Exercise week #7
Time-dependent RC circuits

Problem 1 (in class):

Consider the electrical circuit shown in Fig. 1 with the following parameters:

$R = 33.3\ \Omega$, $C = 150\ \mu$F, $U_1 = +200$ V, $U_2 = -200$ V. U_1 and U_2 are constant over time.

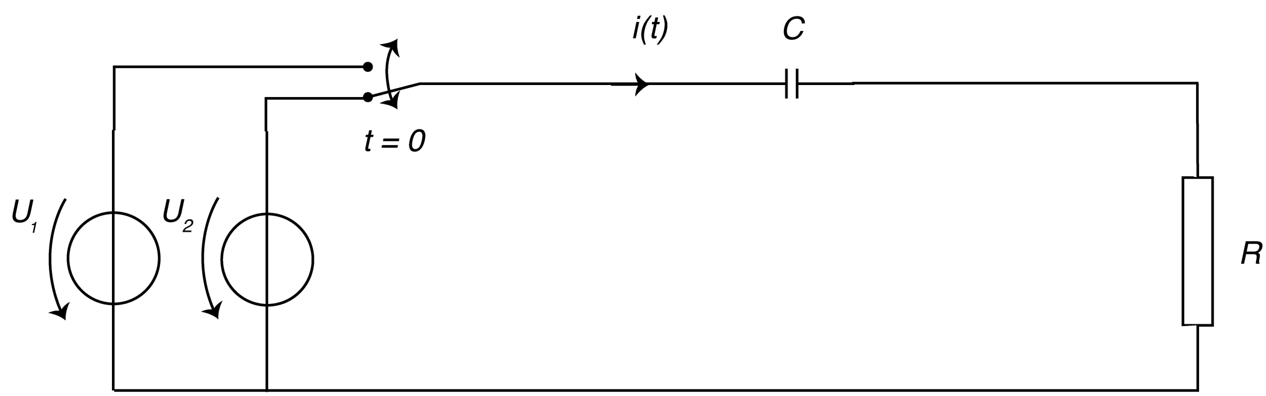

Figure 1: Electrical circuit.

At times $t < 0$, the switch is connected to voltage source U_2, the system is in steady-state and the current flowing through the circuit is $i(t < 0) = 0$.

At time $t = 0$, the switch is toggled to connect voltage source U_1 with the circuit, and then continues toggling between sources U_1 and U_2 every 3 ms. The total period for toggling back and forth between the two voltage sources is thus $T = 6$ ms.

a) Determine the current $i(t)$ and the voltage drop across the capacitor $u_C(t)$ as a function of time until $t = T$. Compute their values at times $t = 0$, $t = \frac{T}{2}$, $t = T$.

b) Plot $i(t)$ and $u_C(t)$ as a function of time t until $t = T$ using a software of your choice. What can you observe?

Problem 2 (self-study):

Consider the same scenario as in problem 1, but now with $T = 50\ ms$. The switch now toggles between sources U_1 and U_2 every 25 ms.

a) Before performing the calculation, explain what you expect to happen?

b) Determine the current $i(t)$ and the voltage drop across the capacitor $u_C(t)$ as a function of time until $t = T$. Compute their values at times $t = 0$, $t = \frac{T}{2}$, $t = T$.

c) Plot $i(t)$ and $u_C(t)$ as a function of time t until $t = T$ using a software of your choice. What can you observe?

Solution 1:

Reminder: As per our convention, small letters u, i denote time-dependent physical quantities.

First, we notice that the resistor and the capacitor are connected in series. Consequently, the current flowing through the resistor is the same as the current flowing through the capacitor. We denote this current i. Furthermore, we denote u_R the voltage drop across the resistor and the u_C voltage drop across the capacitor, see Fig. 2.

