Recap #1
Frequency dependency of impedances

Problem 1:

Consider the electronic circuit shown in Fig. 1.

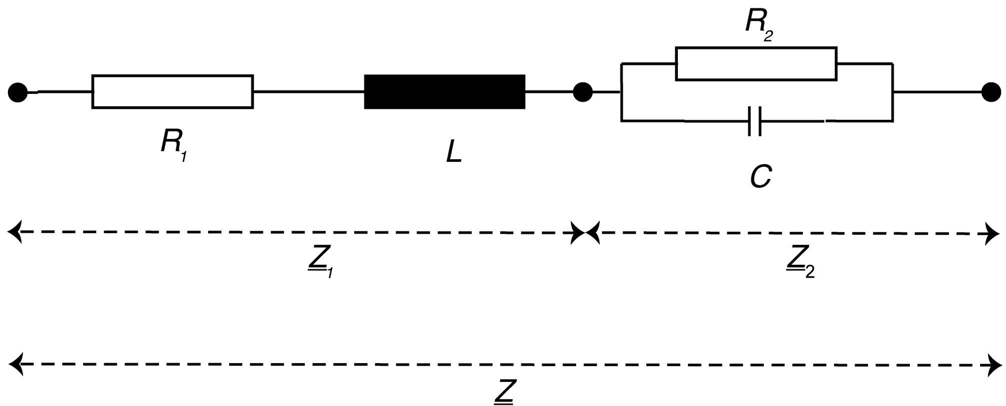

Figure 1: Electronic circuit to be considered.

with the following parameters: $L = 6.5$ mH, $C = 12\ \mu$F, $R_1 = 10\ \Omega$ and $R_2 = 33\ \Omega$.

a) Draw the possible values of the total impedance $\underline{Z}$ for all frequencies $\omega = 2\pi \times f$ in the complex plane.
Hint: Start by considering the admittance $\underline{Y}_2 = \frac{1}{\underline{Z}_2}$ *first.*

b) For which values of f has the circuit an inductive, resistive or capacitive character?

Solution 1:

a) The admittance $\underline{Y}_2$ corresponds to a parallel circuit of a resistor with resistance R_2 and a capacitor with a capacitance C, so we have

$$\underline{Y}_2 = \frac{1}{\underline{Z}_2} = \frac{1}{R_2} + j\omega C. \tag{1}$$

In the complex plane, the corresponding values of $\underline{Y}_2$ lie on a line that is in the upper half of the complex plane, parallel to the y-axis and has a constant value of $\text{Re}(\underline{Y}_2) = \frac{1}{R}$. Since $\underline{Z}_2 = \frac{1}{\underline{Y}_2}$, in order to find the values of $\underline{Z}_2$, we must take the inverse of this line, which corresponds to applying the inverse complex function $w(\underline{z}) = \frac{1}{\underline{z}}$ with z comprising all values of $\underline{Y}_2$. We know that the conformal mapping of a line that does not go through the origin through the function $w(\underline{z}) = \frac{1}{\underline{z}}$ is a circle, and in our case this is a half-circle (because the line describing the values of $\underline{Y}_2$ only exists in the upper half of the complex plane). We can now find a few points through which this half-circle is passing by evaluating $\underline{Z}_2(\omega = 0)$ and $\underline{Z}_2(\omega = \infty)$:

$$\underline{Z}_2(\omega = 0) = \frac{1}{\underline{Y}_2(\omega = 0)} = \frac{1}{1/R_2} = R_2 = 33\ \Omega \tag{2}$$

$$\underline{Z}_2(\omega = \infty) = \frac{1}{\underline{Y}_2(\omega = \infty)} = \frac{1}{j\infty} = -j0. \tag{3}$$

The circle will thus start at $\underline{Z}_2(\omega = 0) = R_2$ and end at $\underline{Z}_2(\omega = \infty) = -j0$, at zero but approaching from the side of $-j$. The radius of the circle is thus $\frac{R_2}{2}$. But let's convince ourselves that this makes sense. At zero frequency, the capacitor is equivalent to an open circuit, and hence all current will flow through the resistor alone. Consequently, the equivalent impedance $\underline{Z}_2(\omega = 0) = R_2$ makes perfect sense. With frequency increasing towards $\omega = \infty$, the capacitor tends towards being shorted while maintaining an imaginary impedance $\underline{Z}_2(\omega = \infty) = -j0$. Consequently, all current will end up flowing through this shorted capacitance. At intermediate currents, the current will split between the resistor and the capacitor.

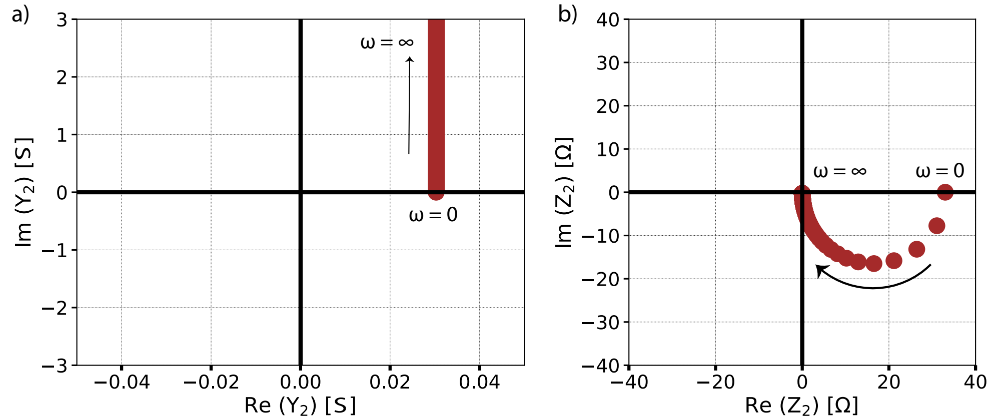

Figure 2: Possible values for $\underline{Z}_2$ as frequency $\omega = 2\pi f$ increases.

