Exercise week #6
Unbalanced three-phase systems

Problem 1 (in class):

Consider the electrical circuit shown in Fig. 1 with the following parameters:

$U_l = 380$ V, $\alpha = 0$ and $R = \omega L = \frac{1}{\omega C} = 10\ \Omega$. The neutral line is not connected to point C.

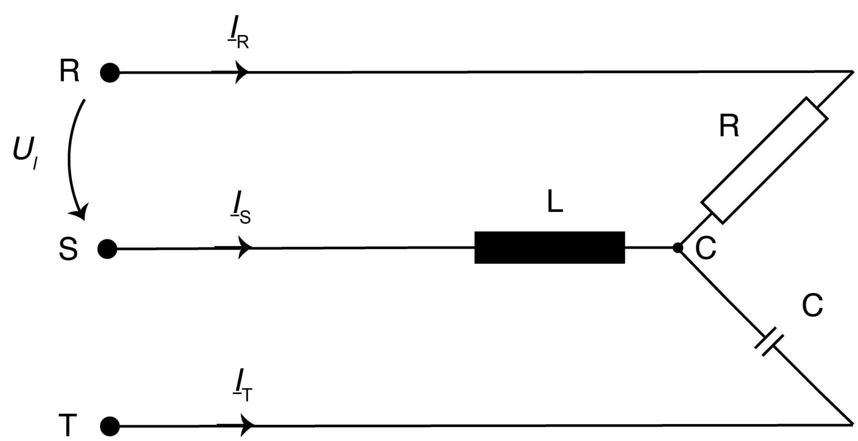

Figure 1: Unbalanced three-phase system.

a) Determine the three line currents $\underline{I}_R$, $\underline{I}_S$ and $\underline{I}_T$. What can you tell about their sum?

b) Compute $\underline{S}$, P and Q.

Problem 2 (self-study):

Consider the electrical circuits shown below with the following parameters:

$U_l = 400\ V$, $\alpha = 0$, $\underline{Z} = R + j\omega L$, $R = \omega L = 50\ \Omega$. The neutral line is not connected to point C.

The impedance on line T is short-circuited, as shown in Fig. 2.

a) With which other point is point C connected?

b) Calculate all line currents in the form $Ie^{j\beta}$.

In a second setting, the impedance connected to line T is disconnected from the circuit, as shown in Fig. 3.

a) Calculate all line currents in the form $Ie^{j\beta}$.

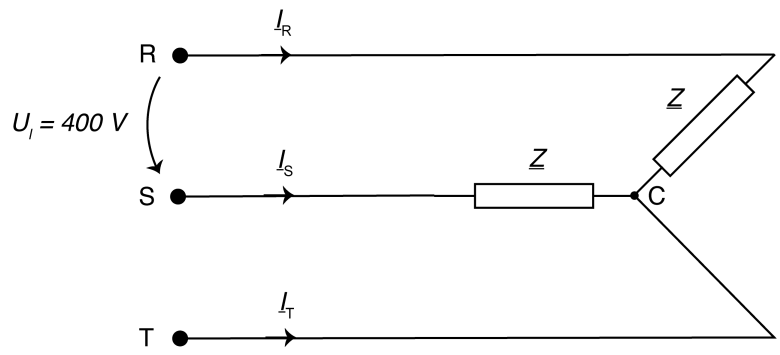

Figure 2: Electrical circuit featuring a shorted impedance on line T.

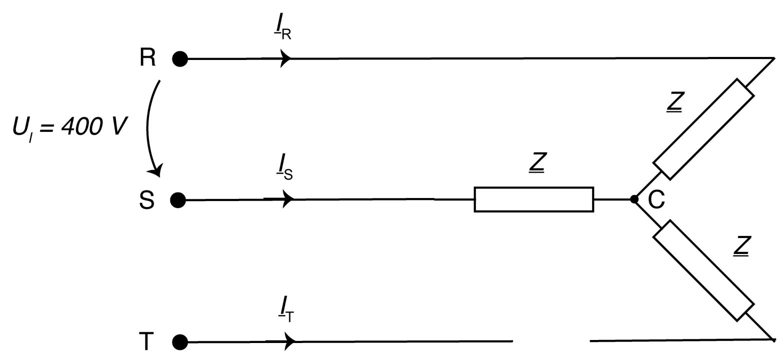

Figure 3: Electrical circuit featuring an open circuit on line T.