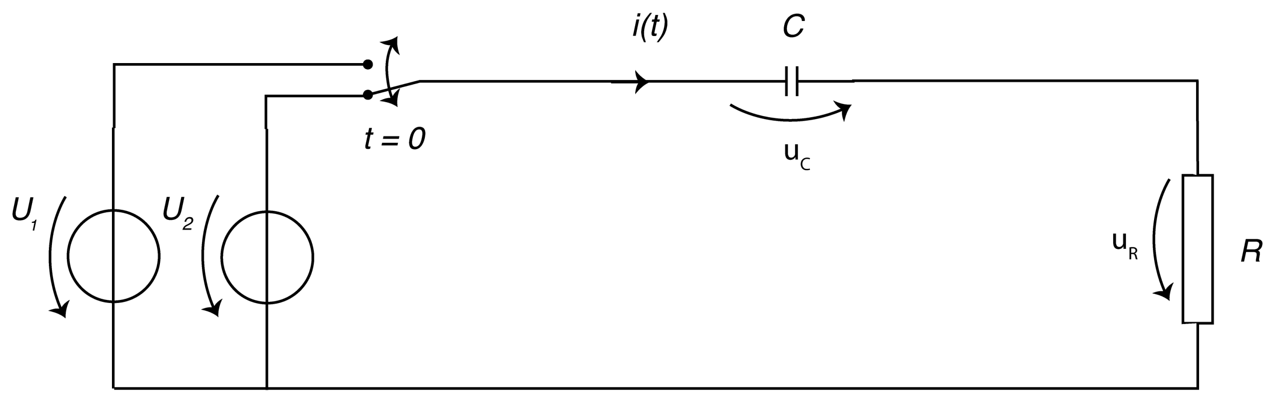

Figure 2: Electrical circuit.

Let's now understand the properties of the system at times $t \leq 0$. Since the current flowing through the circuit is $i(t \to 0, t < 0) = 0$, we have that

$$u_R(t \to 0, t < 0) = 0 \tag{1}$$
$$u_C(t \to 0, t < 0) = U_2. \tag{2}$$

We see therefore that the capacitor is charged - it has electrons on its plates that create a voltage equal to U_2.

At time $t = 0$, the switch is connected to voltage source U_1. Consequently, it must hold that:

$$U_1 = i(t = 0)R + u_C(t = 0). \tag{3}$$

The voltage at the capacitor is determined by the number of charges accumulated on its plates. A change in this voltage can only be introduced by bringing more charges, which takes time. Because of this, a capacitor does not respond instantaneously to a change in voltage.

Consequently, at $t = 0$, the voltage across the capacitor is the same as just before toggling the switch, and hence $u_C(t = 0) = u_C(t \to 0, t < 0) = U_2$.

From here, we can compute the current flowing through the circuit at $t = 0$:

$$i(t = 0) = \frac{U_1 - u_C(t = 0)}{R} = \frac{U_1 - U_2}{R} = \frac{200 + 200}{33.3}\,A = 12\,A. \tag{4}$$

We expect this initial current to lead to a build-up of charge across the capacitor that will lead to a change of voltage across the capacitor as well. Consequently, we expect the voltage drop across the capacitor to not be constant over time.

We can calculate what happens for $0 \leq t < \frac{T}{2}$, knowing that for this time-window we must fulfill that

$$U_1 = u_C + u_R \tag{5}$$

Knowing that $u_R = iR$, multiplying both sides with C and taking the time-derivative of both sides $\frac{d}{dt}$ we get to the equation

$$C \cdot \frac{dU_1}{dt} = C \cdot \frac{du_C}{dt} + RC \cdot \frac{di}{dt} \tag{6}$$

Since $U_1 = \text{const.}$, its time-derivative is zero $\frac{dU_1}{dt} = 0$ and we obtain

$$0 = i + RC\frac{di}{dt} \tag{7}$$

Solving this differential equation of first order in t, we find that the current flowing through the system is hence

$$i = i(t = 0)e^{-\frac{t}{RC}} \tag{8}$$

We can now compute the so-called "RC-time constant" $\tau_{RC} = 5$ ms, which is the time after which the current has decayed by a factor $1/e$ compared to its value at $t = 0$.

Having found the current flowing through the circuit, we can compute the time-dependent voltage drop across a capacitor using the formula $i = C\frac{du_C}{dt}$, and consequently $u_C(t) - u_C(0) = \frac{1}{C}\int_0^t i(t')dt'$:

$$u_C(t) = u_C(0) + \frac{1}{C}\int_0^t i(t')dt' \tag{9}$$

$$= u_C(0) + \frac{-RC}{C} i(t = 0)(e^{-\frac{t}{RC}} - 1) \tag{10}$$

At $t = \frac{T}{2}$, the switch connects again with voltage source U_2 with the circuit. We notice that by that time, the current flowing through the circuit will not have decayed to zero (since $\tau_{RC} > \frac{T}{2}$), but will have a value of $i(t \to \frac{T}{2}, t < \frac{T}{2}) = i(t = 0)e^{-\frac{T}{2RC}} = 6.58$ A.

