EPFL

Exercise 12: Rehearsal

EXO A: Lagrangian formalism applied to a flexible linear stage

Through **Lagrange methodology**, determine the **motion equation** of the system shown on Figure 1. Then, based on the motion equation, identify the system **eigenfrequency** and determine the **sag due to gravity** (vertical static position) by expressing the dynamic condition corresponding to the static case. The **Young modulus** is **200 GPa**, the **length L** is **50 mm**, the **width b** is **10 mm** and the **thickness h** is **300 um**. The **mass m** of the stage is **1 kg**.

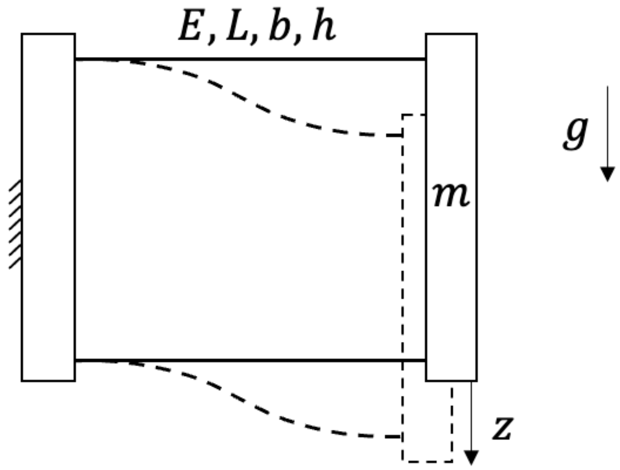

Figure 1 : Linear stage with gravity acting on the mobile shuttle.

Solution A:

First, we describe the lagrangian L of the system. It has two components, the kinematic energy T and the potential energy V:

$$L = T - V$$

The kinetic energy is expressed as follows:

$$T = \frac{1}{2} m\dot{z}^2$$

The potential energy has two contributors, the spring potential energy and the gravitational potential energy:

$$V = \frac{1}{2}Kz^2 - mgz$$

Thus the lagrangian is:

$$L = \frac{1}{2}m\dot{z}^2 - \frac{1}{2}Kz^2 + mgz$$

We now have to calculate the components of the Euler-Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = 0$$

We calculate the partial derivative of L by z:

$$\frac{\partial L}{\partial z} = mg - Kz$$

We calculate also the time derivative of the partial derivative of L by $\dot{z}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{z}} = m\ddot{z}$$

We obtain the equation of motion:

$$m\ddot{z} - mg + Kz = 0$$

By dividing by m:

$$\ddot{z} - g + \frac{K}{m}z = 0$$

Where $\frac{K}{m} = \omega_0^2$, thus the eigenfrequency of the system is:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{K}{m}} = 4.68\ Hz$$

With:

$$K = \frac{2Ebh^3}{L^3} = 864\ \frac{N}{m}$$

The sag is calculated by describing the static condition:

$$\dot{z} = 0 \quad \text{and} \quad \ddot{z} = 0$$

We then obtain the sag position:

$$z_{sag} = \frac{mg}{K} = 11.4\ mm$$

EXO B: Mercurian optomechanical payload rotative stage

In the frame of a solar system exploration mission, a **rotative flexure stage** is designed to be used in an optomechanical system on Mercury to study its atmosphere. The **gravity** of Mercury equals **3.7 m/s²**. You must size the width **b** and the thickness **h** of the blades of an overconstrained flexure pivot (Figure 2) to respect the following constraints: the **sag** due to Mercurian gravity must be equal to **645 nm** and the **rotative eigenfrequency** must be equal to **10 Hz**. The optical payload has a **mass of 0.8 kg** and an **inertia J = 1.72e-4 kg*m^2**. The length of the pivot's blades **length** is imposed (**80 mm**) and the material is **Titanium** (Young modulus **E = 116 GPa**). Also, the inner roots of the blades are considered colinear with the pivot rotation axis (**p = 0**) and no radius is considered at the roots of the blades.