Solution 1:

a) We recognise that we have phase voltage of $U = 220$ V and hence, assuming that $\alpha = 0$ we have

$$\underline{U}_{RS} = \sqrt{3}Ue^{j\frac{\pi}{6}} = U_l e^{j\frac{\pi}{6}} \tag{1}$$

and similarly $\underline{U}_{ST} = \underline{U}_{RS}e^{-j\frac{2\pi}{3}}$ and $\underline{U}_{TR} = \underline{U}_{RS}e^{-j\frac{4\pi}{3}}$.
From this starting point, we can set up the system of linearly dependent equations

$$\underline{U}_{RS} = U_l e^{j\frac{\pi}{6}} = \underline{I}_R R - \underline{I}_S j\omega L \tag{2}$$

$$\underline{U}_{ST} = U_l e^{j\frac{\pi}{6}} e^{-j\frac{2\pi}{3}} = \underline{I}_S j\omega L - \underline{I}_T \frac{1}{j\omega C} \tag{3}$$

$$\underline{U}_{TR} = U_l e^{j\frac{\pi}{6}} e^{-j\frac{4\pi}{3}} = \underline{I}_T \frac{1}{j\omega C} - \underline{I}_R R \tag{4}$$

$$\underline{I}_R + \underline{I}_S + \underline{I}_T = 0. \tag{5}$$

We find therefore that $\underline{I}_T = -\underline{I}_R - \underline{I}_S$ and from eq. 4

$$(-\underline{I}_R - \underline{I}_s)\frac{1}{j\omega C} - \underline{I}_R R = U_l e^{-j\frac{7\pi}{6}} = -U_l e^{-j\frac{\pi}{6}} \tag{6}$$

$$\underline{I}_R(R + \frac{1}{j\omega C}) + \underline{I}_S \frac{1}{j\omega C} = U_l e^{-j\frac{\pi}{6}} \tag{7}$$

We multiply the last equation by $j\omega C$ on both sides and combine it with eq. 2 that we divide by $j\omega L$ into one system of equations

$$\underline{I}_R(j\omega CR + 1) + \underline{I}_S = U_l j\omega C e^{-j\frac{\pi}{6}} \tag{8}$$

$$\underline{I}_R \frac{R}{j\omega L} - \underline{I}_S = \frac{U_l}{j\omega L} e^{j\frac{\pi}{6}} \tag{9}$$

We add up the two equations ad find

$$\underline{I}_R = U_l \frac{j\omega C e^{-j\frac{\pi}{6}} + \frac{e^{j\frac{\pi}{6}}}{j\omega L}}{1 + j\omega CR + \frac{R}{j\omega L}} \tag{10}$$

We use now the fact that $R = \omega L = \frac{1}{\omega C}$ and hence $\omega CR = \frac{R}{\omega L} = 1$ and simplify:

$$\underline{I}_R = U_l j \frac{e^{-j\frac{\pi}{6}} - e^{j\frac{\pi}{6}}}{R(1 + j - j)} = \frac{U_l}{R} j(-2j)\sin(\frac{\pi}{6}) = \frac{U_l}{R} = 38 \text{ A.} \tag{11}$$

From here we compute

$\underline{I}_S = \underline{I}_R \frac{R}{j\omega L} - \frac{U_l}{j\omega L} e^{j\frac{\pi}{6}} = (-j38 + j38e^{j\frac{\pi}{6}})$ A $= (-19 + j38(-1 + \frac{\sqrt{3}}{2}))$ A and

$\underline{I}_T = -\underline{I}_R - \underline{I}_S = (-19 - j38(-1 + \frac{\sqrt{3}}{2}))$ A.