We calculate the voltage drop across the capacitor as time is approaching $t = \frac{T}{2}$ (before the switch toggles) and find

$$u_C(t \to \frac{T}{2}, t < \frac{T}{2}) = u_C(0) + Ri(t = 0)(1 - e^{-\frac{T}{2RC}}) = -19.4\ V \tag{11}$$

Following the same argument as before, since $u_C(t = \frac{T}{2})$ can not change instantaneously when the switch toggles from U_1 back to U_2 at $t = \frac{T}{2}$, we have that $u_C(t = \frac{T}{2}) = u_C(t \to \frac{T}{2}, t < \frac{T}{2})$ and hence

$$u_C(t = \frac{T}{2}) = u_C(t \to \frac{T}{2}, t < \frac{T}{2}) = -19.4\ V \tag{12}$$

From here, we can now compute what happens at $t = \frac{T}{2}$. In order to adapt to the new voltage, the current i needs to have a jump such that

$$U_2 = i(t = \frac{T}{2})R + u_C(t = \frac{T}{2}) \tag{13}$$

and consequently $i(t = \frac{T}{2}) = \frac{U_2 - u_C(t = \frac{T}{2})}{R} = -5.42\ A$.

From here, we can compute what happens for $\frac{T}{2} \leq t < T$ by setting up a system of equations that is analogous to eq. 8:

$$U_2 = iR + u_C \tag{14}$$

$$C \cdot \frac{dU_2}{dt} = C \cdot \frac{du_C}{dt} + RC \cdot \frac{di}{dt} \tag{15}$$

$$0 = i + RC\frac{di}{dt} \tag{16}$$

$$i = i(t = \frac{T}{2})e^{-\frac{t-\frac{T}{2}}{RC}} \tag{17}$$

From here we can compute the current at $i(t \to T, t < T) = i(t = \frac{T}{2})e^{-\frac{T}{2RC}} = -2.97A$.

Applying $i = C\frac{du_C}{dt}$ for the time window $\frac{T}{2} \leq t < T$, we find $u_C(t) - u_C(\frac{T}{2}) = \frac{1}{C}\int_{\frac{T}{2}}^{t} i(t')dt'$ and consequently

$$u_C(t \to T, t < T) = u_C(\frac{T}{2}) + Ri(t = \frac{T}{2})(1 - e^{-\frac{T}{2RC}}) = -100.9\ V \tag{18}$$

b) This same procedure can be continued numerically e.g. in python for a few cycles, yielding a time dependency as shown in Fig. 3.

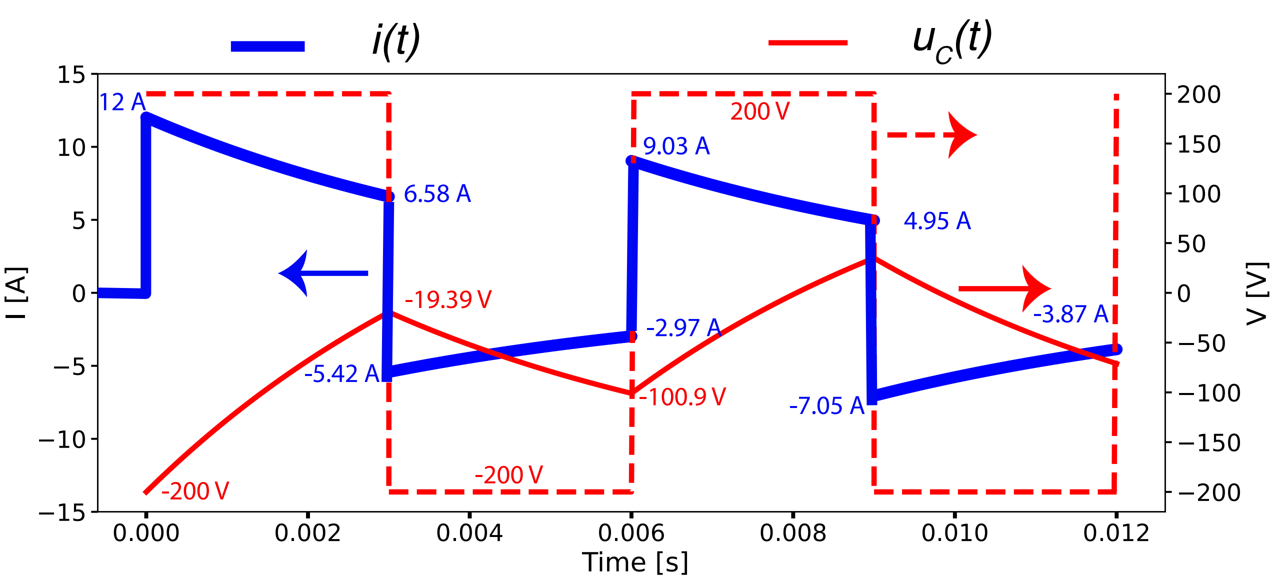

Figure 3: Time-dependency of currents and voltages. Current (blue), voltage drop across capacitor (red full line) and voltage applied to the RC-circuit (dashed red line).

We make a few interesting observations:

1. the current flowing through the systems experiences a few jumps, always when the voltage applied to the circuit is toggled between the two sources.

