COM of solids 1 and 3 thanks to two identical supplementary masses m_{bal}. These two masses are located by the distance l_{bal}.

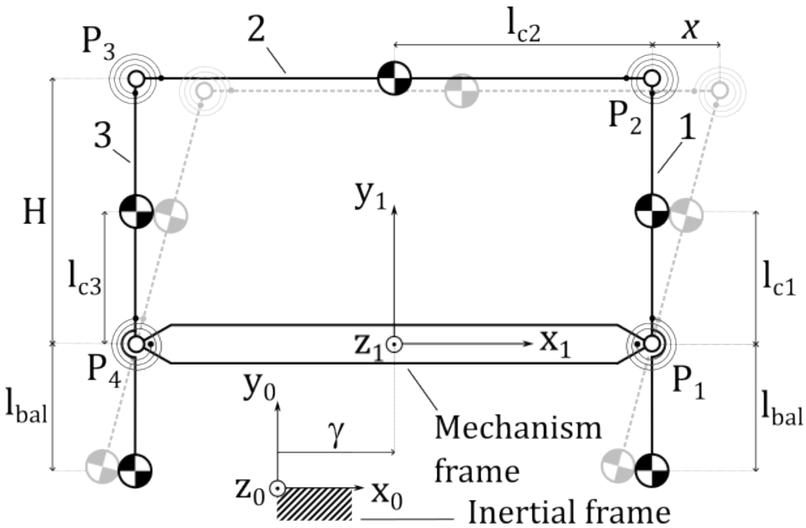

Fig. 2: Flexure-based parallel stage Pseudo-Rigid-Body-Model after its force balancing.

Numerical Values:

- $H = 20\text{e-}3$ [m]
- m_1=1.7138708e-02 [kg]
- m_2=6.8122300e-02 [kg]
- $m_3 = m_1$
- m_{bal}=3.5565044e-02 [kg]
- l_{c1}=1.0411598e-2 [m]
- l_{c2} : not needed
- $l_{c3} = l_{c1}$
- l_{bal} : to be found
- $\bar{\bar{J}}_1 = \begin{bmatrix} 5.3559891\text{e-}6 & 0 & 0 \\ 0 & 1.9478338\text{e-}6 & 0 \\ 0 & 0 & 5.3569824\text{e-}6 \end{bmatrix}$ [kg. m^2]
- $\bar{\bar{J}}_2$: not needed
- $\bar{\bar{J}}_3 = \bar{\bar{J}}_1$
- $\bar{\bar{J}}_{\text{bal}} = \begin{bmatrix} 2.0034975\text{e-}6 & 0 & 0 \\ 0 & 2.5379149\text{e-}6 & 0 \\ 0 & 0 & 1.1271682\text{e-}6 \end{bmatrix}$ [kg. m^2]

1. Using the Euler-Lagrange formalism, calculate the equation of motion of the parallel stage as a function of F, x and γ, assuming small deformations around the rest position. The equation of motion is of the form:

$$F = m_{\mathrm{eq}}\ddot{x} + k_{\mathrm{eq}}x + \tau_{\mathrm{coupling}}\ddot{\gamma}$$

Reminder: The Euler-Lagrange equation that needs to be solved is

$$F = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x}$$

Where $L = K - V$ is the Lagrangian; K and V are the kinetic and potential energies of the system, respectively.

SOLUTION:

We introduce the parameter θ to simplify the writing of the equations. Parameter θ, considered positive **anticlockwise**, represents the angle of solid 1 (respectively, solid 3) in its deformed position relative to its vertical rest position.

Using parameter θ, the kinetic and potential energies of the total mechanism can be expressed as

$$K = \frac{1}{2}m_{1,\mathrm{tot}}\dot{c}_{1,\mathrm{tot}}^2 + \frac{1}{2}m_2\dot{c}_2^2 + \frac{1}{2}m_{3,\mathrm{tot}}\dot{c}_{3,\mathrm{tot}}^2 + \frac{1}{2}J_{1,\mathrm{tot,zz}}\dot{\theta}^2 + \frac{1}{2}J_{3,\mathrm{tot,zz}}\dot{\theta}^2$$

$$V = \frac{1}{2}(k_1 + k_2 + k_3 + k_4)\theta^2$$

Where $m_{1,\mathrm{tot}}$ and $\dot{c}_{1,\mathrm{tot}}$ are the mass and the COM linear velocity vector of solid 1 equipped with its balancing mass; $\dot{c}_2$ is the COM linear velocity vector of solid 2; $m_{3,\mathrm{tot}}$ and $\dot{c}_{3,\mathrm{tot}}$ are the mass and the COM linear velocity vector of solid 3 equipped with its balancing mass; $\dot{\theta}$ is the angular velocity vector of solids 1 and 3; $J_{1,\mathrm{tot,zz}}$ and $J_{3,\mathrm{tot,zz}}$ are the moment of inertia at COM of solids 1 and 3 equipped with their respective balancing mass.