We plot these in Fig. 4 and find that their sum adds up to zero. The currents do not have the same magnitudes despite the magnitude of the impedances being the same.

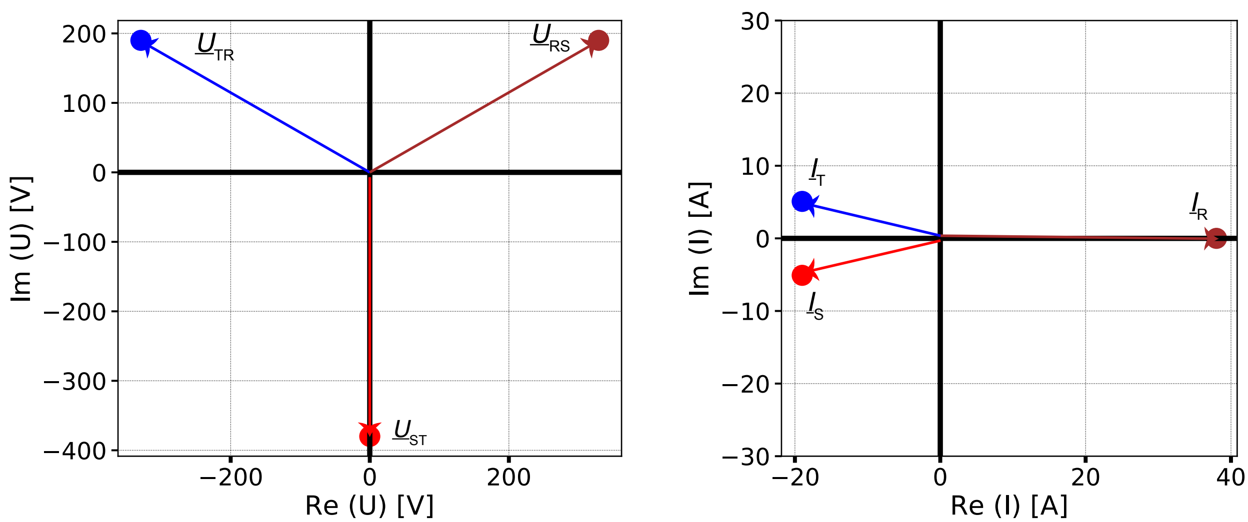

Figure 4: Phasors of line voltages and line currents.

b) The total power can be computed by summing up the contributions to the different loads:

$$\underline{S} = \underline{U}_{RC}\underline{I}_R^* + \underline{U}_{SC}\underline{I}_S^* + \underline{U}_{TC}\underline{I}_T^* \tag{12}$$

$$\underline{S} = R|\underline{I}_R|^2 + j\omega L|\underline{I}_S|^2 + \frac{1}{j\omega C}|\underline{I}_T|^2 \tag{13}$$

We notice that the two reactive powers cancel each other since $|I_T|^2 = |I_S|^2$ and $\omega L = \frac{1}{\omega C}$, yielding

$$\underline{S} = R|\underline{I}_R|^2 = 14440\ VA \tag{14}$$

and, consequently, $P = 14.44\ kW$ and $Q = 0$.

Solution 2:

For the shorted circuit

a) We start by remembering the relationship between phase voltage and line voltages: if $\underline{U}_{RN} = Ue^{j\alpha}$ and $\alpha = 0$ then we have $\underline{U}_{RS} = \sqrt{3}\underline{U}_{RN}e^{j\frac{\pi}{6}}$, $\underline{U}_{ST} = \sqrt{3}\underline{U}_{RN}e^{j(\frac{\pi}{6}-\frac{2\pi}{3})}$ and $\underline{U}_{TR} = \sqrt{3}\underline{U}_{RN}e^{j(\frac{\pi}{6}-\frac{4\pi}{3})}$. Since $\sqrt{3}U = U_l$, we then have $\underline{U}_{RS} = U_l e^{j\frac{\pi}{6}}$, $\underline{U}_{ST} = -jU_l$ and $\underline{U}_{TR} = -U_l e^{-j\frac{\pi}{6}}$.

