Last Name:

First Name:

SCIPER:

Sample Test 2 solutions – Microengineering 110

Spring 2024

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- **During the actual test, you would be entering your answers on moodle and on the paper. For this sample test, only the paper version is being provided. During the actual test you MUST enter your answers BOTH on this paper version and on the moodle. The paper version will be used as a backup in case your moodle version is corrupted, etc.**
- **No communications are allowed with others. Therefore, YOU MUST EXIT ALL COMMUNICATION PROGRAMS (Chat windows, etc.). Should a proctor notice that you have an open communication program, it will be considered to be a violation of academic integrity standards.**

1. The joint probability distribution of the number X of cars and the number Y of buses per signal cycle at a traffic signal for a right-turn lane is displayed in the accompanying joint probability table.

$p(x, y)$		y = 0	1	2
	0	.025	.015	.010
	1	.050	.030	.020
	2	.125	.075	.050
x	3	.150	.090	.060
	4	.100	.060	.040
	5	.050	.030	.020

a. What is the probability that there is exactly one car and exactly one bus during a cycle?

$p(1,1) = .030.$

b. What is the probability that there is at most one car and at most one bus during a cycle?

$P(X \leq 1 \text{ and } Y \leq 1) = p(0,0) + p(0,1) + p(1,0) + p(1,1) = .120.$

c. What is the probability that there is exactly one car during a cycle? Exactly one bus?

$P(X = 1) = p(1,0) + p(1,1) + p(1,2) = .100;\ P(Y = 1) = p(0,1) + \ldots + p(5,1) = .300.$

d. Suppose the right-turn lane is to have a capacity of five cars, and that one bus is equivalent to three cars. What is the probability of an overflow during a cycle?

$P(\text{overflow}) = P(X + 3Y > 5) = 1 - P(X + 3Y \le 5) = 1 - P((X,Y)=(0,0)$ or …or $(5,0)$ or $(0,1)$ or $(1,1)$ or $(2,1)) = 1 - .620 = .380$.

e. Are X and Y independent rv's? Explain.

The marginal probabilities for X (row sums from the joint probability table) are $p_X(0) = .05$, $p_X(1) = .10$, $p_X(2) = .25$, $p_X(3) = .30$, $p_X(4) = .20$, $p_X(5) = .10$; those for Y (column sums) are $p_Y(0) = .5$, $p_Y(1) = .3$, $p_Y(2) = .2$. It is now easily verified that for every (x,y), $p(x,y) = p_X(x) \cdot p_Y(y)$, so X and Y are independent.

2. You have two lightbulbs for a particular lamp. Let X = the lifetime of the first bulb and Y = the lifetime of the second bulb (both in 1000s of hours). Suppose that X and Y are independent and that each has an exponential distribution with parameter l = 1.

a. What is the joint pdf of X and Y?

$$f(x,y) = f_X(x) \cdot f_Y(y) = \begin{cases} e^{-x-y} & x \ge 0, y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

b. What is the probability that each bulb lasts at most 1000 hours (i.e., X <= 1 and Y >0 1)?

By independence, $P(X \le 1 \text{ and } Y \le 1) = P(X \le 1) \cdot P(Y \le 1) = (1 - e^{-1})\,(1 - e^{-1}) = .400$.

c. What is the probability that the total lifetime of the two bulbs is at most 2?

$$P(X + Y \le 2) = \int_0^2 \int_0^{2-x} e^{-x-y}\,dy\,dx = \int_0^2 e^{-x}\left[1 - e^{-(2-x)}\right]dx = \int_0^2 (e^{-x} - e^{-2})\,dx = 1 - e^{-2} - 2e^{-2} = .594.$$

d. What is the probability that the total lifetime is between 1 and 2?

$$P(X + Y \le 1) = \int_0^1 e^{-x}\left[1 - e^{-(1-x)}\right]dx = 1 - 2e^{-1} = .264,$$

so $P(1 \le X + Y \le 2) = P(X + Y \le 2) - P(X + Y \le 1) = .594 - .264 = .330$.

3. An instructor has given a short quiz consisting of two parts. For a randomly selected student, let X = the number of points earned on the first part and Y = the number of points earned on the second part. Suppose that the joint pmf of X and Y is given in the accompanying table.