With

$$m_{1,\mathrm{tot}} = m_1 + m_{\mathrm{bal}} \quad \text{and} \quad m_{3,\mathrm{tot}} = m_3 + m_{\mathrm{bal}}$$

$$\dot{c}_{1,\mathrm{tot}} = \left(\dot{\gamma} - l_{c1,\mathrm{tot}}\cos\theta\,\dot{\theta}\right)x_0 - l_{c1,\mathrm{tot}}\sin\theta\,\dot{\theta}y_0$$

$$\dot{c}_2 = \left(\dot{\gamma} - H\cos\theta\,\dot{\theta}\right)x_0 - H\sin\theta\,\dot{\theta}y_0$$

$$\dot{c}_{3,\mathrm{tot}} = \left(\dot{\gamma} - l_{c3,\mathrm{tot}}\cos\theta\,\dot{\theta}\right)x_0 - l_{c3,\mathrm{tot}}\sin\theta\,\dot{\theta}y_0$$

$$l_{\text{c1,tot}} = \frac{m_{\text{bal}} l_{\text{bal}} - m_1 l_{c1}}{m_{1,\text{tot}}} \quad \text{and} \quad l_{\text{c3,tot}} = \frac{m_{\text{bal}} l_{\text{bal}} - m_3 l_{c3}}{m_{3,\text{tot}}}$$

$$J_{1,\text{tot,zz}} = J_{1,\text{zz}} + J_{\text{bal,zz}} + \left(l_{c1,\text{tot}} + l_{c1}\right)^2 m_1 + \left(l_{bal} - l_{c1,\text{tot}}\right)^2 m_{bal}$$

$$J_{3,\text{tot,zz}} = J_{3,\text{zz}} + J_{\text{bal,zz}} + \left(l_{c3,\text{tot}} + l_{c3}\right)^2 m_3 + \left(l_{bal} - l_{c3,\text{tot}}\right)^2 m_{bal}$$

The kinetic energy simplifies and can then written as

$$\begin{aligned} K = \frac{1}{2} m_{1,\text{tot}}\left(\dot{\gamma}^2 + l_{\text{c1,tot}}^2 \dot{\theta}^2 - 2 l_{\text{c1,tot}} \dot{\gamma}\dot{\theta}\cos\theta\right) \\ + \frac{1}{2} m_2\left(\dot{\gamma}^2 + H^2\dot{\theta}^2 + 2H\dot{\gamma}\dot{\theta}\cos\theta\right) \\ + \frac{1}{2} m_{3,\text{tot}}\left(\dot{\gamma}^2 + l_{\text{c3,tot}}^2 \dot{\theta}^2 - 2 l_{\text{c3,tot}} \dot{\gamma}\dot{\theta}\cos\theta\right) + \frac{1}{2} J_{1,\text{tot,zz}}\dot{\theta}^2 \\ + \frac{1}{2} J_{3,\text{tot,zz}}\dot{\theta}^2 \end{aligned}$$

We also have the following simplifications:

$$m_1 = m_3, \quad l_{c1} = l_{c3}, \quad J_{1,\text{zz}} = J_{3,\text{zz}} \quad \text{and} \quad k_1 = k_2 = k_3 = k_4.$$

$$x = H\sin\theta \approx H\theta \quad \text{and} \quad \cos\theta \approx 1$$

Using these simplifications, we can rewrite K and V so that:

$$\begin{aligned} K \approx \frac{1}{2}\left(2m_{1,\text{tot}} + m_2\right)\dot{\gamma}^2 + \frac{1}{2}\frac{2J_{1,\text{tot,zz}} + 2m_{1,\text{tot}} l_{\text{c1,tot}}^2 + m_2 H^2}{H^2}\dot{x}^2 \\ + \frac{m_2 H - 2m_{1,\text{tot}} l_{c1}}{H}\dot{\gamma}\dot{x} \end{aligned}$$

$$V \approx 2\frac{k_1}{H^2}x^2$$

We then find

$$F = \frac{2J_{1,\text{tot,zz}} + 2m_{1,\text{tot}} l_{\text{c1,tot}}^2 + m_2 H^2}{H^2}\ddot{x} + 4\frac{k_1}{H^2}x + \frac{m_2 H - 2m_{1,\text{tot}} l_{c1,\text{tot}}}{H}\ddot{\gamma}$$

So that

$$m_{eq} = \frac{2J_{1,\text{tot,zz}} + 2m_{1,\text{tot}} l_{\text{c1,tot}}^2 + m_2 H^2}{H^2}$$

$$k_{eq} = 4\frac{k_1}{H^2}$$

$$\tau_{coupling} = \frac{m_2H - 2m_{1,\text{tot}}l_{c1,\text{tot}}}{H}$$

2. Identify the dynamic coupling term τ_{coupling} that couples the mechanism motion to its base acceleration $\ddot{y}$.

SOLUTION:

$$\tau_{coupling} = \frac{m_2H - 2m_{1,\text{tot}}l_{c1,\text{tot}}}{H}$$

3. Find the length l_{bal} that cancels τ_{coupling}.

SOLUTION:

$$l_{\text{bal}} = \frac{2m_1l_{c1} + m_2H}{2m_{\text{bal}}} = 24.1716\ [\text{mm}]$$

4. Compute the linear equivalent masses $m_{\text{eq,init}}$ and $m_{\text{eq,tot}}$ of the parallel stage before and after it is forced balanced.

SOLUTION:

$$m_{\text{eq,init}} = \frac{2\left(J_{1,zz} + m_1l_{c1}^2\right) + m_2H^2}{H^2} \approx 104.2\ [\text{g}]$$

$$m_{\text{eq,tot}} = \frac{2\left(J_{1,\text{tot},zz} + m_{1,\text{tot}}l_{c1,\text{tot}}^2\right) + m_2H^2}{H^2} \approx 213.7\ [g]$$

5. Considering $F = 0$, write the analytical transfer functions $T_{\text{init}}(\omega)$ and $T_{\text{final}}(\omega)$ of the parallel stage before and after it is forced balanced:

$$T(\omega) = \frac{X(\omega)}{(j\omega)^2\Gamma(\omega)}.$$

SOLUTION:

$$T(\omega) = \frac{\tau_{\text{coupling}}}{k_{\text{eq}}}\frac{1}{\left(\frac{\omega}{\omega_0}\right)^2 - 1}$$

With

$$\omega_0^2 = \frac{k_{\text{eq}}}{m_{\text{eq}}}$$