Furthermore, we have $\underline{Z} = R + j\omega L = R(1+j) = R\sqrt{2}e^{j\frac{\pi}{4}}$.

If the impedance on the T line is short-circuited, point C coincides with point T. From here, we can write down the dependencies between line currents and phase voltages:

$$\underline{U}_{ST} = \underline{I}_S\underline{Z} \tag{15}$$

and hence $\underline{I}_S = \frac{\underline{U}_{ST}}{\underline{Z}} = \frac{-jU_l}{R\sqrt{2}e^{j\frac{\pi}{4}}} = \frac{U_l}{R\sqrt{2}}e^{-j\frac{3\pi}{4}} = 4\sqrt{2}e^{-j\frac{3\pi}{4}}\ A$.

We also have that

$$\underline{U}_{RT} = \underline{I}_R\underline{Z} \tag{16}$$

and consequently $\underline{I}_R = \frac{\underline{U}_{RT}}{\underline{Z}} = -\frac{\underline{U}_{TR}}{\underline{Z}} = \frac{U_l e^{-j\frac{\pi}{6}}}{R\sqrt{2}e^{j\frac{\pi}{4}}} = \frac{U_l e^{-j\frac{5\pi}{12}}}{R\sqrt{2}} = 4\sqrt{2}e^{-j\frac{5\pi}{12}}\ A$

Finally, we have that $\underline{I}_T = -\underline{I}_s - \underline{I}_R = -4\sqrt{2}(e^{-j\frac{3\pi}{4}} + e^{-j\frac{5\pi}{12}})\ A = 9.8e^{j\frac{5\pi}{12}}$. All currents are plotted in Fig. 5 (left panel).

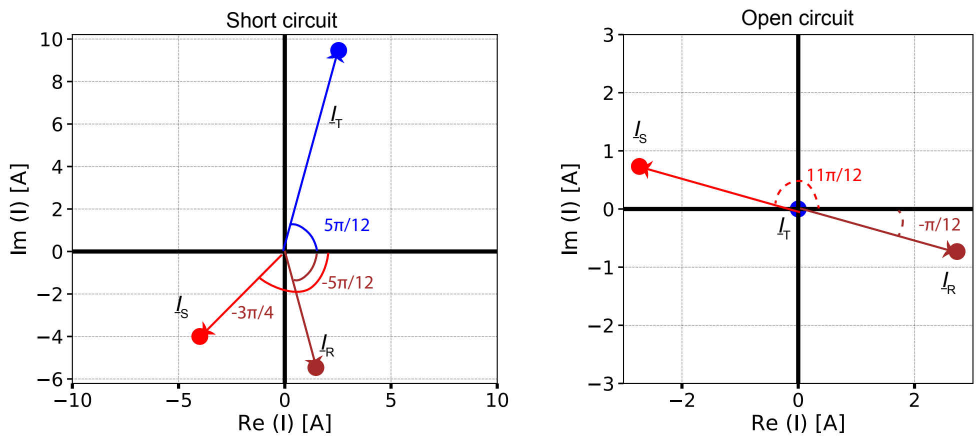

Figure 5: Currents for the shorted and open circuit.

For the open circuit

a) In the case of the open circuit, current $\underline{I}_T = 0$. Consequently, $\underline{I}_R = -\underline{I}_S$ and $\underline{U}_{RS} = \underline{I}_R\underline{Z} - \underline{I}_S\underline{Z} = 2\underline{I}_R\underline{Z}$. We can now compute the currents:

$$\underline{I}_R = -\underline{I}_S = \frac{\underline{U}_{RS}}{2\underline{Z}} = \frac{U_l e^{j\frac{\pi}{6}}}{2\sqrt{2}Re^{j\frac{\pi}{4}}} = 2\sqrt{2}e^{-j\frac{\pi}{12}}. \tag{17}$$

All currents are plotted in Fig. 5 (right panel). *Note:* A minus sign manifests itself as a phase difference of π.