2. the current flowing through the circuit never reaches zero, because the RC-time constant is longer than the time span between two toggling events.

3. the voltage across the capacitor never reaches the voltage of the source before a toggling event happens. The system is therefore resilient to sudden jumps in voltage, especially for those that happen on time-scales shorter than the RC-time constant.

Solution 2

a) We recognize that in this case the time between two switching events ($\frac{T}{2}$) is larger than the RC-time constant since $\frac{T}{2} = 5\tau_{RC}$. Consequently, analyzing Eq. 8, we expect that after each switching event, there is sufficient time for the current to drop to zero. This behavior is accompanied by the fact that the capacitor slowly charges up, until the point where the voltage across the capacitor is equal to the value of the voltage source connected to the circuit.

b) We use the formulas we found in problem 1 to compute the currents and voltages:

$$u_C(t=0) = u_C(t \to 0, t < 0) = U_2 \tag{19}$$
$$i_C(t \to 0, t < 0) = 0 \tag{20}$$
$$i_C(t=0) = 12.012\ A \tag{21}$$
$$i_C(t \to \frac{T}{2}, t < \frac{T}{2}) = 0.08\ A \tag{22}$$
$$u_C(t \to \frac{T}{2}, t < \frac{T}{2}) = 197.31\ V \tag{23}$$
$$i_C(t = \frac{T}{2}) = -11.93\ A \tag{24}$$
$$u_C(t = \frac{T}{2}) = 197.31\ V \tag{25}$$
$$i_C(t \to T, t < T) = -0.08\ A \tag{26}$$
$$u_C(t \to T, t < T) = -197.31\ V \tag{27}$$

c) The currents and voltages are plotted in Fig. 4.

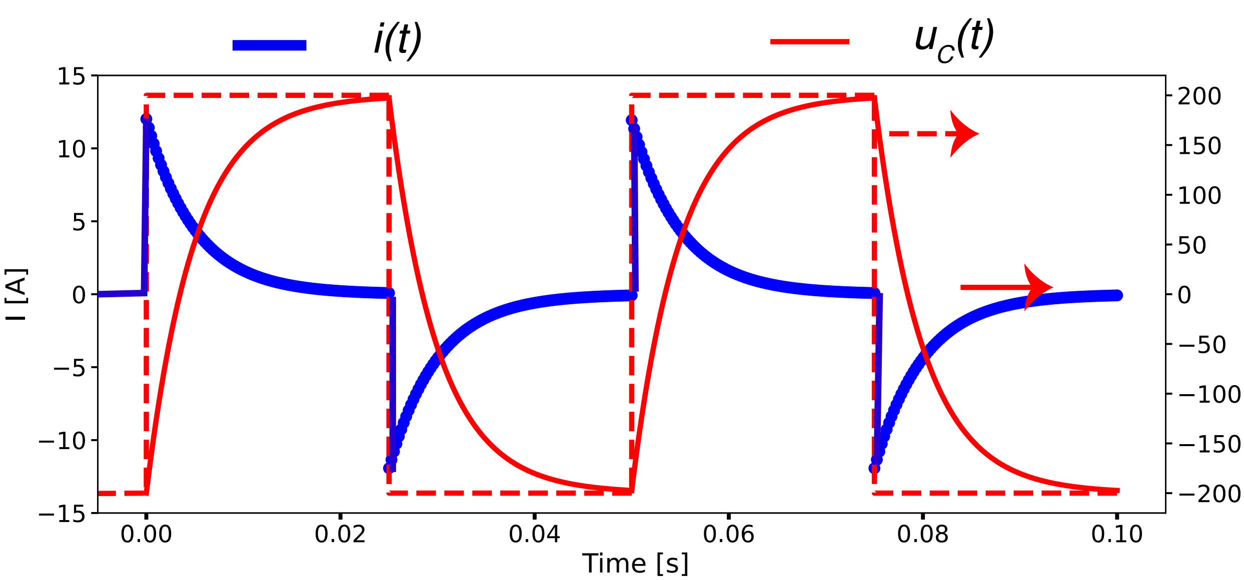

Figure 4: Time-dependency of currents and voltages. Current (blue), voltage drop across capacitor (red full line) and voltage applied to the RC-circuit (dashed red line).

We make a few interesting observations:

1. the current flowing through the systems experiences a few jumps, always when the voltage applied to the circuit is toggled between the two sources.
2. the current flowing through the circuit reaches zero, because the RC-time constant is shorter than the time span between two toggling events.

3. the voltage across the capacitor reaches the voltage of the source before a toggling event happens. The system is therefore smoothens out sudden jumps in voltage.