$p(x, y)$		y = 0	5	10	15
	0	.02	.06	.02	.10
x	5	.04	.15	.20	.10
	10	.01	.15	.14	.01

a. If the score recorded in the grade book is the total number of points earned on the two parts, what is the expected recorded score E(X + Y)?

$E(X+Y) = \sum\sum (x+y)p(x,y) = (0+0)(.02) + (5+0)(.04) + \ldots + (10+15)(.01) = 14.10.$

Note: It can be shown that $E(X + Y)$ always equals $E(X) + E(Y)$, so in this case we could also work out the means of X and Y from their marginal distributions: $E(X) = 5.55$, $E(Y) = 8.55$, so $E(X + Y) = 5.55 + 8.55 = 14.10$.

b. If the maximum of the two scores is recorded, what is the expected recorded score?

For each coordinate, we need the maximum; e.g., max(0,0) = 0, while max(5,0) = 5 and max(5,10) = 10. Then calculate the sum: $E(\max(X,Y)) = \sum\sum \max(x,y)\cdot p(x,y) =$ $\max(0,0)(.02) + \max(5,0)(.04) + \ldots + \max(10,15)(.01) = 0(.02) + 5(.04) + \ldots + 15(.01) = 9.60.$

c. Compute the covariance for X and Y

$E(X) = 5.55$, $E(Y) = 8.55$, $E(XY) = (0)(.02) + (0)(.06) + \ldots + (150)(.01) = 44.25$, so $\text{Cov}(X,Y) = 44.25 - (5.55)(8.55) = -3.20$.

4. Suppose that for aluminum alloy sheets of a particular type, mean value and standard deviation of the Young's modulus are 70 GPa and 1.6 GPa, respectively.

a. If Xbar is the sample mean Young's modulus for a random sample of n = 16 sheets, where is the sampling distribution of Xbar centered, and what is the standard deviation of the Xbar distribution?

The sampling distribution of $\bar{X}$ is centered at $E(\bar{X}) = \mu = 70$ GPa, and the standard deviation of the $\bar{X}$ distribution is $\sigma_{\bar{X}} = \frac{\sigma_{X}}{\sqrt{n}} = \frac{1.6}{\sqrt{16}} = 0.4$ GPa.

b. Answer the questions posed in part (a) for a sample size of n = 64 sheets.

With $n = 64$, the sampling distribution of $\bar{X}$ is still centered at $E(\bar{X}) = \mu = 70$ GPa, but the standard deviation of the $\bar{X}$ distribution is $\sigma_{\bar{X}} = \frac{\sigma_{X}}{\sqrt{n}} = \frac{1.6}{\sqrt{64}} = 0.2$ GPa.

c. For which of the two random samples, the one of part (a) or the one of part (b), is Xbar more likely to be within 1 GPa of 70 GPa? Explain your reasoning.

$\bar{X}$ is more likely to be within 1 GPa of the mean (70 GPa) with the second, larger, sample. This is due to the decreased variability of $\bar{X}$ that comes with a larger sample size.

5. Let X represent the error in making a measurement of a physical characteristic or property (e.g., the boiling point of a particular liquid). It is often reasonable to assume that E(X) = 0 and that X has a normal distribution. Thus the pdf of any particular measurement error is

$$f(x;\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-x^2/2\theta} \qquad -\infty < x < \infty$$

(where we have used θ in place of σ^2). Now suppose that n independent measurements are made, resulting in measurement errors $X_1 = x_1$, $X_2 = x_2$, … , $X_n = x_n$. Obtain the mle of θ.

Determine the joint pdf (aka the likelihood function), take a logarithm, and then use calculus:

$$f(x_1,\dots,x_n \mid \theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta}} e^{-x_i^2/2\theta} = (2\pi\theta)^{-n/2} e^{-\sum x_i^2/2\theta}$$

$$\ell(\theta) = \ln[f(x_1,\dots,x_n \mid \theta)] = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\theta) - \sum x_i^2/2\theta$$

$$\ell'(\theta) = 0 - \frac{n}{2\theta} + \sum x_i^2/2\theta^2 = 0 \Rightarrow -n\theta + \sum x_i^2 = 0$$

Solving for θ, the maximum likelihood estimator is $\hat{\theta} = \frac{1}{n}\sum X_i^2$.