In a second step, we will evaluate $\underline{Z} = \underline{Z}_1 + \underline{Z}_2$. We find that $\underline{Z}_1 = R_1 + j\omega L$ which is now in the *impedance plane* also a line that exists in the upper half of the complex plane and starts at $R_1 = 10\ \Omega$ and is parallel to the y-axis. To find the total impedance, we must add this half line to the half circle shown in Fig. 2 b describing the values of Z_2. For this, we must keep in mind that we need to sum up the points on each curve corresponding to the same frequency.

As usual, we begin by evaluating the impedances at the two extreme frequencies, $\omega = 0$ and $\omega = \infty$ and find:

$$\underline{Z}(\omega = 0) = R_1 + R_2 \tag{4}$$

$$\underline{Z}(\omega = \infty) = R_1 + j\infty. \tag{5}$$

The start of the curve representing possible values of $\underline{Z}$ starts therefore at $\underline{Z}(\omega = 0) = R_1 + R_2$ and ends at $\underline{Z}(\omega = \infty) = R_1 + j\infty$. Since we're adding up a half circle to a half line we anticipate that, at intermediate frequencies ω, the impedance values will follow a rounded curve. The question is whether this curve goes below the x-axis or not. To answer this question, we can proceed mathematically and search whether there exists any frequency ω_x for which the imaginary part of $\Im\mathfrak{m}(\underline{Z}(\omega_x)) = 0$. Since

$$\underline{Z} = \underline{Z}_1 + \underline{Z}_2 = R_1 + j\omega L + \frac{R_2}{1 + j\omega C R_2} \tag{6}$$

$$\underline{Z} = R_1 + j\omega L + \frac{R_2}{1 + \omega^2 C^2 R_2^2} - j\omega \frac{C R_2^2}{1 + \omega^2 C^2 R_2^2} \tag{7}$$

Therefore, ω_x must satisfy $L = \frac{CR_2^2}{1+\omega_x^2 C^2 R_2^2}$ and hence $\omega_x = \sqrt{\frac{CR_2^2 - L}{C^2 R_2^2 L}} = 2\pi \times 404$ Hz. This equation has one (and a single) real solution only if $CR_2^2 \geq L$, which is the case given the parameters considered here.

The curve representing possible values of $\underline{Z}$ goes first below the x axis, to then cross it when $\omega = \omega_x$, where the impedance takes a value of $\underline{Z} = R_1 + \frac{R_2}{1+\omega^2 C^2 R_2^2}$.

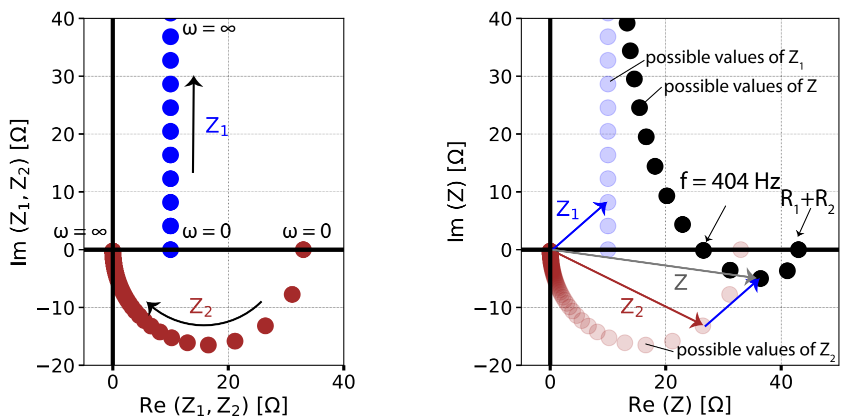

Figure 3: Possible values for $\underline{Z}$ as frequency $\omega = 2\pi f$ increases.

Hint: The math lovers among you can check out the optional section 4.6 in the written lecture notes describing tricks on how to map from the impedance plane to the admittance plane and vice-versa using conformal mapping through $f(\underline{z}) = \frac{1}{\underline{z}}$.

b) In conclusion, the total impedance $\underline{Z}$ has resistive values at $\omega = 2\pi \times 0$ Hz and $\omega = 2\pi \times 404$ Hz. At lower frequencies $\omega < 2\pi \times 404$ Hz, the possible values of $\underline{Z}$ are in the lower half of the complex plane and thus of capacitive nature. At frequencies higher than that $\omega > 2\pi \times 404$ Hz, the possible values of $\underline{Z}$ are in the upper half of the complex plane and $\underline{Z}$ has therefore inductive character.