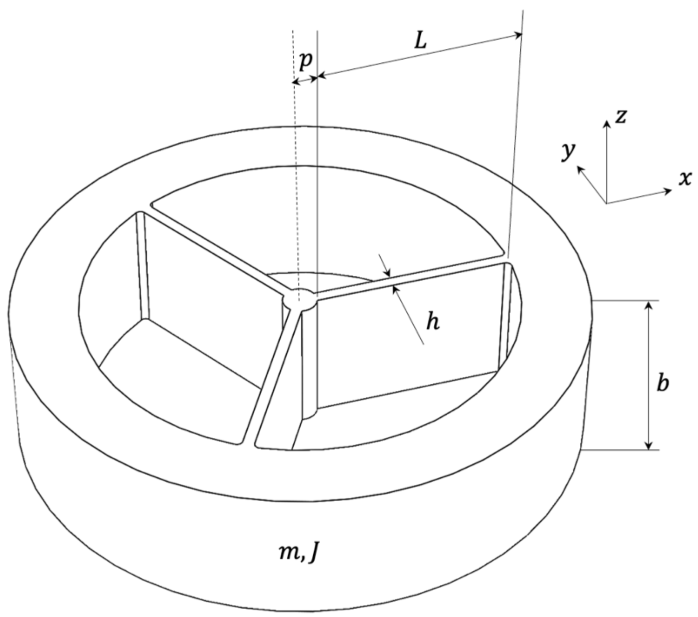

Figure 2 : Flexible rotation stage for orientation of an optical payload aboard a rover on Mercury.

Solution B:

The vertical stiffness of the stage Kz value of the stage is determined through the sag value, the Mercurian gravity, and the mobile mass of the stage:

$$K_z = \frac{m \cdot g_{Mer}}{sag} = 4.59 \frac{N}{\mu m} \quad (1)$$

The rotation stiffness is determined by the rotation frequency and the inertia:

$$K_\theta = 4 \cdot \pi^2 \cdot f_\theta^2 \cdot J = 0.68 \frac{Nm}{rad} \quad (2)$$

Then, we express the vertical stiffness:

$$K_z = \frac{3 \cdot E \cdot b^3 \cdot h}{L^3} \quad (3)$$

It comes:

$$h = \frac{K_z \cdot L^3}{3 \cdot E \cdot b^3} \quad (4)$$

And the rotation stiffness (RCC formula with 3 blades and p = 0):

$$K_\theta = \frac{E \cdot b \cdot h^3}{L} \quad (5)$$

By replacing h formula given in (4), we obtain:

$$b = \sqrt[8]{\frac{K_z^3}{27 \cdot K_\theta \cdot E^2}} \cdot L = 30\ mm \quad (6)$$

Finally, through (4):

$$h = 250\ \mu m$$

EXO C: Crank-linear stage linkage geometric model

In a machine, a crank-linear stage is implemented using flexures. The system geometry is modeled as illustrated in Figure 3. The **input** of the mechanism is **θ** angle. It's motion ranges from -15° to +15°. The length L of both connecting rods is 50 mm. Calculate the **maximum deflections** of angles **α** and **β** as well as the linear motion range of the linear stage (***z*** **translation**)

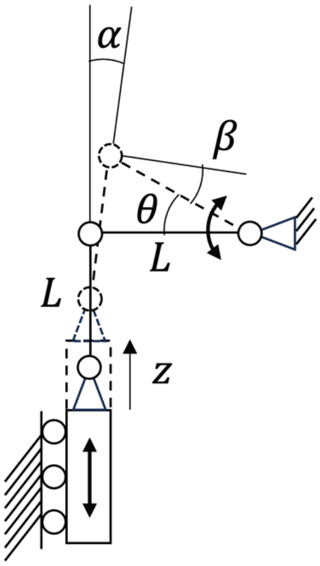

Figure 3: Geometric model of a crank-linear stage linkage.

The joints of the structure are then implemented in flexible elements. The pivots have separated crossed blades, whose **length** is **15 mm**. Calculate their thickness ***h_alpha, h_beta*** and ***h_theta*** to reach the fatigue stress σ_D = 100 MPa with maximal deflection for each pivot. The material is Aluminum Avional. The Young modulus is E = 73 GPa.

Solution C:

The analysis of the system geometry leads to the following relations:

$$L \cdot \sin\alpha + L \cdot \cos\theta = L$$

$$\alpha = \text{asin}\,(1 - \cos\theta)$$

And:

$$L \cdot \sin\theta + L = L \cdot \cos\alpha + z$$

So:

$$z = L(\sin\theta - \cos\alpha + 1)$$

Also:

$$\beta = \alpha - \theta$$

For both extreme values of angle θ, the maximal deflections are calculated on each joint:

$\alpha_{max} = 1.95°$
$\beta_{max} = 16.95°$
$z_{max} = 12.97$ mm

Note that the absolute values are considered. Finally, the thicknesses of the pivot's blades are calculated using:

$$h_\alpha = \frac{2 \cdot \sigma_D \cdot L}{E \cdot \alpha} = 1206\ um$$

$$h_\beta = \frac{2 \cdot \sigma_D \cdot L}{E \cdot \beta} = 139\ um$$

$$h_\theta = \frac{2 \cdot \sigma_D \cdot L}{E \cdot \theta} = 157\ um$$